

## T-Axi (algorithmic)

Wojciech Kołowski  
Mateusz Pyzik

## Algorithmic terms

## Terms:

e :=

```

(e : A) | x | λ x. e | e1 e2 |
box e | let box x = e1 in e2 |
(e1, e2) | let (x, y) = e1 in e2 |
inl e | inr e | case e of {x.e1; y.e2} |
unit | let unit = e1 in e2 |
Empty-elim e |
letr x = e1 in e2 |
Λ a. e | e A |
Λ {a}. e | e @A |
choose p | choose-witness x h for p in

```

The algorithmic terms are almost the same to the declarative ones, but there are far fewer annotations.

## Recovering annotated terms

We can recover some of the original terms by combining algorithmic terms with annotations.

- $\text{let}_A \text{ unit} = e_1 \text{ in } e_2 \equiv \text{let unit} = e_1 \text{ in } (e_2 : A)$
  - $\text{Empty-elim}_A e_1 \equiv (\text{Empty-elim } e_1 : A)$
  - $\text{let}_r x : A = e_1 \text{ in } e_2 \equiv \text{let}_r x = (e_1 : A) \text{ in } e_2$

However, as long as annotations must be types (and not partial types, for example), we cannot recover more, like  $\text{inl}_A e$ , because we don't have partial annotations.

## Algorithmic prooftypes

Proofterms ( $P, Q$  are propositions,  $e$  are terms,  $h$  are variables):

*p, q ::=*

*h* | assumption | trivial | absurd | *p*

**assume**  $h$  **in**  $q$  | **apply**  $p_1\ p_2$  |

**both**  $p_1$   $p_2$  | **and-left**  $p$  | **and-right**  $p$

**or-left**  $p$  | **or-right**  $p$  | **cases**  $p_1\ p_2\ p_3$

**lemma**  $h : P$  **by**  $p$  **in**  $q$  | **proving**  $P$  **by**  $p$  |

suffices  $P$  by  $q$  in  $p$  |

**pick-any**  $x$  in  $e$  | **instantiate**  $p$  with  $e$

witness  $e$  such that  $p \mid \text{pick-witness } x \cdot h \text{ for } p_1 \text{ in } p_2 \mid$

**refl** | **rewrite**  $p_1$  **at**  $x.P$  **in**  $p_2$  | **funext**  $x$  **in**  $p$

by-contradiction  $h$  in  $q$  |

**choose-spec**  $p$  | **choose-witness**  $x\ h$  for  $p$  in  $q$

# Recovering annotated proofterms

We can recover some of the original proofterms by combining algorithmic proofterms with annotations.

- **refl**  $e \equiv \text{proving } e = e \text{ by refl}$
  - **by-contradiction**  $h : \neg P \text{ in } q \equiv \text{proving } P \text{ by by-contradiction } h \text{ in } q$

# Judgements

Kinding inference judgement:  $\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma'$

Type conversion judgements:  $\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma'$  (all types),  
 $\Gamma \vdash A \hat{\equiv} B \Rightarrow \text{Type}_r \dashv \Gamma'$  (types in whnf)

Type checking judgement:  $\Gamma \vdash; e \Leftarrow A \dashv \Gamma'$

Type inference judgement:  $\Gamma \vdash; e \Rightarrow A \dashv \Gamma'$

Term conversion judgements:  $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$  (all terms),  
 $\Gamma \vdash e_1 \hat{\equiv} e_2 \Leftarrow A \dashv \Gamma'$  (whnfs),  $\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \dashv \Gamma'$  (neutral terms),  
 $\Gamma \vdash n_1 \hat{\equiv} n_2 \Rightarrow A \dashv \Gamma'$  (neutral terms, type in whnf)

Proposition checking judgement:  $\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma'$

Proposition conversion judgements:  $\Gamma \vdash P \equiv Q \dashv \Gamma'$  (all propositions),  
 $\Gamma \vdash P \hat{\equiv} Q \dashv \Gamma'$  (whnfs)

# Auxiliary judgements

Kind checking judgement:  $\Gamma \vdash A \Leftarrow \text{Type}_r \dashv \Gamma'$

Type conversion (checking) judgement:  $\Gamma \vdash A \equiv B \Leftarrow \text{Type}_r \dashv \Gamma'$

## Subtraction of quantities

$r_1 - r_2$  is the least  $r'$  such that  $r_1 \sqsubseteq r' + r_2$ .

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

## Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$  holds when  $r_2 - r_1$  is defined.

Explicitly:  $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$

## Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$  holds when  $r_2 - 1 = r_1$ .

$*$   $\leq_{\text{dec}}$   $+$

$$\overline{0 \leq_{\text{dec}} 1}$$

$\overline{0 \leq_{\text{dec}} ?}$

## Arithmetic order on quantities

The arithmetic order on quantities is  $0 \leq 1 \leq ? \leq + \leq *$ . The idea is to compare the quantities by how “big” they are.

## Division with remainder

$a/b = (q, r)$  when  $a = b \cdot q + r$ , with  $q$  as large as possible and  $r$  being as small as possible according to the arithmetic order. Note that  $a/b = q$  means that  $r = 0$ .

$r_1/r_2$	0	1	?	+	*
0	*	0	0	0	0
1	(*, 1)	1	(0, 1)	(0, 1)	(0, 1)
?	(*, ?)	?	?	(0, ?)	(0, ?)
+	(*, +)	+	(*, 1)	+	(*, 1)
*	(*, *)	*	*	*	*

## Decrement variable in context

$\cdot - x = \text{undefined}$

$$(\Gamma, r \times : A) - x = \Gamma, (r - 1) \times : A$$

$$(\Gamma, r\,y : A) - x = \Gamma - x, r\,y : A$$

$$(\Gamma, r \, x : A := e) - x = \Gamma, (r - 1) \, x : A := e$$

$$(\Gamma, r\,y : A := e) - x = \Gamma - x, r\,y : A := e$$

$$(\Gamma, h : P) - x = \Gamma - x, h : P$$

$$(\Gamma, a : \text{Type}_r) - x = \Gamma - x, a : \text{Type}_r$$

## Context division with remainder

$$\overline{\cdot/r} = \cdot$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, rx : A)/q = ((\Gamma_1, r_1 x : A), (\Gamma_2, r_2 x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r \times :A := e)/q = ((\Gamma_1, r_1 \times :A := e), (\Gamma_2, r_2 \times :A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma / q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \text{Type}_r) / q = ((\Gamma_1, a : \text{Type}_r), (\Gamma_2, a : \text{Type}_r))}$$

## Kind inference

$$\Gamma \vdash \text{Unit} \Rightarrow \text{Type} \dashv \Gamma \quad \Gamma \vdash \text{Empty} \Rightarrow \text{Type} \dashv \Gamma$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash r A \rightarrow B \Rightarrow \text{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash !_0 A \Rightarrow \text{Type} \dashv \Gamma'} \quad \frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma' \quad r \neq 0}{\Gamma \vdash !_r A \Rightarrow \text{Type}_{r,s} \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \otimes B \Rightarrow \text{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \oplus B \Rightarrow \text{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

# Kind inference

$$\frac{(a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \Rightarrow \text{Type}_r \dashv \Gamma}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall a : \text{Type}_r. B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall \{a : \text{Type}_r\}. B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

## Kind checking

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \Leftarrow \text{Type}_r \dashv \Gamma'}$$

# Context clean-up

$\Gamma, rx : A \vdash; e \Leftarrow A \dashv \Gamma', 0x : A$  is a shorthand for  
 $\Gamma, rx : A \vdash; e \Leftarrow A \dashv \Gamma', r'x : A$  with the additional condition  
 $r' \sqsubseteq 0$  when  $i = c$

$\Gamma, rx : A \vdash; e \Rightarrow A \dashv \Gamma', 0x : A$  is a shorthand for  
 $\Gamma, rx : A \vdash; e \Rightarrow A \dashv \Gamma', r'x : A$  with the additional condition  
 $r' \sqsubseteq 0$  when  $i = c$

## Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2} \text{ANNOT}$$

## Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_c x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{\text{nc}} x \rightarrow A \dashv \Gamma}$$

# Functions

$$\frac{\Gamma, r_A^i x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0x : A}{\Gamma \vdash_i \lambda x. e \Leftarrow rA \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1/r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f \ a \Rightarrow B \dashv \Gamma_3 + r \Gamma_4}$$

## Box

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \text{box } e \Leftarrow !_r A \dashv \Gamma_2 + r \Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0 x : A}{\Gamma \vdash_i \text{let box } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash_i e \Leftarrow \text{Empty} \dashv \Gamma'}{\Gamma \vdash_i \text{Empty-elim } e \Leftarrow A \dashv \Gamma'}$$

# Unit

$$\frac{}{\Gamma \vdash; \text{unit} \Leftarrow \text{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash; \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

## Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1_A^i x : A, 1_B^i y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0x : A, 0y}{\Gamma \vdash_i \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

# Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash_i \text{inl } e \Leftarrow A \oplus B \dashv \Gamma'} \quad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash_i \text{inr } e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \frac{\Gamma_1, 1_A^i x : A \vdash_i e_1 \Leftarrow C \dashv \Gamma_2, 0 x : A \quad \Gamma_1, 1_B^i y : B \vdash_i e_2 \Leftarrow C \dashv \Gamma_3, 0 x : A}{\Gamma \vdash_i \text{case } e \text{ of } \{x.e_1; y.e_2\} \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \frac{\Gamma_1, 1_A^i x : A \vdash_i e_1 \Rightarrow C \dashv \Gamma_2, 0 x : A \quad \Gamma_1, 1_B^i y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_3, 0 x : A}{\Gamma \vdash_i \text{case } e \text{ of } \{x.e_1; y.e_2\} \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}}$$

# Let

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash; e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r \Gamma_3), r_A^i x : A \vdash; e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash; \text{let}_r x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_4}$$

# Polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda a. e \Leftarrow \forall a : \text{Type}_r. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e A \Rightarrow B[a := A] \dashv \Gamma_2}$$

# Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda \{a\}. e \Leftarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e @ A \Rightarrow B [a := A] \dashv \Gamma_2}$$

# Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash P \equiv Q \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow Q \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{proving}~P~\mathbf{by}~p \Rightarrow P \dashv \Gamma_2} \text{ANNOT}$$

# Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \quad \frac{(h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \mathbf{assume} \ h \ \mathbf{in} \ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{apply} \ q \ p \Rightarrow Q \dashv \Gamma_2}$$

# Propositional logic

$$\frac{}{\Gamma \vdash \text{trivial} \Leftarrow \top \dashv \Gamma}$$

$$\frac{\Gamma \vdash p \Leftarrow \perp \dashv \Gamma'}{\Gamma \vdash \text{absurd } p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \text{both } p \ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \text{and-left } pq \Rightarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \text{and-right } pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \text{or-left } p \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \text{or-right } q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \text{cases } pq \ r_1 \ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

# Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \mathbf{destruct} \ pq \ \mathbf{as} \ h_1 \ h_2 \ \mathbf{in} \ r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

# Utilities

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2 \quad \Gamma_2, h : P \vdash q \Leftarrow Q \dashv \Gamma_3, h : P}{\Gamma \vdash \mathbf{lemma} \ h : P \ \mathbf{by} \ p \ \mathbf{in} \ q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P \dashv \Gamma_3}{\Gamma \vdash \mathbf{suffices} \ P \ \mathbf{by} \ q \ \mathbf{in} \ p \Leftarrow Q \dashv \Gamma_2}$$

# Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P[x := y] \dashv \Gamma'}{\Gamma \vdash \text{pick-any } y \text{ in } p \Leftarrow \forall x : A. P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A. P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \text{stantiate } p \text{ with } e \Rightarrow P[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P[x := e] \dashv \Gamma_2}{\Gamma \vdash \text{witness } e \text{ such that } p \Leftarrow \exists x : A. P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \text{pick-witness } y \ h \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2}$$

## Equality

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \mathbf{refl} \Leftarrow e_1 =_A e_2 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \dashv \Gamma_1 \quad \Gamma_1, x : A \vdash P \Leftarrow \text{prop} \dashv \Gamma_2, x : A}{\Gamma_2 \vdash p \Leftarrow P[x := e_2] \dashv \Gamma_3} \quad \frac{}{\Gamma \vdash \text{rewrite } q \text{ at } x.P \text{ in } p \Rightarrow P[x := e_1] \dashv \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \dashv \Gamma', x : A}{\Gamma \vdash \mathbf{funext} \, x \, \mathbf{in} \, p \Leftarrow f =_{rA \rightarrow B} g \dashv \Gamma'}$$

# Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma', h : \neg P}{\Gamma \vdash \text{by-contradiction } h \text{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} \text{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash \text{choose-spec } p \Rightarrow P[x := \text{choose } p] \dashv \Gamma'}$$

# Classical logic

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P [x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \text{choose } p, h$$

---

$$\Gamma \vdash \text{choose-witness } y \ h \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2$$

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P [x := y] \vdash_{\text{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \text{choose } p,$$

---

$$\Gamma \vdash_{\text{nc}} \text{choose-witness } y \ h \text{ for } p \text{ in } e \Rightarrow B \dashv \Gamma_2$$

## Type conversion

$$\frac{\Gamma \vdash A \equiv B \Rightarrow \text{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \equiv B \Leftarrow \text{Type}_r \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \triangleq B \Rightarrow \text{Type}_r \dashv \Gamma'}{\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma'}$$

# Type conversion

$$\frac{}{\Gamma \vdash \text{Unit} \triangleq \text{Unit} \Rightarrow \text{Type} \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash \text{Empty} \triangleq \text{Empty} \Rightarrow \text{Type} \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash r_1 A_1 \rightarrow B_1 \triangleq r_2 A_2 \rightarrow B_2 \Rightarrow \text{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash !_0 A_1 \triangleq !_0 A_2 \Rightarrow \text{Type} \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad r_1 \neq 0 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash !_r A_1 \triangleq !_r A_2 \Rightarrow \text{Type}_{r_1 \cdot s} \dashv \Gamma'}$$

## Type conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \stackrel{\triangle}{=} A_2 \otimes B_2 \Rightarrow \text{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \stackrel{\triangle}{=} A_2 \oplus B_2 \Rightarrow \text{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

## Type conversion

$$\frac{(a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \hat{=} a \Rightarrow \text{Type}_r \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \text{Type}_{r_1} \vdash B_1 \equiv B_2 [a_2 := a_1] \Rightarrow \text{Type}_s \dashv \Gamma', a_1 : \text{Type}_{r_1}}{\Gamma \vdash \forall a_1 : \text{Type}_{r_1}. B_1 \hat{\equiv} \forall a_2 : \text{Type}_{r_2}. B_2 \Rightarrow \text{Type}_s \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \text{Type}_{r_1} \vdash B_1 \equiv B_2 [a_2 := a_1] \Rightarrow \text{Type}_s \dashv \Gamma', a_1 : \text{Type}_{r_1}}{\Gamma \vdash \forall \{a_1 : \text{Type}_{r_1}\}. B_1 \hat{\equiv} \forall \{a_2 : \text{Type}_{r_2}\}. B_2 \Rightarrow \text{Type}_s \dashv \Gamma'}$$

## Proposition conversion

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma'}{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma'}$$

# Proposition conversion

$$\frac{}{\Gamma \vdash \top \hat{=} \top \dashv \Gamma} \quad \frac{}{\Gamma \vdash \perp \hat{=} \perp \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \hat{=} P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \hat{=} P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \hat{=} P_2 \vee Q_2 \dashv \Gamma_2}$$

## Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2 [x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1. P_1 \cong \forall x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2 [x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1. P_1 \cong \exists x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \xrightarrow{\text{Type}_r} \Gamma_1 \quad \begin{array}{c} \Gamma_1 \vdash e_1 \equiv e_2 \xleftarrow{\text{A}_1} \Gamma_2 \\ \Gamma_2 \vdash e'_1 \equiv e'_2 \xleftarrow{\text{A}_1} \Gamma_3 \end{array}}{\Gamma \vdash e_1 =_{A_1} e'_1 \triangleq e_2 =_{A_2} e'_2 \dashv \Gamma_3}$$

# Whnf's and neutral terms

Whnf's:

$\hat{e} ::=$

$$\begin{aligned} n &| \lambda x. e &| \text{box } e &| (e_1, e_2) &| \text{inl } e &| \text{inr } e &| \text{unit} \\ \Lambda a : \text{Type}_r. e &| \Lambda \{a : \text{Type}_r\}. e \end{aligned}$$

Neutral forms:

$n ::=$

$$\begin{aligned} x &| n\ e &| n\ A &| n @A &| \text{Empty-elim}_A\ e &| \\ \text{let}_A \text{ unit} = n \text{ in } e &| \text{let}_A \text{ box } x = n \text{ in } e &| \\ \text{let}_A (x, y) = n \text{ in } e &| \text{case}_A\ n \text{ of } \{x.e_1; y.e_2\} &| \\ \text{choose } p & \end{aligned}$$

## Computation

$e_1 \rightarrow e'_1$  – single step  
 $e_1 \rightarrow^* e'_1$  – multi step

## Term conversion – checking, all terms

$\Gamma \vdash e_1 \equiv e_2 \Leftarrow \text{Unit} \dashv \Gamma$

$\Gamma \vdash e_1 \equiv e_2 \Leftarrow \text{Empty} \dashv \Gamma$

$$\frac{A \neq \text{Unit} \quad A \neq \text{Empty} \quad e_1 \longrightarrow^* e'_1 \quad e_2 \longrightarrow^* e'_2 \quad \Gamma \vdash e'_1 \stackrel{\hat{=}}{\equiv} e'_2 \xleftarrow{A \dashv \Gamma'} }{ \Gamma \vdash e_1 \equiv e_2 \xleftarrow{A \dashv \Gamma'} }$$

# Term conversion – checking whnfs

$$\frac{\Gamma, x : A \vdash f \ x \equiv g \ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \ \hat{=} \ g \Leftarrow r A \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash \text{let box } x = e_1 \text{ in } x \equiv \text{let box } x = e_2 \text{ in } x \Leftarrow A \dashv \Gamma'}{\Gamma \vdash e_1 \ \hat{=} \ e_2 \Leftarrow !_r A \dashv \Gamma'}$$

$$\frac{\begin{array}{l} \Gamma \vdash \text{let } (x, y) = e_1 \text{ in } x \equiv \text{let } (x, y) = e_2 \text{ in } x \Leftarrow A \dashv \Gamma_1 \\ \Gamma_1 \vdash \text{let } (x, y) = e_1 \text{ in } y \equiv \text{let } (x, y) = e_2 \text{ in } y \Leftarrow B \dashv \Gamma_2 \end{array}}{\Gamma \vdash e_1 \ \hat{=} \ e_2 \Leftarrow A \otimes B \dashv \Gamma_2}$$

# Term conversion – checking whnfs

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \text{inl } e_1 \hat{\equiv} \text{inl } e_2 \Leftarrow A \oplus B \dashv \Gamma}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow B \dashv \Gamma'}{\Gamma \vdash \text{inr } e_1 \hat{\equiv} \text{inr } e_2 \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f \ a \equiv g \ a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \hat{\equiv} g \Leftarrow \forall a : \text{Type}_r. B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f @a \equiv g @a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \hat{\equiv} g \Leftarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma'}$$

## Term conversion – switch mode

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 \hat{\equiv} n_2 \Leftarrow B \dashv \Gamma_2}$$

# Term conversion – infer neutrals

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \equiv x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1 \ e_1 \equiv n_2 \ e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 \ A_1 \equiv n_2 \ A_2 \Rightarrow B[a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 @A_1 \equiv n_2 @A_2 \Rightarrow B[a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma'}{\Gamma \vdash \text{Empty-elim}_{A_1} e_1 \equiv \text{Empty-elim}_{A_2} e_2 \Rightarrow A_1 \dashv \Gamma'}$$

## Term conversion – infer neutrals

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \text{let}_{A_1} \text{ unit} = n_1 \text{ in } e_1 \equiv \text{let}_{A_2} \text{ unit} = n_2 \text{ in } e_2 \Rightarrow A_1 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_s \dashv \Gamma_1 \\ \Gamma \vdash n_1 \hat{=} n_2 \Rightarrow !_r A \dashv \Gamma_2 \end{array} \quad \Gamma_2, x : A \vdash e_1 \equiv e_2 \Leftarrow B_1 \dashv \Gamma_3, x : A}{\Gamma \vdash \text{let}_{B_1} \text{ box } x = n_1 \text{ in } e_1 \equiv \text{let}_{B_2} \text{ box } x = n_2 \text{ in } e_2 \Rightarrow B_1 \dashv \Gamma_3}$$

$$\frac{\begin{array}{c} \Gamma \vdash C_1 \equiv C_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \\ \Gamma_1 \vdash n_1 \hat{=} n_2 \Rightarrow A \otimes B \dashv \Gamma_2 \end{array} \quad \Gamma_2, x : A, y : B \vdash e_1 \equiv e_2 \Leftarrow C_1 \dashv \Gamma_3, x : A}{\Gamma \vdash \text{let}_{C_1} (x, y) = n_1 \text{ in } e_1 \equiv \text{let}_{C_2} (x, y) = n_2 \text{ in } e_2 \Rightarrow C_1 \dashv \Gamma_3}$$

$$\frac{\begin{array}{c} \Gamma \vdash C_1 \equiv C_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \\ \Gamma_1 \vdash e_1 \equiv e_2 \Rightarrow A \oplus B \dashv \Gamma_2 \end{array} \quad \Gamma_2, x : A \vdash f_1 \equiv f_2 \Leftarrow C \dashv \Gamma_3, x : A \\ \Gamma_2, y : B \vdash g_1 \equiv g_2 \Leftarrow C \dashv \Gamma_4, y : B \end{array} \quad \Gamma \vdash \text{case}_{C_1} e_1 \text{ of } \{x_1.f_1; y_1.g_1\} \equiv \text{case}_{C_2} e_2 \text{ of } \{x_2.f_2; y_2.g_2\} \Rightarrow C_1 \dashv \Gamma_3$$

# Term conversion (choice)

$$\frac{\Gamma \vdash p_1 \Rightarrow \exists x : A_1. P_1 \dashv \Gamma_1 \quad \Gamma_1 \vdash p_2 \Rightarrow \exists x : A_2. P_2 \dashv \Gamma_2 \quad \Gamma_2 \vdash \exists x : A_1. P_1 \equiv \exists x : A_1. P_2 \dashv \Gamma_3}{\Gamma \vdash \text{choose } p_1 \equiv \text{choose } p_2 \Rightarrow A_1 \dashv \Gamma_3}$$