Hint-based typing for STLC with Subtyping

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Types, contexts and judgements

Types:
$A, B ::= A \rightarrow B \mid A \times B \mid A + B \mid \top \mid \bot$

Typing contexts:
$\Gamma ::= \cdot \mid \Gamma, x : A$

Judgements:
$\Gamma \vdash e : A$

Subtyping relation

$$\overline{\bot <: A} \quad \overline{A <: \top}$$

$$\frac{A' <: A \quad B <: B'}{A \to B <: A' \to B'}$$

$$\frac{A <: A' \quad B <: B'}{A \times B <: A' \times B'}$$

$$\frac{A <: A' \quad B <: B'}{A + B <: A' + B'}$$

The subtyping relation is the reflexive-transitive closure of the above rules.

Subtyping – properties

Subtyping is a partial order with top and bottom, congruent with
type constructors.

- Reflexivity: $A <: A$
- Transitivity: $A <: B \implies B <: C \implies A <: C$
- Antisymmetry: $A <: B \implies B <: A \implies A = B$

Terms

Terms:

$e ::=$

$\quad x \mid$

$\quad \lambda x.\, e \mid e_1\ e_2 \mid$

$\quad (e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$

$\quad \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$

$\quad \text{unit} \mid \text{elim}_0\ e$

Note: unit is the term of the $\top$ type, whereas elim_0 is the eliminator of the $\bot$ type.

Declarative typing – differences

$$\frac{\Gamma \vdash e : A \quad A <: B}{\Gamma \vdash e : B}\text{Sub}$$

Comments

The only difference between Extrinsic STLC with Subtyping and
the original Extrinsic STLC is the addition of the subsumtpion rule.
We invoke this rule every time we need subtyping to happen.
However, this rule is stationary, and thus non-algorithmic, so we
need to seek a better system.

Bidirectional typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\text{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A}\text{Annot}$$

$$\frac{\Gamma \vdash e \Rightarrow A \quad A <: B}{\Gamma \vdash e \Leftarrow B}\text{Sub}$$

Metatheory

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Hints

We no longer need holes – because of $\top$ and $\bot$, types suffice. We also don't need a separate order on hints, because subtyping suffices.

Least upper bound of two types

$$\bot \sqcup H = H$$
$$H \sqcup \bot = H$$
$$\top \sqcup H = \top$$
$$H \sqcup \top = \top$$
$$(H_1 \to H_2) \sqcup (H_1' \to H_2') = (H_1 \sqcap H_1') \to (H_2 \sqcup H_2')$$
$$(H_1 \times H_2) \sqcup (H_1' \times H_2') = (H_1 \sqcup H_1') \times (H_2 \sqcup H_2')$$
$$(H_1 + H_2) \sqcup (H_1' + H_2') = (H_1 \sqcup H_1') + (H_2 \sqcup H_2')$$

Greatest lower bound of two types

$$\bot \sqcap H = \bot$$
$$H \sqcap \bot = \bot$$
$$\top \sqcap H = H$$
$$H \sqcap \top = H$$
$$(H_1 \to H_2) \sqcap (H_1' \to H_2') = (H_1 \sqcup H_1') \to (H_2 \sqcap H_2')$$
$$(H_1 \times H_2) \sqcap (H_1' \times H_2') = (H_1 \sqcap H_1') \times (H_2 \sqcap H_2')$$
$$(H_1 + H_2) \sqcap (H_1' + H_2') = (H_1 \sqcap H_1') + (H_2 \sqcap H_2')$$

Hints for term constructors

$$\mathtt{hint}(\lambda x.\, e) = \bot \to \top$$
$$\mathtt{hint}((e_1, e_2)) = \top \times \top$$
$$\mathtt{hint}(\mathtt{inl}\ e) = \top + \bot$$
$$\mathtt{hint}(\mathtt{inr}\ e) = \bot + \top$$

Subtyping
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Extrinsic STLC with Subtyping
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Bidirectional STLC with Subtyping
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Hinting
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Terms and judgements

Terms:

$e ::=$

$\quad x \mid (e : A) \mid$

$\quad \lambda x.\, e \mid e_1\ e_2 \mid$

$\quad (e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\quad \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\quad \texttt{unit} \mid \texttt{elim}_0\ e$

Note: terms are the same as in Bidirectional STLC, i.e. annotations are types, not hints.

Judgements:

$\Gamma \vdash e \Leftarrow A \Rightarrow B$ – in context Γ, term e checks with type A and infers type B

Subtyping
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Extrinsic STLC with Subtyping
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Bidirectional STLC with Subtyping
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Hinting
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Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\textsc{Var}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash (e : A) : A}\textsc{Annot}$$

$$\frac{\Gamma \vdash e : A \quad A <: B}{\Gamma \vdash e : B}\textsc{Sub}$$

Subtyping
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Algorithmic typing – basic rules

$$\frac{(x : B) \in \Gamma \quad B <: A}{\Gamma \vdash x \Leftarrow A \Rightarrow B} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow C \quad B <: A}{\Gamma \vdash (e : B) \Leftarrow A \Rightarrow C} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \Leftarrow \text{hint}(e) \Rightarrow A \quad e \text{ constructor}}{\Gamma \vdash e \Leftarrow \top \Rightarrow A} \text{HOLE}$$

Note that the rule HOLE can only be applied once, because $\text{hint}(e)$ can never be $\top$. After applying HOLE, the only applicable rules are the type-directed ones.

Algorithmic typing – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \rightarrow B \Rightarrow A \rightarrow B'}$$

$$\frac{\Gamma \vdash f \Leftarrow \bot \rightarrow B \Rightarrow A \rightarrow B' \quad \Gamma \vdash a \Leftarrow A \Rightarrow A'}{\Gamma \vdash f\, a \Leftarrow B \Rightarrow B'}$$

$$\frac{\Gamma \vdash a \Leftarrow A \Rightarrow A' \quad \Gamma \vdash b \Leftarrow B \Rightarrow B'}{\Gamma \vdash (a, b) \Leftarrow A \times B \Rightarrow A' \times B'}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times \top \Rightarrow A' \times B}{\Gamma \vdash \mathtt{outl}\, e \Leftarrow A \Rightarrow A'} \qquad \frac{\Gamma \vdash e \Leftarrow \top \times B \Rightarrow A \times B'}{\Gamma \vdash \mathtt{outr}\, e \Leftarrow B \Rightarrow B'}$$

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Algorithmic typing – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow A \Rightarrow A'}{\Gamma \vdash \texttt{inl } e \Leftarrow A + B \Rightarrow A' + B}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow B'}{\Gamma \vdash \texttt{inr } e \Leftarrow A + B \Rightarrow A + B'}$$

$$\frac{\Gamma \vdash e \Leftarrow \top + \top \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f \Leftarrow A \rightarrow C_1 \Rightarrow A' \rightarrow C_2 \\ \Gamma \vdash g \Leftarrow B \rightarrow C_2 \Rightarrow B' \rightarrow C_3 \end{array}}{\Gamma \vdash \texttt{case } e \texttt{ of } (f, g) \Leftarrow C_1 \Rightarrow C_3}$$

$$\frac{}{\Gamma \vdash \texttt{unit} \Leftarrow \top \Rightarrow \top} \qquad \frac{\Gamma \vdash e \Leftarrow \bot \Rightarrow \bot}{\Gamma \vdash \texttt{elim}_0 \ e \Leftarrow A \Rightarrow A}$$

Algorithmic rules – alternative rules

$$\dfrac{\Gamma \vdash a \Leftarrow \top \Rightarrow A \quad \Gamma \vdash f \Leftarrow A \to B \Rightarrow A' \to B'}{\Gamma \vdash f\ a \Leftarrow B \Rightarrow B'} \text{AltApp}$$

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Metatheory

- If $\Gamma \vdash e \Leftarrow A \Rightarrow A'$, then $A' <: A$