Оглавление

Глава 1

Определённый интеграл

1.1 Интегрирование по Риману

Утверждение 1. $[a, b], \quad P = \{x_k\}_{k=1}^{\infty}, \quad f$

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1})$$

Где $M_k := \sup\limits_{x \in [x_{k-1}, x_k]} \{ f(x) \}, \quad m_k := \inf\limits_{x \in [x_{k-1}, x_k]} \{ f(x) \}$

$$0 \leq M_k - m_k = \omega_k = \omega_f([x_{k-1}, x_k])$$

$$f \in \mathcal{R}([a, b]) \iff \forall \varepsilon > 0 \exists P$$

$$\sum_{k=1}^{n} \omega_k(x_k - x_{k-1}) < \varepsilon$$

Теорема 1 (Критерий интегрируемости по Риману)**.** $c \in (a, b)$

$$f \in \mathcal{R}([a, b]) \iff \begin{cases} f \in \mathcal{R}([a, c]) \\ f \in \mathcal{R}([c, b]) \end{cases}$$

Доказательство.

- $\implies$

$$P = \{x_k\}_{k=0}^{n}, \qquad x_0 = a, \quad x_n = b, \qquad c \in P$$

$$\forall \varepsilon > 0 \quad \sum_{k=1}^{n} \omega_k(x_k - x_{k-1}) < \varepsilon$$

$$P_1 = \{x_k\}_{k=0}^{l}, \qquad P_2 = \{x_k\}_{k=l}^{n}$$

$$\sum_{k=1}^{n} \ldots = \sum_{k=1}^{l} \ldots + \sum_{k=l+1}^{n} \ldots$$

$$\sum_{k=1}^{l} \omega_k(x_k - x_{k-1}) < \varepsilon, \qquad \sum_{k=l+1}^{n} \omega_k(x_k - x_{k-1}) < \varepsilon$$

- $\impliedby$

$$\widetilde{P_1} = \{x_k\}_{k=0}^{m} \quad x_0 = 0, \quad x_m = c, \qquad \widetilde{P_2} = \{k_l\}_{l=m}^{m+q}$$

$$\sum_{k=1}^{m} \omega_k(x_k - x_{k-1}) < \frac{\varepsilon}{2}, \qquad \sum_{k=m+1}^{m+q} \omega_k(x_k - x_{k-1}) < \frac{\varepsilon}{2}$$

$$\widetilde{P} = \widetilde{P_1} \cup \widetilde{P_2}$$

$$\sum_{k=1}^{m+q} \omega_k (x_k - x_{k-1}) = \sum_{k=1}^{m} \omega_k (x_k - x_{k-1}) + \sum_{k=m+1}^{m+1} \omega_k (x_k - x_{k-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Утверждение 2. $\sup\{ f+g \} \le \sup\{ f \} + \sup\{ g \}, \qquad \inf\{ f+g \} \ge \inf\{ f \} + \inf\{ g \}$

Доказательство.

$$\left.\begin{array}{l} \forall \delta > 0 \quad \exists x_* \in [\alpha, \beta] : f(x_*) + g(x_*) > \sup\{ f+g \} - \delta \\ \sup\{ f \} + \sup\{ g \} \ge f(x_*) + g(x_*) \end{array}\right\} \implies$$

$$\implies \sup\{ f \} + \sup\{ g \} > \sup\{ f+g \} - \delta \implies \sup\{ f \} + \sup\{ g \} \ge \sup\{ f+g \}$$

□

Свойства.

1. $f \in \mathcal{R}([a,b])$

$$c \ne 0 \implies cf \in \mathcal{R}([a,b])$$

Доказательство.

- $c > 0$

$$\forall [x_{k-1}, x_k] \subset [a,b] \quad \omega_{cf}([x_{k-1}, x_k]) = c\omega_f([x_{k-1}, x_k])$$

$$\sum_{k=1}^{n} \omega_{cf}([x_{k-1}, x_k])(x_k - x_{k-1}) = c \sum_{k=1}^{n} \omega_f([x_{k-1}, x_k])(x_k - x_{k-1}) < c\varepsilon$$

- $c < 0 \qquad c = (-1) \cdot |c|$ (см. следующе свойство)

□

2. $\omega_{-f}([x_{k-1}, x_k]) = \omega_f([x_{k-1}, x_k])$

Доказательство.

$$\sup\{ -f \} = -\inf\{ f \}, \qquad \inf\{ -f \} = -\sup\{ f \}$$

$$\sup\{ -f \} - \inf\{ -f \} = \sup\{ f \} - \inf\{ f \}$$

□

3. $f \in \mathcal{R}([a,b]), \qquad g \in \mathcal{R}([a,b]) \qquad \implies \qquad f+g \in \mathcal{R}([a,b])$

Доказательство.

$$\left.\begin{array}{l} \sup\{ f+g \} \le \sup f + \sup\{ g \} \\ \inf\{ f+g \} \ge \in \{ f \} + \inf\{ g \} \end{array}\right\} \implies \omega_{f+g}([x_{k-1}, x_k]) \le \omega_f([x_{k-1}, x_k]) + \omega_g([x_{k-1}, x_k])$$

(см. утверждение 2)

$$\sum_{k=1}^{n} \omega_{f+g}([x_{k-1}, x_k])(x_k - x_{k-1}) \le$$

$$\le \sum_{k=1}^{n} \omega_f([x_{k-1}, x_k])(x_k - x_{k-1}) + \sum_{k=1}^{n} \omega_g([x_{k-1}, x_k])(x_k - x_{k-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

4. $\omega_{|f|}([x_{k-1}, x_k]) \le \omega_f([x_{k-1}, x_k])$

5. $f \in \mathcal{R}([a,b]) \implies |f| \in \mathcal{R}([a,b])$ (в обратную сторону **неверно**)

Доказательство. Следует из свойства 4 $\square$

6. $f \in \mathcal{R}([a,b]) \implies f^2 \in \mathcal{R}([a,b])$

Доказательство. $f^2 = |f|^2$
Будем считать, что $f(x) \geq 0, f \in \mathcal{R}([a,b])$, значит, $\forall x \in [a,b] \quad f(x) \leq M$
Положим $[\alpha,\beta] \subset [a,b]$
Возьмём $x_1, x_2 \in [\alpha,\beta] : f(x_2) \geq f(x_1) \geq 0$

$$f^2(x_2) - f^2(x_1) = \Big(f(x_2) - f(x_1)\Big)\Big(f(x_2) + f(x_1)\Big) \leq 2M\Big(f(x_2) - f(x_1)\Big)$$

$$\left.\begin{aligned} f(x_2) &\leq \sup\{\, f \,\} \\ f(x_1) &\geq \inf\{\, f \,\} \end{aligned}\right\} \implies 2M\Big(f(x_2) - f(x_1)\Big) \leq 2M\omega_f([\alpha,\beta]) \tag{1.1}$$

$$\forall \delta > 0 \quad f^2(x_2) > \sup_{[\alpha,\beta]}\{\, f^2 \,\} - \delta, \qquad \forall \delta > 0 \quad f^2(x_2) < \inf_{[\alpha,\beta]}\{\, f^2 \,\} + \delta$$

$$2M\omega_f([\alpha,\beta]) \underset{(1.1)}{\geq} f^2(x_2) - f^2(x_1) > \sup\{\, f^2 \,\} - \inf\{\, f^2 \,\} - \delta$$

$$\omega_{f^2}([\alpha,\beta]) < 2M\omega_f([\alpha,\beta]) + 2\delta \implies \omega_{f^2}([\alpha,\beta]) \leq 2M\omega_f([\alpha,\beta]) \implies$$

$$\implies \sum_{k=1}^{n} \omega_{f^2}([x_k - x_{k-1}])(x_k - x_{k-1}) < 2M\varepsilon$$

$\square$

7. $f, g \in \mathcal{R}([a,b]) \implies fg \in \mathcal{R}([a,b])$

Очевидно, что. $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$
Все части этого выражения интегрируемы по Риману $\square$

Свойства.

1. $f \in C([a,b]) \implies f \in \mathcal{R}([a,b])$

Доказательство. По теореме Кантора, f равномерно непрерывна на $[a,b]$. Значит,

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \underset{0 < x_2 - x_1 < \delta}{x_1, x_2 \in [a,b]} \quad |f(x_2) - f(x_1)| < \varepsilon$$

$$x_k = a + \frac{b-a}{n}k, \qquad k = 0, 1, \dots, n$$

$$x_k - x_{k-1} < \delta$$

По второй теореме Веейрштрасса,

$$\begin{cases} \exists x_k^+ \in [x_{k-1}, x_k] : \forall x \in [x_{k-1}, x_k] \quad f(x) \leq f(x_k^+) \\ \exists x_k^- \in [x_{k-1}, x_k] : \forall x \in [x_{k-1}, x_k] \quad f(x) \geq f(x_k^-) \end{cases}$$

$$\omega_f([x_{k-1}, x_k]) = f(x_k^+) - f(x_k^-)$$

$$|x_k^+ - x_k^-| \leq x_k - x_{k-1} < \delta \implies f(x_k^+) - f(x_k^-) < \varepsilon$$

$$\sum_{k=1}^{n} \omega_f([x_{k-1}, x_k])(x_k - x_{k-1}) < \sum_{k=1}^{n} \varepsilon(x_k - x_{k-1}) = \varepsilon(b-a)$$

$\square$