Quantities

Quantities:
$r ::= 0 \mid 1 \mid ? \mid + \mid *$

$*$ is the default quantity, so when there's nothing to indicate quantity, it means it's $*$.

Quantities – explanation

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced "few") means a resource must be used at most once.
- $+$ (pronounced "many") means a resource must be used at least once.
- $*$ (pronounced "any") means no restrictions on usage.

Subusage ordering

$r_1 \sqsubseteq r_2$ means that a resource with quantity r_1 may be used when quantity r_2 is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$
\begin{gathered}
* \sqsubseteq ? \\
* \sqsubseteq + \\
+ \sqsubseteq 1 \\
? \sqsubseteq 1 \\
? \sqsubseteq 0
\end{gathered}
$$

We will denote the greatest lower bound in this order with $r_1 \sqcap r_2$ and the least upper bound (if it exists) with $r_1 \sqcup r_2$.

Addition of quantities

When we have two quantities of the same resource, we can sum the quantities.

$$
\begin{aligned}
0 + r &= r \\
r + 0 &= r \\
?+? &= * \\
? + * &= * \\
*+? &= * \\
* + * &= * \\
\_ + \_ &= +
\end{aligned}
$$

Multiplication of quantities

When we have a quantity r_1 of resource A that contains quantitty r_2 of resource B, then we in fact have quantity $r_1 \cdot r_2$ of resource B.

$$
\begin{aligned}
0 \cdot r &= 0 \\
r \cdot 0 &= 0 \\
1 \cdot r &= r \\
r \cdot 1 &= r \\
? \cdot ? &= ? \\
+ \cdot + &= + \\
\_ \cdot \_ &= *
\end{aligned}
$$

The algebra of quantities

Quantities $\mathcal{Q}$ form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$ is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$ is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.
- If $r_1 \cdot r_2 = 0$, then $r_1 = 0$ or $r_2 = 0$.

Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

Subtraction order on quantities

$r_1 \leq_{\mathtt{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\mathtt{sub}} 1 \leq_{\mathtt{sub}} ? \leq_{\mathtt{sub}} + \leq_{\mathtt{sub}} * \leq_{\mathtt{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\mathtt{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\frac{}{* \leq_{\mathtt{dec}} *}$$

$$\frac{}{* \leq_{\mathtt{dec}} +}$$

$$\frac{}{0 \leq_{\mathtt{dec}} 1}$$

$$\frac{}{0 \leq_{\mathtt{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how "big" they are.

Trait-checking Division

$$r_1/r_2 = \sup\{s \in \mathcal{Q} \mid s \cdot r_2 \sqsubseteq r_1\}$$

r_1/r_2	0	1	$?$	$+$	$*$
0	0	0	0	0	0
1		1	1	1	1
$?$		$?$	1	$?$	1
$+$		$+$	$+$	1	1
$*$		$*$	$+$	$?$	1

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	$?$	$+$	$*$
0	$*$	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
$?$	$(*, ?)$	$?$	$?$	$(0, ?)$	$(0, ?)$
$+$	$(*, +)$	$+$	$(*, 1)$	$+$	$(*, 1)$
$*$	$(*, *)$	$*$	$*$	$*$	$*$