

Complete and Easy Quantitative Contextual Typing

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Types

Types:

$A, B ::=$

$$A \multimap B \mid !_r A \mid A \otimes B \mid A \oplus B \mid \text{Unit} \mid \text{Empty} \mid a \mid \tilde{a} \mid \hat{a} \mid \forall @a : \text{Type}_r. A \mid \forall @a. A \mid \forall a : \text{Type}_r. A \mid \forall a. A$$

Monotypes:

$T :=$

$$\begin{array}{l} \tau_1 \multimap \tau_2 \mid !_r \tau \mid \tau_1 \otimes \tau_2 \mid \tau_1 \oplus \tau_2 \mid \text{Unit} \mid \text{Empty} \mid \\ a \mid \hat{a} \mid \forall @a : \text{Type}_r. \tau \mid \forall @a. \tau \end{array}$$

Terms

Terms:

$e ::=$

$$\begin{aligned} & (e : A) \mid x \mid \lambda x. e \mid \lambda x : A. e \mid e_1 \ e_2 \mid e @A \mid \\ & \Lambda a. e \mid \Lambda a : \text{Type}_r. e \mid e \ A \mid \\ & (e_1, e_2) \mid \text{let } (x, y) = e_1 \ \text{in } e_2 \mid \\ & \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } \{x.e_1; \ y.e_2\} \mid \\ & \text{unit} \mid \text{let unit} = e_1 \ \text{in } e_2 \mid \\ & \text{Empty-elim } e \mid \\ & \text{let } x = e_1 \ \text{in } e_2 \mid \text{let } x : A = e_1 \ \text{in } e_2 \mid \end{aligned}$$

Contexts

Contexts:

$\Gamma ::=$

- $\cdot \mid \Gamma, r x : A \mid \Gamma, a : \text{Type}_r \mid \Gamma, \tilde{a} : \text{Type}_r \mid$
- $\Gamma, \hat{a} \mid \Gamma, \hat{a} = \tau \mid \Gamma, \blacktriangleright_a \mid$
- $\Gamma, \square \mid \Gamma, \square : A \mid \Gamma, \square e \mid \Gamma, \square @A \mid \Gamma, \square A$

Context operations and notations

- Γ, \square, Δ is a notation which means that Δ does not contain any hints. It works not only for \square , but for all kinds of hints.
 - $\Gamma[\hat{a}]$ is a notation for Γ, \hat{a}, Γ' . We use it for checking variable presence and for updating existential variables in the context.
 - $\langle \Gamma \rangle A$ substitutes all equations $\hat{a} = \tau$ from Γ in A .

Judgements

- $\Gamma \vdash r \Rightarrow A \Rightarrow s \dashv \Gamma'$ – kinding
- $\Gamma \vdash e \Rightarrow A \dashv \Gamma'$ – typing
- $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$ – matching
- $\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$ – subtyping
- $\Gamma \vdash r \Rightarrow \hat{a} := A \Rightarrow s \dashv \Gamma'$ – sub-instantiation
- $\Gamma \vdash r \Rightarrow A =: \hat{a} \Rightarrow s \dashv \Gamma'$ – super-instantiation

Kinding judgement

$\Gamma \vdash r \Rightarrow A \Rightarrow s \dashv \Gamma'$ – in context Γ , check if type A is of kind $Type_r$. If it is, s will be 1. If it is not, s indicates how much A is missing to be in $Type_r$, i.e. A is in $Type_{r/s}$. Return output context Γ' .

Note: Γ and r are inputs. A is subject. s and Γ' are outputs.

$\Gamma \vdash r \Rightarrow A \dashv \Gamma'$ – kind checking is a notation for
 $\Gamma \vdash r \Rightarrow A \Rightarrow 1 \dashv \Gamma'$.

$\Gamma \vdash A \Rightarrow r \dashv \Gamma'$ – kind inference is a notation for
 $\Gamma \vdash * \Rightarrow A \Rightarrow s \dashv \Gamma'$ with $r = */s$

Kinding – variables and quantifiers

$$\Gamma[a : \text{Type}_s] \vdash r \xrightarrow{} a \Rightarrow r/s \dashv \Gamma[a : \text{Type}_s]$$

$$\Gamma[\tilde{a} : \text{Type}_s] \vdash r \xrightarrow{\text{blue}} \tilde{a} \xrightarrow{\text{red}} r/s \dashv \Gamma[\tilde{a} : \text{Type}_s]$$

$$\overline{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{b}, \hat{a} = !_r \hat{b}]} \quad \text{---}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow \forall a : \text{Type}_s. A \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow \forall @a : \text{Type}_s. A \Rightarrow t \dashv \Gamma'}$$

Kinding – functions and boxes

$$\frac{\Gamma \vdash 1 \Rightarrow A \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A \multimap B \Rightarrow r \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow !_0 A \Rightarrow 1 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash r/s \Rightarrow A \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow !_s A \Rightarrow t \dashv \Gamma'}$$

Kinding – remaining type formers

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Unit} \Rightarrow 1 \dashv \Gamma} \quad \frac{}{\Gamma \vdash r \Rightarrow \text{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash r \Rightarrow A \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A \otimes B \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash r \Rightarrow A \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A \oplus B \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

Typing judgement

$\Gamma \vdash e \Rightarrow A \dashv \Gamma'$ – in context Γ , term e infers type A , returning output context Γ'

Note: Γ is input. e is subject. A and Γ' are outputs.

Note: “inference mode” is $\Gamma, \square \vdash e \Rightarrow A \dashv \Gamma'$, as it requires the empty hint.

Note: “checking mode” is $\Gamma, \square : A \vdash e \Rightarrow _ \dashv \Gamma'$, as it requires a type hint $\square : A$, but does not use the inferred type.

Typing – variables and annotations

$$\frac{\Gamma(x) = A \quad 0\Gamma + x \vdash \langle 0\Gamma + x \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_c x \Rightarrow B \dashv \Gamma' + \Gamma}$$

$$\frac{\Gamma(x) = A \quad \Gamma \vdash \langle \Gamma \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} x \Rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow r \dashv \Gamma_1 \quad 0\Gamma_1, \square : A \vdash e \Rightarrow _ \dashv \Gamma_2 \quad \Gamma_2 \vdash \langle \Gamma_2 \rangle A \rightsquigarrow B \dashv \Gamma_3}{\Gamma \vdash (e : A) \Rightarrow B \dashv \Gamma_3 + \Gamma_1}$$

Typing – stationary rules

$$\frac{\Gamma, \Delta, a : \text{Type}_r, \square : A \vdash e \Rightarrow _ \dashv \Gamma', a : \text{Type}_r, \Theta}{\Gamma, \square : \forall a : \text{Type}_r. A, \Delta \vdash e \Rightarrow \forall a : \text{Type}_r. A \dashv \Gamma'}$$

$$\frac{0\Gamma, 0\Delta, \Box : A \vdash e \Rightarrow _ \dashv \Gamma'}{\Gamma, \Box : !_r A, \Delta \vdash e \Rightarrow !_r A \dashv r \Gamma' + (\Gamma, \Delta)}$$

Typing – application(s)

$$\frac{\Gamma, \Box e_2 \vdash e_1 \Rightarrow A \dashv \Gamma'}{\Gamma \vdash e_1 \ e_2 \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma, \Box @A \vdash e \Rightarrow B \dashv \Gamma'}{\Gamma \vdash e @A \Rightarrow B \dashv \Gamma'}$$

Typing – function abstraction

$$\frac{\Gamma, \Delta, 0x : A, \Box : B \vdash e \Rightarrow _ \dashv \Gamma_1, rx : A, \Theta}{\begin{array}{c} \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle A \Rightarrow 1 \dashv \Gamma_2 \\ \hline \Gamma, \Box : A \multimap B, \Delta \vdash \lambda x. e \Rightarrow A \multimap B \dashv \Gamma_2 \end{array}}$$

$$\frac{\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \multimap \hat{a}_2], \Delta, 0x : \hat{a}_1, \Box : \hat{a}_2 \vdash e \Rightarrow _ \dashv \Gamma_1, rx : \hat{a}_1, \Theta}{\begin{array}{c} \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle \hat{a}_1 \Rightarrow 1 \dashv \Gamma_2 \\ \hline \Gamma[\hat{a}], \Box : \hat{a}, \Delta \vdash \lambda x. e \Rightarrow \hat{a} \dashv \Gamma_2 \end{array}}$$

$$\frac{\Gamma, \Box \vdash e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta, 0x : A \vdash e \Rightarrow B \dashv \Gamma_2, rx : A, \Theta}{\begin{array}{c} \Gamma_2 \vdash r \Rightarrow \langle \Gamma_2 \rangle A \Rightarrow 1 \dashv \Gamma_3 \\ \hline \Gamma, \Box e', \Delta \vdash \lambda x. e \Rightarrow B \dashv \Gamma_3 \end{array}}$$

$$\Gamma, \Delta \blacktriangleright_{\hat{a}}, \hat{a}, 0x : \hat{a}, \Box \vdash e \Rightarrow A \dashv \Gamma_1, rx : \hat{a}, \Gamma_4$$

$$\Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle \hat{a} \Rightarrow 1 \dashv \Gamma_2, \blacktriangleright_{\hat{a}}, \Gamma_3$$

$$B = \forall \text{unsolved}(\Gamma_3). \langle \Gamma_3 \rangle (\hat{a} \multimap \forall \text{unsolved}(\Gamma_4). \langle \Gamma_4 \rangle A)$$

$$\Gamma, \Box, \Delta \vdash \lambda x. e \Rightarrow B \dashv \Gamma_2$$

Typing – smarter hint rule for functions

$$\frac{0\Gamma, \square \vdash e' \Rightarrow A \dashv \Gamma_1 \\ (0\Gamma_1 + \Gamma), \Delta, 0x : A \vdash e \Rightarrow B \dashv \Gamma_2, rx : A, \Theta \\ \Gamma_2 \vdash r \Rightarrow \langle \Gamma_2 \rangle A \Rightarrow s \dashv \Gamma_3}{\Gamma, \square e', \Delta \vdash \lambda x. e \Rightarrow B \dashv \Gamma_3 + s\Gamma_1}$$

Note: this rule is equivalent to the rule for let expressions which automatically takes care of boxing.

Typing – functions with annotation

$$\frac{\Gamma, \Box : A \vdash e' \Rightarrow _ \dashv \Gamma_1 \\ \Gamma_1, \Delta, 0x : A \vdash e \Rightarrow B \dashv \Gamma_2, rx : A, \Theta \\ \Gamma_2 \vdash r \Rightarrow \langle \Gamma_2 \rangle A \Rightarrow 1 \dashv \Gamma_3}{\Gamma, \Box e', \Delta \vdash \lambda x : A. e \Rightarrow B \dashv \Gamma_3}$$

$$\frac{\Gamma, \Delta, 0x : A, \Box \vdash e \Rightarrow B \dashv \Gamma_1, rx : A, \Theta \\ \Gamma_1 \vdash r \Rightarrow A \Rightarrow 1 \dashv \Gamma_2}{\Gamma, \Box, \Delta \vdash \lambda x : A. e \Rightarrow A \multimap B \dashv \Gamma_2}$$

Typing – explicit quantification

$$\frac{\Gamma, \Delta, a : \text{Type}_r, \square : A \vdash e \Rightarrow _ \dashv \Gamma'}{\Gamma, \square : \forall @a : \text{Type}_r. A, \Delta \vdash \Lambda a. e \Rightarrow \forall @a : \text{Type}_r. A \dashv \Gamma'}$$

$$\frac{\Gamma, \Delta, a : \text{Type}_r, \square \vdash e \Rightarrow A \dashv \Gamma'}{\Gamma, \square, \Delta \vdash \Lambda a : \text{Type}_r. e \Rightarrow \forall @a : \text{Type}_r. A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash * \xrightarrow{B} r \dashv \Gamma_1 \quad \Gamma_1, \Delta, a : \text{Type}_{*/r} \vdash e \Rightarrow A \dashv \Gamma_2}{\Gamma, \square B, \Delta \vdash \Lambda a. e \Rightarrow \forall @a : \text{Type}_{*/r}. A \dashv \Gamma_2}$$

$$\frac{\Gamma, \square A \vdash e \Rightarrow B \dashv \Gamma'}{\Gamma \vdash e A \Rightarrow B \dashv \Gamma'}$$

Typing – Unit

$$\frac{\Gamma \vdash \text{Unit} \rightsquigarrow _ \dashv \Gamma'}{\Gamma \vdash \text{unit} \Rightarrow \text{Unit} \dashv \Gamma'}$$

$$\frac{\Gamma, \square : \text{Unit} \vdash e_1 \Rightarrow _ \dashv \Gamma_1 \quad \Gamma_1 \vdash e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Typing – Empty

$$\frac{\Gamma, \Delta, \square : \text{Empty} \vdash e \Rightarrow _ \dashv \Gamma'}{\Gamma, \square : A, \Delta \vdash \text{Empty-elim } e \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma, \square \vdash e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}{\Gamma, \square \ e', \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}{\Gamma, \square @A, \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}{\Gamma, \square A, \Delta \vdash \text{Empty-elim } e \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash \text{Empty} \multimap \forall a. a \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash \text{Empty-elim} \Rightarrow C \dashv \Gamma'} \text{DUBIOUS}$$

Typing – Product

$$\frac{\Gamma, \Delta, \square : A \vdash e_1 \Rightarrow _ \dashv \Gamma_1 \quad \Gamma_1, \square : B \vdash e_2 \Rightarrow _ \dashv \Gamma_2}{\Gamma, \square : A \otimes B, \Delta \vdash (e_1, e_2) \Rightarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2], \Delta, \square : \hat{a}_1 \vdash e_1 \Rightarrow _ \dashv \Gamma_1 \quad \Gamma_1, \square : \hat{a}_2 \vdash e_2 \Rightarrow _ \dashv \Gamma_2}{\Gamma[\hat{a}], \square : \hat{a}, \Delta \vdash (e_1, e_2) \Rightarrow \hat{a} \dashv \Gamma_2}$$

$$\frac{\Gamma, \Delta, \square \vdash e_1 \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \square \vdash e_2 \Rightarrow B \dashv \Gamma_2}{\Gamma, \square, \Delta \vdash (e_1, e_2) \Rightarrow A \otimes B \dashv \Gamma_2}$$

$$\Gamma, \square \vdash e_1 \Rightarrow A \otimes B \dashv \Gamma_1$$

$$\Gamma_1, 0x : A, 0y : B \vdash e_2 \Rightarrow C \dashv \Gamma_2, r_A x : A, r_B y : B$$

$$\Gamma_2 \vdash r_A \Rightarrow A \Rightarrow 1 \dashv \Gamma_3 \quad \Gamma_3 \vdash r_B \Rightarrow B \Rightarrow 1 \dashv \Gamma_4$$

$$\frac{}{\Gamma \vdash \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_4}$$

Typing – alternative rules for products

$$\frac{\Gamma \vdash \forall a. a \multimap \forall b. b \multimap a \otimes b \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash \text{pair} \Rightarrow C \dashv \Gamma'} \text{DUBIOUS}$$

$$0\Gamma, \square \vdash e_1 \Rightarrow A \otimes B \dashv \Gamma_1$$

$$(0\Gamma_1 + \Gamma), 0x : A, 0y : B \vdash e_2 \Rightarrow C \dashv \Gamma_2, r_A x : A, r_B y : B$$

$$\Gamma_2 \vdash r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_3$$

$$\Gamma_3 \vdash r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_4$$

$$\Gamma \vdash \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_4 + (s_A \sqcap s_B) \Gamma_1$$

Typing – Sum

$$\frac{\Gamma, \Delta, \square : A \vdash e \Rightarrow _ \dashv \Gamma'}{\Gamma, \square : A \oplus B, \Delta \vdash \text{inl } e \Rightarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma, \Delta, \square : B \vdash e \Rightarrow _ \dashv \Gamma'}{\Gamma, \square : A \oplus B, \Delta \vdash \text{inr } e \Rightarrow A \oplus B \dashv \Gamma'}$$

$$0\Gamma, \square \vdash e \Rightarrow A \oplus B \dashv \Gamma_1$$

$$0\Gamma_1, 0x : A \vdash e_1 \Rightarrow C_1 \dashv \Gamma_2, r_A x : A$$

$$0\Gamma_1, 0y : B \vdash e_2 \Rightarrow C_2 \dashv \Gamma_3, r_B y : B$$

$$\Gamma_4 = \Gamma_2 \sqcap \Gamma_3 \quad C_1 = C_2$$

$$\Gamma_4 \vdash r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_5$$

$$\Gamma_5 \vdash r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_6$$

$$\frac{}{\Gamma \vdash \text{case } e \text{ of } \{x.e_1; y.e_2\} \Rightarrow C_1 \dashv \Gamma_6 + (s_A \sqcap s_B) \Gamma_1 + \Gamma}$$

Typing – alternative rules for sums

$$\frac{\Gamma \vdash \forall a. a \multimap \forall b. a \oplus b \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash \text{inl } \Rightarrow C \dashv \Gamma'} \text{DUBIOUS}$$

$$\frac{\Gamma \vdash \forall b. b \multimap \forall a. a \oplus b \rightsquigarrow C \dashv \Gamma'}{\Gamma \vdash \text{inr } \Rightarrow C \dashv \Gamma'} \text{DUBIOUS}$$

$$0\Gamma, \square \vdash e \Rightarrow A \oplus B \dashv \Gamma_1$$

$$0\Gamma_1, 0x : A \vdash e_1 \Rightarrow C \dashv \Gamma_2, r_A x : A$$

$$0\Gamma_2, 0y : B, \square : C \vdash e_2 \Rightarrow _ \dashv \Gamma_3, r_B y : B$$

$$\Gamma_4 = \Gamma_2 \sqcap \Gamma_3$$

$$\Gamma_4 \vdash r_A \Rightarrow A \Rightarrow s_A \dashv \Gamma_5$$

$$\Gamma_5 \vdash r_B \Rightarrow B \Rightarrow s_B \dashv \Gamma_6$$

$$\frac{}{\Gamma \vdash \text{case } e \text{ of } \{x.e_1; y.e_2\} \Rightarrow C \dashv \Gamma_6 + (s_A \sqcap s_B) \Gamma_1 + \Gamma}$$

Typing – let

$$\frac{\begin{array}{c} 0\Gamma, \square \vdash e_1 \Rightarrow A \dashv \Gamma_1 \\ (0\Gamma_1 + \Gamma), 0x : A \vdash e_2 \Rightarrow B \dashv \Gamma_2, rx : A \\ \Gamma_2 \vdash r \Rightarrow A \Rightarrow s \dashv \Gamma_3 \end{array}}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_3 + s\Gamma_1}$$

$$\frac{\begin{array}{c} 0\Gamma, \square : A \vdash e_1 \Rightarrow _ \dashv \Gamma_1 \\ (0\Gamma_1 + \Gamma), 0x : A \vdash e_2 \Rightarrow B \dashv \Gamma_2, rx : A \\ \Gamma_2 \vdash r \Rightarrow A \Rightarrow s \dashv \Gamma_3 \end{array}}{\Gamma \vdash \text{let } x : A = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_3 + s\Gamma_1}$$

Matching judgement

Matching: $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$ – type A matches context Γ , with output type B and output context Γ'

Note: Γ and A are inputs. B and Γ' are outputs.

Matching – inference and checking

$$\frac{}{\Gamma, \square, \Delta \vdash A \rightsquigarrow A \dashv \Gamma, \Delta}$$

$$\frac{\Gamma, \Delta \vdash 1 \Rightarrow A <: B \Rightarrow r \dashv \Gamma'}{\Gamma, \square : B, \Delta \vdash A \rightsquigarrow B \dashv r \Gamma'}$$

Matching – term argument hints

$$\frac{\Gamma, \Box : A \vdash e \xrightarrow{-} _\dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash B \rightsquigarrow C \dashv \Gamma_2}{\Gamma, \Box e, \Delta \vdash A \multimap B \rightsquigarrow C \dashv \Gamma_2}$$

$$\frac{\Gamma, \Box e, \Delta \vdash A \rightsquigarrow C \dashv \Gamma'}{\Gamma, \Box e, \Delta \vdash !_r A \rightsquigarrow C \dashv \Gamma'}$$

$$\frac{\Gamma, \Box e, \Delta, \hat{a} \vdash A[a := !_r \hat{a}] \rightsquigarrow C \dashv \Gamma'}{\Gamma, \Box e, \Delta \vdash \forall a : \text{Type}_r. A \rightsquigarrow C \dashv \Gamma'}$$

Matching – implicit type argument hints

$$\frac{\Gamma \vdash r \Rightarrow B \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash A[a := B] \rightsquigarrow C \dashv \Gamma_2}{\Gamma, \Box @B, \Delta \vdash \forall a : \text{Type}_r. A \rightsquigarrow C \dashv \Gamma_2}$$

$$\frac{\Gamma, \Box @B, \Delta \vdash A \rightsquigarrow C \dashv \Gamma'}{\Gamma, \Box @B, \Delta \vdash !_r A \rightsquigarrow C \dashv \Gamma'}$$

Matching – explicit type argument hints

$$\frac{\Gamma \vdash r \Rightarrow B \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash A[a := B] \rightsquigarrow C \dashv \Gamma_2}{\Gamma, \Box B, \Delta \vdash \forall @a : \text{Type}_r. A \rightsquigarrow C \dashv \Gamma_2}$$

$$\frac{\Gamma, \Box B, \Delta \vdash A \rightsquigarrow C \dashv \Gamma'}{\Gamma, \Box B, \Delta \vdash !_r A \rightsquigarrow C \dashv \Gamma'}$$

$$\frac{\Gamma, \Box B, \Delta, \hat{a} \vdash A[a := !_r \hat{a}] \rightsquigarrow C \dashv \Gamma'}{\Gamma, \Box B, \Delta \vdash \forall a : \text{Type}_r. A \rightsquigarrow C \dashv \Gamma'}$$

Subtyping judgement

$\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$ – in context Γ , A is a subtype of $!_{r/s} B$, with output context Γ' .

Note: Γ , r , A and B are inputs. s and Γ' are outputs.

Note: If A is a subtype of $!_r B$, then s is 1. If it is not, then s indicates how much is missing for the subtyping to be the case.

Subtyping – variables

$$\frac{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: \hat{a}}{r \dashv \Gamma[\hat{a}]}$$

$$\frac{\Gamma[a : \text{Type}_s] \vdash r \Rightarrow a <: a}{r/s \dashv \Gamma[a : \text{Type}_s]}$$

$$\frac{\Gamma[\tilde{a} : \text{Type}_s] \vdash r \Rightarrow \tilde{a} <: \tilde{a}}{r/s \dashv \Gamma[\tilde{a} : \text{Type}_s]}$$

Subtyping – quantifiers

$$\frac{\Gamma, \tilde{a} : \text{Type}_s \vdash r \Rightarrow A[a := \tilde{a}] <: B[a := \tilde{a}] \Rightarrow t \dashv \Gamma', \tilde{a}, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \text{Type}_s. A <: \forall a : \text{Type}_s. B \Rightarrow t \dashv \Gamma'}$$

$$\frac{a \in \text{FV}(A) \quad B \neq \forall _. B' \quad \Gamma, \blacktriangleright_{\hat{a}}, \hat{a} \vdash r \Rightarrow A[a := !_s \hat{a}] <: B \Rightarrow t \dashv \Gamma', \blacktriangleright_{\hat{a}}, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \text{Type}_s. A <: B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \text{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow \forall @a : \text{Type}_s. A <: \forall @a : \text{Type}_s. B \Rightarrow t \dashv \Gamma'}$$

Subtyping – boxes

$$\frac{\Gamma \vdash r \cdot s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow A <: !_s B \Rightarrow t \dashv \Gamma'}$$

$$\frac{B \neq !_r B' \quad \Gamma \vdash r/s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow !_s A <: B \Rightarrow t \dashv \Gamma'}$$

Subtyping – type formers

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Unit} <: \text{Unit} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Empty} <: \text{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \otimes B_1 <: A_2 \otimes B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \oplus B_1 <: A_2 \oplus B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A_2 <: A_1 \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B_1 <: \langle \Gamma_1 \rangle B_2 \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \multimap B_1 <: A_2 \multimap B_2 \Rightarrow r \dashv \Gamma_2}$$

Subtyping – switch to instantiation

$$\frac{\hat{a} \notin \text{FV}(A) \quad \Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := A \Rightarrow s \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'}$$

$$\frac{\hat{a} \notin \text{FV}(A) \quad \Gamma[\hat{a}] \vdash r \Rightarrow A =: \hat{a} \Rightarrow s \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'}$$

Instantiation judgements

$\Gamma \vdash r \Rightarrow \hat{a} := A \Rightarrow s \dashv \Gamma'$ – in context Γ , instantiate the existential variable \hat{a} to a subtype of $!_{r/s} A$, with output context Γ' .

$\Gamma \vdash r \Rightarrow A =: \hat{a} \Rightarrow s \dashv \Gamma'$ – in context Γ , instantiate the existential variable \hat{a} , so that $A <: !_{r/s} \hat{a}$, with output context Γ .

Note: in both cases, Γ , r , \hat{a} and A are inputs, whereas s and Γ' are outputs.

Sub-instantiation

$$\frac{\Gamma \vdash r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \hat{a} := \tau \Rightarrow 1 \dashv \Gamma_1, \hat{a} = !_s \tau, \Gamma'}$$

$$\frac{}{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{a} := \hat{b} \Rightarrow r \dashv \Gamma[\hat{a}][\hat{b} = \hat{a}]}$$

$$\frac{}{\Gamma[b : \text{Type}_s][\hat{a}] \vdash r \Rightarrow \hat{a} := b \Rightarrow 1 \dashv \Gamma[b : \text{Type}_s][\hat{a} = !_r b]}$$

$$\frac{\Gamma[\hat{a}] \vdash r \cdot s \Rightarrow \hat{a} := A \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := !_s A \Rightarrow t \dashv \Gamma'}$$

Note: no rule for implicit quantifier.

Sub-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := \text{Unit} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := \text{Empty} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Empty}]}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 := A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 := \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := A \otimes B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow \hat{a}_1 := A \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \hat{a}_2 := \langle \Gamma_1 \rangle B \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := A \oplus B \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = !_r(\hat{a}_1 \multimap \hat{a}_2)] \vdash 1 \Rightarrow A =: \hat{a}_1 \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \hat{a}_2 := \langle \Gamma_1 \rangle B \Rightarrow 1 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow \hat{a} := A \multimap B \Rightarrow 1 \dashv \Gamma_2}$$

Super-instantiation

$$\frac{\Gamma \vdash r \Rightarrow \tau \Rightarrow s \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \tau =: \hat{a} \Rightarrow s \dashv \Gamma_1, \hat{a} = \tau, \Gamma'}$$

$$\overline{\Gamma[\hat{a}][\hat{b}] \vdash r \Rightarrow \hat{b} =: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a}][\hat{b} = !_r \hat{a}]}$$

$$\overline{\Gamma[b : \text{Type}_s][\hat{a}] \vdash r \Rightarrow b =: \hat{a} \Rightarrow r/s \dashv \Gamma[b : \text{Type}_s][\hat{a} = b]}$$

$$\frac{b \in \text{FV}(B) \quad \Gamma[\hat{a}], \blacktriangleright_{\hat{b}}, \hat{b} \vdash r \Rightarrow B[b := \hat{b}] =: \hat{a} \Rightarrow s \dashv \Gamma', \blacktriangleright_{\hat{b}}, \Theta}{\Gamma[\hat{a}] \vdash r \Rightarrow \forall b. B =: \hat{a} \Rightarrow s \dashv \Gamma'}$$

$$\frac{\Gamma[\hat{a}] \vdash r/s \Rightarrow A =: \hat{a} \Rightarrow t \dashv \Gamma'}{\Gamma[\hat{a}] \vdash r \Rightarrow !_s A =: \hat{a} \Rightarrow t \dashv \Gamma'}$$

Super-instantiation

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \text{Unit} =: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Unit}]}$$

$$\frac{}{\Gamma[\hat{a}] \vdash r \Rightarrow \text{Empty} =: \hat{a} \Rightarrow 1 \dashv \Gamma[\hat{a} = \text{Empty}]}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \otimes \hat{a}_2] \vdash r \Rightarrow A =: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B =: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \otimes B =: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \oplus \hat{a}_2] \vdash r \Rightarrow A =: \hat{a}_1 \Rightarrow s_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash r \Rightarrow \langle \Gamma_1 \rangle B =: \hat{a}_2 \Rightarrow s_2 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \oplus B =: \hat{a} \Rightarrow s_1 \sqcap s_2 \dashv \Gamma_2}$$

$$\frac{\begin{array}{c} \Gamma[\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \multimap \hat{a}_2] \vdash 1 \Rightarrow \hat{a}_1 := A \Rightarrow 1 \dashv \Gamma_1 \\ \Gamma_1 \vdash 1 \Rightarrow \langle \Gamma_1 \rangle B =: \hat{a}_2 \Rightarrow 1 \dashv \Gamma_2 \end{array}}{\Gamma[\hat{a}] \vdash r \Rightarrow A \multimap B =: \hat{a} \Rightarrow r \dashv \Gamma_2}$$

Monotypes and explicit quantifiers

Note: we use the monotype rule to handle the difficult case when $\tau = \forall @a : \text{Type}_s. A$. If we had a separate rule for this, then the monotype rule would be admissible. Alternatively, we can write down separate rules to handle the explicit quantifier in a naive way.

$$\frac{\Gamma \vdash r \Rightarrow \forall @a : \text{Type}_s. \tau \Rightarrow t \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \hat{a} := \forall @a : \text{Type}_s. \tau \Rightarrow 1 \dashv \Gamma_1, \hat{a} = !_t \forall @a : \text{Type}_s. \tau, \Gamma'}$$

$$\frac{\Gamma \vdash r \Rightarrow \forall @a : \text{Type}_s. \tau \Rightarrow t \dashv \Gamma_1}{\Gamma, \hat{a}, \Gamma' \vdash r \Rightarrow \forall @a : \text{Type}_s. \tau =: \hat{a} \Rightarrow t \dashv \Gamma_1, \hat{a} = \forall @a : \text{Type}_s. \tau, \Gamma'}$$