

# Judgements

- Typing:  $\Gamma \vdash e \Rightarrow A \dashv \Gamma'$  – in context  $\Gamma$ , term  $e$  infers type  $A$ , with output context  $\Gamma'$
- Matching:  $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$  – in context  $\Gamma$ , type  $A$  matches (output) type  $B$ , with output context  $\Gamma'$
- Subtyping:  $\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ ,  $s$  copies of subtype  $A$  are needed to produce  $r$  copies of supertype  $B$ , with output context  $\Gamma'$ .
- Sub-instantiation:  $\Gamma \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ , instantiate the existential variable  $\hat{a}$  to a subtype of  $\mathbf{!}_r A$ , lacking  $s$ , returning context  $\Gamma'$
- Super-instantiation:  $\Gamma \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ , instantiate the existential variable  $\hat{a}$ , so that  $A <: \mathbf{!}_r \hat{a}$ , lacking  $s$ , returning context  $\Gamma$

# New kinding

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} r \Rightarrow A \Rightarrow 1 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash_{\text{c}} r \Rightarrow A \Rightarrow r/s \dashv \Gamma'}$$

# Variables and annotations

$$\frac{\Gamma(x) = A \quad 0\Gamma + x \vdash \langle 0\Gamma + x \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_c x \Rightarrow B \dashv \Gamma' + \Gamma}$$

$$\frac{\Gamma(x) = A \quad \Gamma \vdash \langle \Gamma \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{nc} x \Rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \begin{array}{c} 0\Gamma_1, \square : A \vdash_i e \Rightarrow A' \dashv \Gamma_2 \\ \Gamma_2 \vdash \langle \Gamma_2 \rangle A' \rightsquigarrow B \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i (e : A) \Rightarrow B \dashv \Gamma_3 + \Gamma_1}$$

# Stationary rules

$$\frac{\Gamma, \Delta, a, \Box : A \vdash_i e \Rightarrow A' \dashv \Gamma', a, \Theta}{\Gamma, \Box : \forall a. A, \Delta \vdash_i e \Rightarrow \forall a. A \dashv \Gamma'}$$

$$\frac{0\Gamma, 0\Delta, \Box : A \vdash_i e \Rightarrow A' \dashv \Gamma'}{\Gamma, \Box : !_r A, \Delta \vdash_i e \Rightarrow !_r A \dashv r \Gamma' + (\Gamma, \Delta)}$$

# Application

$$\frac{\Gamma, \Box e_2 \vdash_i e_1 \Rightarrow A \dashv \Gamma'}{\Gamma \vdash_i e_1 \ e_2 \Rightarrow A \dashv \Gamma'}$$

# Functions

$$\frac{\Gamma, \Delta, 0x : A, \square : B \vdash_i e \Rightarrow B' \dashv \Gamma_1, rx : A, \Theta \quad \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle A \Rightarrow 1 \dashv \Gamma_2}{\Gamma, \square : A \rightarrow B, \Delta \vdash_i \lambda x. e \Rightarrow A \rightarrow B \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma \vdash_i e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta, 0x : A \vdash_i e \Rightarrow B \dashv \Gamma_2, rx : A, \Theta \\ \Gamma_2 \vdash_i r \Rightarrow \langle \Gamma_2 \rangle A \Rightarrow 1 \dashv \Gamma_3 \end{array}}{\Gamma, \square e', \Delta \vdash_i \lambda x. e \Rightarrow B \dashv \Gamma_3}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \rightarrow \hat{a}_2], \Delta, 0x : \hat{a}_1, \square : \hat{a}_2 \vdash_i e \Rightarrow A \dashv \Gamma_1, rx : \hat{a}_1, \Theta \\ \Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle \hat{a}_1 \Rightarrow 1 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}], \square : \hat{a}, \Delta \vdash_i \lambda x. e \Rightarrow \hat{a} \dashv \Gamma_2}$$

$$\Gamma, \Delta \blacktriangleright_{\hat{a}}, \hat{a}, 0x : \hat{a}, \square \vdash_i e \Rightarrow A \dashv \Gamma_1, rx : \hat{a}, \Gamma_4$$

$$\Gamma_1 \vdash_i r \Rightarrow \langle \Gamma_1 \rangle \hat{a} \Rightarrow 1 \dashv \Gamma_2, \blacktriangleright_{\hat{a}}, \Gamma_3$$

$$B = \forall \text{unsolved}(\Gamma_3). \langle \Gamma_3 \rangle (\hat{a} \rightarrow \forall \text{unsolved}(\Gamma_4). \langle \Gamma_4 \rangle A)$$

$$\Gamma, \square, \Delta \vdash_i \lambda x. e \Rightarrow B \dashv \Gamma_2$$

# Matching

$$\frac{}{\Gamma, \square, \Delta \vdash A \rightsquigarrow A \dashv \Gamma, \Delta}$$

$$\frac{\Gamma, \Delta \vdash 1 \Rightarrow A <: B \Rightarrow r \dashv \Gamma'}{\Gamma, \square : B, \Delta \vdash A \rightsquigarrow B \dashv r \Gamma'}$$

$$\frac{\Gamma, \square : A \vdash_i e \Rightarrow A' \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash B \rightsquigarrow B' \dashv \Gamma_2}{\Gamma, \square e, \Delta \vdash A \rightarrow B \rightsquigarrow B' \dashv \Gamma_2}$$

# Matching

$$\frac{\Gamma, \Box e, \Delta \vdash A \rightsquigarrow B \dashv \Gamma'}{\Gamma, \Box e, \Delta \vdash !_r A \rightsquigarrow B \dashv \Gamma'}$$

$$\frac{\Gamma, \Box e, \Delta, \hat{a} \vdash A[a := !_r \hat{a}] \rightsquigarrow B \dashv \Gamma'}{\Gamma, \Box e, \Delta \vdash \forall a : \text{Type}_r. A \rightsquigarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i r \Rightarrow B \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash A[a := B] \rightsquigarrow C \dashv \Gamma_2}{\Gamma, \Box B, \Delta \vdash \forall @a : \text{Type}_r. A \rightsquigarrow C \dashv \Gamma_2}$$

# Subtyping

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Unit} <: \text{Unit} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Empty} <: \text{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \otimes B_1 <: A_2 \otimes B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \oplus B_1 <: A_2 \oplus B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A_2 <: A_1 \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 \Rightarrow r \dashv \Gamma_2}$$

# Subtyping

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \hat{a} \Rightarrow r \dashv \Gamma[\hat{a}]}$$

$$\frac{}{\Gamma[a : \text{Type}_s] \vdash r \Rightarrow a <: a \Rightarrow r/s \dashv \Gamma[a : \text{Type}_s]}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \text{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow A <: \forall a : \text{Type}_s. B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, \blacktriangleright_{\hat{a}}, \hat{a} \vdash r \Rightarrow A[a := !_s \hat{a}] <: B \Rightarrow t \dashv \Gamma', \blacktriangleright_{\hat{a}}, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \text{Type}_s. A <: B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \text{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow \forall @a : \text{Type}_s. A <: \forall @a : \text{Type}_s. B \Rightarrow t \dashv \Gamma'}$$

# Subtyping

$$\frac{\Gamma \vdash r \cdot s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow A <: !_s B \Rightarrow t \dashv \Gamma'}$$

$$\frac{B \neq !_{\_} B' \quad \Gamma \vdash r/s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow !_s A <: B \Rightarrow t \dashv \Gamma'}$$

# Subtyping – switch to instantiation

$$\frac{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma' \quad \hat{a} \text{ fresh in } A}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'}$$

$$\frac{\Gamma[\hat{a}] \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma' \quad \hat{a} \text{ fresh in } A}{\Gamma[\hat{a}] \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'}$$

## Sub-instantiation

$$\frac{\Gamma \vdash_i r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \hat{a} <: \tau \Rightarrow 1 \dashv \Gamma_1, \hat{a} = !_s \tau, \Gamma'}$$

$$\frac{}{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{a} <: \hat{b} \Rightarrow r \dashv \Gamma[\hat{a}][\hat{b} = \hat{a}]}$$

$$\frac{}{\Gamma[b : \text{Type}_s][\hat{a}] \vdash r \Rightarrow \hat{a} <: b \Rightarrow 1 \dashv \Gamma[b : \text{Type}_s][\hat{a} = !_{r/s} b]}$$

$$\frac{\Gamma[\hat{a}] \vdash r \cdot s \Rightarrow \hat{a} <: A \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: !_s A \Rightarrow t \dashv \Gamma'}$$

Note: no rule for implicit quantifier.

## Sub-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \text{Unit} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \text{Empty} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Empty}]}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 <: A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \otimes B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 <: A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \oplus B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = !_r(\hat{a}_1 \rightarrow \hat{a}_2)] \vdash 1 \Rightarrow A <: \hat{a}_1 \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \hat{a}_2 <: \langle \Gamma_1 \rangle B \Rightarrow 1 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \rightarrow B \Rightarrow 1 \dashv \Gamma_2}$$

# Super-instantiation

$$\frac{\Gamma \vdash_i r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \tau <: \hat{a} \Rightarrow s \dashv \Gamma_1, \hat{a} = \tau, \Gamma'}$$

$$\overline{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{b} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a}][\hat{b} = !_r \hat{a}]}$$

$$\overline{\Gamma[b : \text{Type}_s][\hat{a}] \vdash r \Rightarrow b <: \hat{a} \Rightarrow r/s \dashv \Gamma[b : \text{Type}_s][\hat{a} = b]}$$

$$\frac{\Gamma[\hat{a}], \blacktriangleright_{\hat{b}}, \hat{b} \vdash r \Rightarrow B[b := \hat{b}] <: \hat{a} \Rightarrow s \dashv \Gamma', \blacktriangleright_{\hat{b}}, \Theta}{\Gamma[\hat{a}] \vdash r \Rightarrow \forall b. B <: \hat{a} \Rightarrow s \dashv \Gamma'}$$

$$\frac{\Gamma[\hat{a}] \vdash r/s \Rightarrow A <: \hat{a} \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow !_s A <: \hat{a} \Rightarrow t \dashv \Gamma'}$$

# Super-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \text{Unit} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \text{Empty} <: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Empty}]}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow A <: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \otimes B <: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow A <: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \oplus B <: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \rightarrow \hat{a}_2] \vdash 1 \Rightarrow \hat{a}_1 <: A \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B <: \hat{a}_2 \Rightarrow 1 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \rightarrow B <: \hat{a} \Rightarrow r \dashv \Gamma_2}$$