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T-Axi (algorithmic)

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Syntax

The syntax in the algorithmic system is the same, but we will have far fewer annotations in terms

Judgements

Kind inference judgement: $\Gamma \vdash A \Rightarrow \mathtt{Type}_r \dashv \Gamma'$

Type conversion judgements: $\Gamma \vdash A \equiv B \Rightarrow \mathtt{Type}_r \dashv \Gamma'$ (all types),
$\Gamma \vdash A \triangleq B \Rightarrow \mathtt{Type}_r \dashv \Gamma'$ (types in whnf)

Type checking judgement: $\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$

Type inference judgement: $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

Term conversion judgements: $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$ (all terms),
$\Gamma \vdash e_1 \equiv e_2 \Rightarrow A \dashv \Gamma'$ (all terms), $\Gamma \vdash e_1 \triangleq e_2 \Leftarrow A \dashv \Gamma'$ (whnfs),
$\Gamma \vdash e_1 \triangleq e_2 \Rightarrow A \dashv \Gamma'$ (whnfs)

Proposition checking judgement: $\Gamma \vdash P \Leftarrow \mathtt{prop} \dashv \Gamma'$

Proposition conversion judgements: $\Gamma \vdash P \equiv Q \dashv \Gamma'$ (all
propositions), $\Gamma \vdash P \triangleq Q \dashv \Gamma'$ (whnfs), $\Gamma \vdash P \sim Q \dashv \Gamma'$ (whnfs)

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Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	$+$	$*$
0	0				
1	1	0			
?	?	0	0		
$+$	$+$	$*$	$+$	$*$	$+$
$*$	$*$	$*$	$*$	$*$	$*$

Subtraction order on quantities

$r_1 \leq_{\mathrm{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\mathrm{sub}} 1 \leq_{\mathrm{sub}} ? \leq_{\mathrm{sub}} + \leq_{\mathrm{sub}} * \leq_{\mathrm{sub}} +$

Decrementation order on quantities

$r_1 \leq_{dec} r_2$ holds when $r_2 - 1 = r_1$.

$$\overline{* \leq_{dec} +}$$

$$\overline{0 \leq_{dec} 1}$$

$$\overline{0 \leq_{dec} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how "big" they are.

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	$?$	$+$	$*$
0	$*$	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
$?$	$(*, ?)$	$?$	$?$	$(0, ?)$	$(0, ?)$
$+$	$(*, +)$	$+$	$(*, 1)$	$+$	$(*, 1)$
$*$	$(*, *)$	$*$	$*$	$*$	$*$

Decrement variable in context

$$\cdot - x = \textbf{undefined}$$
$$(\Gamma, r\,x : A) - x = \Gamma, (r-1)\,x : A$$
$$(\Gamma, r\,y : A) - x = \Gamma - x, r\,y : A$$
$$(\Gamma, r\,x : A := e) - x = \Gamma, (r-1)\,x : A := e$$
$$(\Gamma, r\,y : A := e) - x = \Gamma - x, r\,y : A := e$$
$$(\Gamma, h : P) - x = \Gamma - x, h : P$$
$$(\Gamma, a : \texttt{Type}_r) - x = \Gamma - x, a : \texttt{Type}_r$$

Context division with remainder

$$\overline{\cdot/r = \cdot}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\, x : A)/q = ((\Gamma_1, r_1\, x : A), (\Gamma_2, r_2\, x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\, x : A := e)/q = ((\Gamma_1, r_1\, x : A := e), (\Gamma_2, r_2\, x : A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \mathrm{Type}_r)/q = ((\Gamma_1, a : \mathrm{Type}_r), (\Gamma_2, a : \mathrm{Type}_r))}$$

Kind inference

$$\overline{\Gamma \vdash \mathtt{Unit} \Rightarrow \mathtt{Type} \dashv \Gamma} \qquad \overline{\Gamma \vdash \mathtt{Empty} \Rightarrow \mathtt{Type} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash r\,A \to B \Rightarrow \mathtt{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_s \dashv \Gamma'}{\Gamma \vdash !_0\,A \Rightarrow \mathtt{Type} \dashv \Gamma'} \qquad \frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_s \dashv \Gamma' \quad r \neq 0}{\Gamma \vdash !_r\,A \Rightarrow \mathtt{Type}_{r \cdot s} \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \otimes B \Rightarrow \mathtt{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \mathtt{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \mathtt{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \oplus B \Rightarrow \mathtt{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

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Kind inference

$$\frac{(a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \Rightarrow \text{Type}_r \dashv \Gamma}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall a : \text{Type}_r. \, B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall \{a : \text{Type}_r\}. \, B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

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Kind checking

$$\frac{\Gamma \vdash A \Rightarrow \mathrm{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \Leftarrow \mathrm{Type}_r \dashv \Gamma'}$$

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Context clean-up

$\Gamma, r\,x : A \vdash_i e \Leftarrow A \dashv \Gamma', 0\,x : A$ is a shorthand for
$\Gamma, r\,x : A \vdash_i e \Leftarrow A \dashv \Gamma', r'\,x : A$ with the additional condition
$r' \sqsubseteq 0$ when $i = \mathsf{c}$

$\Gamma, r\,x : A \vdash_i e \Rightarrow A \dashv \Gamma', 0\,x : A$ is a shorthand for
$\Gamma, r\,x : A \vdash_i e \Rightarrow A \dashv \Gamma', r'\,x : A$ with the additional condition
$r' \sqsubseteq 0$ when $i = \mathsf{c}$

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Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma_2} \text{\scriptsize SUBSUMPTION}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2} \text{\scriptsize ANNOT}$$

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Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{\mathbf{c}} x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{\mathbf{nc}} x \Rightarrow A \dashv \Gamma}$$

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Functions

$$\frac{\Gamma, r_A^i\, x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0\, x : A}{\Gamma \vdash_i \lambda\, x.\, e \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow r\, A \to B \dashv \Gamma_1 \quad \Gamma_1/r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f\, a \Rightarrow B \dashv \Gamma_3 + r\, \Gamma_4}$$

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Box

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \text{box } e \Leftarrow !_r A \dashv \Gamma_2 + r\,\Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0\,x : A}{\Gamma \vdash_i \text{let box } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

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Empty

$$\frac{\Gamma \vdash_i e \Leftarrow \texttt{Empty} \dashv \Gamma'}{\Gamma \vdash_i \texttt{Empty-elim}\ e \Leftarrow A \dashv \Gamma'}$$

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Unit

$$\overline{\Gamma \vdash_i \text{unit} \Leftarrow \text{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash_i \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1_A^i \, x : A, 1_B^i \, y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0 \, x : A, 0 \, y}{\Gamma \vdash_i \texttt{let } (x, y) = e_1 \texttt{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash_i \text{inl } e \Leftarrow A \oplus B \dashv \Gamma'} \qquad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash_i \text{inr } e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Leftarrow 1 A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Leftarrow 1 B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Rightarrow 1 A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Rightarrow 1 B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

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Let

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r\,\Gamma_3), r_A^i\,x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash_i \mathtt{let}_r\ x = e_1\ \mathtt{in}\ e_2 \Rightarrow B \dashv \Gamma_4}$$

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Polymorphism

$$\frac{\Gamma, a : \mathrm{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \mathrm{Type}_r}{\Gamma \vdash_i \Lambda\, a.\, e \Leftarrow \forall a : \mathrm{Type}_r.\, B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \mathrm{Type}_r.\, B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \mathrm{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e\, A \Rightarrow B\, [a := A] \dashv \Gamma_2}$$

Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \mathrm{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \mathrm{Type}_r}{\Gamma \vdash_i \Lambda\{a\}.\, e \Leftarrow \forall\{a : \mathrm{Type}_r\}.\, B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall\{a : \mathrm{Type}_r\}.\, B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \mathrm{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e\ @A \Rightarrow B\,[a := A] \dashv \Gamma_2}$$

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Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash P \equiv Q \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow Q \dashv \Gamma_2} \text{\textsc{Subsumption}}$$

$$\frac{\Gamma \vdash P \Leftarrow \mathrm{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \textbf{proving } P \textbf{ by } p \Rightarrow P \dashv \Gamma_2} \text{\textsc{Annot}}$$

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Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \qquad \frac{(h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \mathbf{assume}\ h\ \mathbf{in}\ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{apply}\ q\ p \Rightarrow Q \dashv \Gamma_2}$$

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Propositional logic

$$\frac{}{\Gamma \vdash \textbf{trivial} \Leftarrow \top \dashv \Gamma} \qquad \frac{\Gamma \vdash p \Leftarrow \bot \dashv \Gamma'}{\Gamma \vdash \textbf{absurd } p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \textbf{both } p \ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \textbf{and-left } pq \Rightarrow P \dashv \Gamma'} \qquad \frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \textbf{and-right } pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \textbf{or-left } p \Leftarrow P \vee Q \dashv \Gamma'} \qquad \frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \textbf{or-right } q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \textbf{cases } pq \ r_1 \ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \textbf{destruct } pq \textbf{ as } h_1 \ h_2 \textbf{ in } r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

Utilities

$$\frac{\Gamma \vdash P \Leftarrow \mathrm{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2 \quad \Gamma_2, h : P \vdash q \Leftarrow Q \dashv \Gamma_3, h : P}{\Gamma \vdash \textbf{lemma } h : P \textbf{ by } p \textbf{ in } q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \mathrm{prop} \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P \dashv \Gamma_3}{\Gamma \vdash \textbf{suffices } P \textbf{ by } q \textbf{ in } p \Leftarrow Q \dashv \Gamma_2}$$

Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P\,[x := y] \dashv \Gamma'}{\Gamma \vdash \textbf{pick-any } y \textbf{ in } p \Leftarrow \forall x : A.\, P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A.\, P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\mathrm{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e \Rightarrow P\,[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\mathrm{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P\,[x := e] \dashv \Gamma_2}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p \Leftarrow \exists x : A.\, P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P\,[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \textbf{pick-witness } y\; h \textbf{ for } p \textbf{ in } q \Leftarrow Q \dashv \Gamma_2}$$

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Equality

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \textbf{refl} \Leftarrow e_1 =_A e_2 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \dashv \Gamma_1 \qquad \begin{array}{l} \Gamma_1, x : A \vdash P \Leftarrow \mathrm{prop} \dashv \Gamma_2, x : A \\ \Gamma_2 \vdash p \Leftarrow P[x := e_2] \dashv \Gamma_3 \end{array}}{\Gamma \vdash \textbf{rewrite } q \textbf{ at } x.P \textbf{ in } p \Rightarrow P[x := e_1] \dashv \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \dashv \Gamma', x : A}{\Gamma \vdash \textbf{funext } x \textbf{ in } p \Leftarrow f =_{r\,A \to B} g \dashv \Gamma'}$$

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Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \bot \dashv \Gamma', h : \neg P}{\Gamma \vdash \textbf{by-contradiction } h \textbf{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \bot \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \textbf{by-contradiction } h : \neg P \textbf{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} \text{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash \textbf{choose-spec } p \Rightarrow P \, [x := \text{choose } p] \dashv \Gamma'}$$

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Classical logic

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \quad \Gamma, y : A := \texttt{choose } p, h : P\,[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \texttt{choose } p, h}{\Gamma \vdash \textbf{choose-witness } y\ h\ \textbf{for}\ p\ \textbf{in}\ q \Leftarrow Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A.\, P \dashv \Gamma_1 \quad \Gamma, y : A := \texttt{choose } p, h : P\,[x := y] \vdash_{\texttt{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \texttt{choose } p,}{\Gamma \vdash_{\texttt{nc}} \texttt{choose-witness } y\ h\ \texttt{for}\ p\ \texttt{in}\ e \Rightarrow B \dashv \Gamma_2}$$

Type conversion

$$\frac{\Gamma \vdash A \equiv B \Rightarrow \mathtt{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \equiv B \Leftarrow \mathtt{Type}_r \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \triangleq B \Rightarrow \mathtt{Type}_r \dashv \Gamma'}{\Gamma \vdash A \equiv B \Rightarrow \mathtt{Type}_r \dashv \Gamma'}$$

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Type conversion

$$\overline{\Gamma \vdash \texttt{Unit} \triangleq \texttt{Unit} \Rightarrow \texttt{Type} \dashv \Gamma}$$

$$\overline{\Gamma \vdash \texttt{Empty} \triangleq \texttt{Empty} \Rightarrow \texttt{Type} \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \texttt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \texttt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash r_1 A_1 \to B_1 \triangleq r_2 A_2 \to B_2 \Rightarrow \texttt{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \texttt{Type}_s \dashv \Gamma'}{\Gamma \vdash !_0 A_1 \triangleq !_0 A_2 \Rightarrow \texttt{Type} \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad r_1 \neq 0 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \texttt{Type}_s \dashv \Gamma'}{\Gamma \vdash !_{r_1} A_1 \triangleq !_{r_2} A_2 \Rightarrow \texttt{Type}_{r_1 \cdot s} \dashv \Gamma'}$$

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Type conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \triangleq A_2 \otimes B_2 \Rightarrow \mathtt{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \mathtt{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \mathtt{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \triangleq A_2 \oplus B_2 \Rightarrow \mathtt{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

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Type conversion

$$\frac{(a : \mathrm{Type}_r) \in \Gamma}{\Gamma \vdash a \triangleq a \Rightarrow \mathrm{Type}_r \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \mathrm{Type}_{r_1} \vdash B_1 \equiv B_2 \left[a_2 := a_1\right] \Rightarrow \mathrm{Type}_s \dashv \Gamma', a_1 : \mathrm{Type}_{r_1}}{\Gamma \vdash \forall a_1 : \mathrm{Type}_{r_1}. B_1 \triangleq \forall a_2 : \mathrm{Type}_{r_2}. B_2 \Rightarrow \mathrm{Type}_s \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \mathrm{Type}_{r_1} \vdash B_1 \equiv B_2 \left[a_2 := a_1\right] \Rightarrow \mathrm{Type}_s \dashv \Gamma', a_1 : \mathrm{Type}_{r_1}}{\Gamma \vdash \forall \{a_1 : \mathrm{Type}_{r_1}\}. B_1 \triangleq \forall \{a_2 : \mathrm{Type}_{r_2}\}. B_2 \Rightarrow \mathrm{Type}_s \dashv \Gamma'}$$

Proposition conversion

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma'}{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma'}$$

Proposition conversion

$$\overline{\Gamma \vdash \top \triangleq \top \dashv \Gamma} \qquad \overline{\Gamma \vdash \bot \triangleq \bot \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \triangleq P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \triangleq P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \triangleq P_2 \vee Q_2 \dashv \Gamma_2}$$

Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2\left[x_2 := x_1\right] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1.\, P_1 \triangleq \forall x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2\left[x_2 := x_1\right] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1.\, P_1 \triangleq \exists x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \begin{array}{c} \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A_1 \dashv \Gamma_2 \\ \Gamma_2 \vdash e_1' \equiv e_2' \Leftarrow A_1 \dashv \Gamma_3 \end{array}}{\Gamma \vdash e_1 =_{A_1} e_1' \triangleq e_2 =_{A_2} e_2' \dashv \Gamma_3}$$

Whnfs and neutral terms

Whnfs:
$\hat{e} ::=$

$\quad n \mid \lambda_r x. e \mid \text{box}_r\, e \mid (e_1, e_2) \mid \text{inl}\ e \mid \text{inr}\ e \mid \text{unit}$

$\quad \Lambda\, a : \text{Type}_r.\, e \mid \Lambda\, \{a : \text{Type}_r\}.\, e$

Neutral forms:
$n ::=$

$\quad x \mid n\, e \mid n\, A \mid n\, @A \mid$

$\quad \text{let unit} = n \text{ in } e \mid \text{let box } x = n \text{ in } e \mid$

$\quad \text{let } (x, y) = n \text{ in } e \mid \text{case } n \text{ of } (e_1, e_2) \mid$

$\quad \text{choose } p \mid \text{choose-witness } x\ h \text{ for } p \text{ in } n$

Note that Empty-elim e is neither whnf nor neutral.

Computation

$e_1 \longrightarrow e_1'$ – single step
$e_1 \longrightarrow^* e_1'$ – multi step

Term conversion (all terms)

$$\overline{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \mathtt{Unit} \dashv \Gamma}$$

$$\overline{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \mathtt{Empty} \dashv \Gamma}$$

$$\overline{\Gamma \vdash \mathtt{Empty\text{-}elim}\ e_1 \equiv \mathtt{Empty\text{-}elim}\ e_2 \Leftarrow A \dashv \Gamma'}$$

$$\frac{A \neq \mathtt{Unit} \quad A \neq \mathtt{Empty} \quad e_1 \longrightarrow^* e_1' \quad e_2 \longrightarrow^* e_2' \quad \Gamma \vdash e_1' \mathrel{\hat{\equiv}} e_2' \Leftarrow A \dashv}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}$$

Term conversion (whnfs)

$$\frac{\Gamma, x : A \vdash f\ x \equiv g\ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \triangleq g \Leftarrow r\ A \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash \mathtt{let\ box}\ x = e_1\ \mathtt{in}\ x \equiv \mathtt{let\ box}\ x = e_2\ \mathtt{in}\ x \Leftarrow A \dashv \Gamma'}{\Gamma \vdash e_1 \triangleq e_2 \Leftarrow\ !_r\ A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash \mathtt{let}\ (x, y) = e_1\ \mathtt{in}\ x \equiv \mathtt{let}\ (x, y) = e_2\ \mathtt{in}\ x \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash \mathtt{let}\ (x, y) = e_1\ \mathtt{in}\ y \equiv \mathtt{let}\ (x, y) = e_2\ \mathtt{in}\ y \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash e_1 \triangleq e_2 \Leftarrow A \otimes B \dashv \Gamma_2}$$

Term conversion (whnfs)

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \mathtt{inl}\ e_1 \triangleq \mathtt{inl}\ e_2 \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow B \dashv \Gamma'}{\Gamma \vdash \mathtt{inr}\ e_1 \triangleq \mathtt{inr}\ e_2 \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ a \equiv g\ a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \triangleq g \Leftarrow \forall a : \mathtt{Type}_r.\, B \dashv \Gamma'}$$

$$\frac{\Gamma, a : \mathtt{Type}_r \vdash f\ @a \equiv g\ @a \Leftarrow B \dashv \Gamma'}{\Gamma \vdash f \triangleq g \Leftarrow \forall \{a : \mathtt{Type}_r\}.\, B \dashv \Gamma'}$$

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Term conversion (neutral)

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \equiv x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow r\ A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1\ e_1 \equiv n_2\ e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall a : \mathrm{Type}_r.\ B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \mathrm{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1\ A_1 \equiv n_2\ A_2 \Rightarrow B\ [a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \forall \{a : \mathrm{Type}_r\}.\ B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \mathrm{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1\ @A_1 \equiv n_2\ @A_2 \Rightarrow B\ [a := A] \dashv \Gamma_2}$$

Term conversion (neutral)

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow \mathtt{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash \mathtt{let\ unit} = n_1 \mathtt{\ in\ } e_1 \equiv \mathtt{let\ unit} = n_2 \mathtt{\ in\ } e_2 \Rightarrow A \dashv}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, x : A \vdash e_1 \equiv e_2 \Rightarrow B \dashv \Gamma_2, x : A}{\Gamma \vdash \mathtt{let\ box\ } x = n_1 \mathtt{\ in\ } e_1 \equiv \mathtt{let\ box\ } x = n_2 \mathtt{\ in\ } e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \triangleq n_2 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, x : A, y : B \vdash e_1 \equiv e_2 \Rightarrow C \dashv \Gamma_2, x : A, y}{\Gamma \vdash \mathtt{let\ } (x, y) = n_1 \mathtt{\ in\ } e_1 \equiv \mathtt{let\ } (x, y) = n_2 \mathtt{\ in\ } e_2 \Rightarrow C \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash f_1 \equiv f_2 \Rightarrow 1\,A \to C \dashv \Gamma_2 \\ \Gamma_1 \vdash g_1 \equiv g_2 \Rightarrow 1\,B \to C \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathtt{case\ } e_1 \mathtt{\ of\ } (f_1, g_1) \equiv \mathtt{case\ } e_2 \mathtt{\ of\ } (f_2, g_2) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

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Term conversion (choice)

$$\frac{\begin{array}{c} \Gamma \vdash p_1 \Rightarrow \exists x : A_1.\, P_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash p_2 \Rightarrow \exists x : A_2.\, P_2 \dashv \Gamma_2 \end{array} \qquad \Gamma_2 \vdash \exists x : A_1.\, P_1 \equiv \exists x : A_1.\, P_2 \dashv \Gamma_3}{\Gamma \vdash \mathtt{choose}\ p_1 \equiv \mathtt{choose}\ p_2 \Rightarrow A_1 \dashv \Gamma_3}$$