

# T-Axi (declarative)

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# TODO

- Type operators.
- Higher-order quantification.
- Totality checker.
- Computation.
- Inductives and records.
- Algorithmic system.
- Distinguish partial terms.

# Quantities

Quantities:

$$r ::= 0 \mid 1 \mid ? \mid + \mid *$$

\* is the default quantity, so when there's nothing to indicate quantity, it means it's \*.

## Kinds

$U ::= \text{Type}_1 \mid \text{Type}_? \mid \text{Type}_+ \mid \text{Type}_*$

Type is a notation for Type<sub>\*</sub>

## Types

## Types:

*A, B ::=*

$r A \rightarrow B$  |  $!_r A$  |  $A \otimes B$  |  $A \oplus B$  | Unit | Empty |  
 $\forall a : \text{Type}_r. A$  |  $\forall\{a : \text{Type}_r\}. A$

## Terms

## Terms:

$e :=$

$x \mid \lambda_r x : A. e \mid e_1 e_2 \mid$   
 $\text{box}_r e \mid \text{let}_A \text{ box } x = e_1 \text{ in } e_2 \mid$   
 $(e_1, e_2) \mid \text{let}_A (x, y) = e_1 \text{ in } e_2 \mid$   
 $\text{inl}_A e \mid \text{inr}_A e \mid \text{case}_A e \text{ of } (e_1, e_2) \mid$   
 $\text{unit} \mid \text{let}_A \text{ unit } = e_1 \text{ in } e_2 \mid$   
 $\text{Empty-elim}_A e \mid$   
 $\text{let}_r x : A = e_1 \text{ in } e_2 \mid$   
 $\Lambda a : \text{Type}_r. e \mid e A \mid$   
 $\Lambda \{a : \text{Type}_r\}. e \mid e @A \mid$   
 $\text{choose } p \mid \text{choose-witness } x h \text{ for } p \text{ in } e$

choose  $p$  and choose-witness  $x \ h$  for  $p$  in  $e$  are noncomputable terms, whereas all others are computable.

# Propositions

Propositions:

$P, Q ::=$

$\top \mid \perp \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid$

$\forall x : A. P \mid \exists x : A. P \mid$

$e_1 =_A e_2$

Notations:

$\neg P$  stands for  $P \Rightarrow \perp$

$P \Leftrightarrow Q$  stands for  $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

# Proofterms

Proofterms ( $P, Q$  are propositions,  $e$  are terms,  $h$  are variables):

$p, q ::=$

**$h \mid \text{assumption} \mid \text{trivial} \mid \text{absurd } p$**

**$\text{assume } h : P \text{ in } q \mid \text{apply } p_1 \ p_2 \mid$**

**$\text{both } p_1 \ p_2 \mid \text{and-left } p \mid \text{and-right } p \mid$**

**$\text{or-left } p \mid \text{or-right } p \mid \text{cases } p_1 \ p_2 \ p_3 \mid$**

**$\text{lemma } h : P \text{ by } p \text{ in } q \mid \text{proving } P \text{ by } p \mid$**

**$\text{suffices } P \text{ by } q \text{ in } p \mid$**

**$\text{pick-any } x : A \text{ in } e \mid \text{instantiate } p \text{ with } e \mid$**

**$\text{witness } e \text{ such that } p \mid \text{pick-witness } x \ h \text{ for } p_1 \text{ in } p_2 \mid$**

**$\text{refl } e \mid \text{rewrite } p_1 \text{ in } p_2 \mid \text{funext } x : A \text{ in } p$**

**$\text{by-contradiction } h : \neg P \text{ in } q \mid$**

**$\text{choose-spec } p \mid \text{choose-witness } x \ h \text{ for } p \text{ in } q$**

# Contexts

Contexts:

$$\Gamma ::= \cdot \mid \Gamma, r x : A \mid \Gamma, r x : A := e \mid \Gamma, h : P \mid \Gamma, a : \text{Type}_r$$

# Judgements

Well-formed context judgement:  $\Gamma \text{ ctx}_i$ , where  $i$  is either c or nc.

Well-formed type judgement:  $\Gamma \vdash A : \text{Type}_r$ ,

Type conversion judgement:  $\Gamma \vdash A \equiv B : \text{Type}_r$ ,

Typing judgement:  $\Gamma \vdash; e : A$ , where  $i$  is either c or nc

Conversion judgement:  $\Gamma \vdash e_1 \equiv e_2 : A$

Well-formed proposition judgement:  $\Gamma \vdash P \text{ prop}$

Proposition conversion judgement:  $\Gamma \vdash P \equiv Q \text{ prop}$

Proof judgement:  $\Gamma \vdash p : P$

# Sanity checks

We'll set up the system so that:

- If  $\Gamma \vdash e_1 \equiv e_2 : A$ , then  $\Gamma \vdash_{\text{nc}} e_1 : A$  and  $\Gamma \vdash_{\text{nc}} e_2 : A$ .
- If  $\Gamma \vdash_i e : A$ , then  $\Gamma \text{ ctx}_i$  and  $|\Gamma| \vdash A : \text{Type}_{\text{qty}(A)}$ .
- If  $\Gamma \vdash A \equiv B : \text{Type}_r$ , then  $\Gamma \vdash A : \text{Type}_r$  and  $\Gamma \vdash B : \text{Type}_r$ .
- If  $\Gamma \vdash A : \text{Type}_r$ , then  $\Gamma \text{ ctx}_{\text{nc}}$ .
- If  $\Gamma \vdash p : P$ , then  $\Gamma \vdash P \text{ prop}$ .
- If  $\Gamma \vdash P \equiv Q \text{ prop}$ , then  $\Gamma \vdash P \text{ prop}$  and  $\Gamma \vdash Q \text{ prop}$ .
- If  $\Gamma \vdash P \text{ prop}$ , then  $\Gamma \text{ ctx}_{\text{nc}}$ .
- If  $\Gamma \text{ ctx}_{\text{nc}}$ , then  $\Gamma \text{ ctx}_c$ .

# Quantities

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced “few”) means a resource must be used at most once.
- + (pronounced “many”) means a resource must be used at least once.
- \* (pronounced “any”) means no restrictions on usage.

## Subusage ordering

$r_1 \sqsubseteq r_2$  means that a resource with quantity  $r_1$  may be used when quantity  $r_2$  is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$* \sqsubseteq ?$$

$$* \sqsubseteq +$$

$$+ \sqsubseteq 1$$

$$? \sqsubseteq 1$$

$$? \sqsubseteq 0$$

We will denote the greatest lower bound in this order with  $r_1 \sqcap r_2$  and the least upper bound (if it exists) with  $r_1 \sqcup r_2$ .

# Addition of quantities

When we have two quantities of the same resource, we can sum the quantities.

$$0 + r = r$$

$$r + 0 = r$$

$$? + ? = *$$

$$? + * = *$$

$$* + ? = *$$

$$* + * = *$$

$$\_ + \_ = +$$

# Multiplication of quantities

When we have a quantity  $r_1$  of resource  $A$  that contains quantity  $r_2$  of resource  $B$ , then we in fact have quantity  $r_1 \cdot r_2$  of resource  $B$ .

$$0 \cdot r = 0$$

$$r \cdot 0 = 0$$

$$1 \cdot r = r$$

$$r \cdot 1 = r$$

$$?\cdot?=?$$

$$+\cdot+=+$$

$$-\cdot-=*$$

# The algebra of quantities

Quantities  $\mathcal{Q}$  form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$  is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$  is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If  $r_1 + r_2 = 0$ , then  $r_1 = 0$  and  $r_2 = 0$ .
- If  $r_1 \cdot r_2 = 0$ , then  $r_1 = 0$  or  $r_2 = 0$ .

# Operations on contexts

Operations on contexts:

$\Gamma_1 \sqsubseteq \Gamma_2$  – context subusaging

$\Gamma_1 + \Gamma_2$  – context addition

$r\Gamma$  – context scaling

$|\Gamma|$  – cartesianization

$|\Gamma|_x$  – spotlight  $x$

# Context subusage

$$\cdot \overline{\sqsubseteq} \cdot$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, r_1 x : A \sqsubseteq \Gamma_2, r_2 x : A}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, r_1 x : A := e \sqsubseteq \Gamma_2, r_2 x : A := e}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2}{\Gamma_1, h : P \sqsubseteq \Gamma_2, h : P}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad r_1 \sqsubseteq r_2}{\Gamma_1, a : \text{Type}_{r_1} \sqsubseteq \Gamma_2, a : \text{Type}_{r_2}}$$

# Context addition

$$\cdot + \cdot = \cdot$$

$$(\Gamma_1, r_1 x : A) + (\Gamma_2, r_2 x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2) x : A$$

$$(\Gamma_1, r_1 x : A := e) + (\Gamma_2, r_2 x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2) x : A := e$$

$$(\Gamma_1, h : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P$$

$$(\Gamma_1, a : \text{Type}_r) + (\Gamma_2, a : \text{Type}_r) = (\Gamma_1 + \Gamma_2), a : \text{Type}_r$$

# Context scaling

$$\begin{aligned}s \cdot &= . \\ s(\Gamma, rx : A) &= s\Gamma, (s \cdot r)x : A \\ s(\Gamma, rx : A := e) &= s\Gamma, (s \cdot r)x : A := e \\ s(\Gamma, h : P) &= s\Gamma, h : P \\ s(\Gamma, a : \text{Type}_r) &= s\Gamma, a : \text{Type}_r\end{aligned}$$

# Spotlight

$|\cdot|_x = \mathbf{undefined}$

$|\Gamma, r x : A|_x = 0 \Gamma, 1 x : A$

$|\Gamma, r y : A|_x = |\Gamma|_x, 0 y : A$

$|\Gamma, r x : A := e|_x = 0 \Gamma, 1 x : A := e$

$|\Gamma, r y : A := e|_x = |\Gamma|_x, 0 y : A := e$

$|\Gamma, h : P|_x = |\Gamma|_x, h : P$

$|\Gamma, a : \text{Type}_r|_x = |\Gamma|_x, a : \text{Type}_r$

# Cartesianization

Cartesianization turns a context into a context with the same shape but unlimited resources.

$$\begin{aligned} |\cdot| &= \cdot \\ |\Gamma, r x : A| &= |\Gamma|, x : A \\ |\Gamma, r x : A := e| &= |\Gamma|, x : A := e \\ |\Gamma, h : P| &= |\Gamma|, h : P \\ |\Gamma, a : \text{Type}_r| &= |\Gamma|, a : \text{Type}_r \end{aligned}$$

# Well-formed contexts

 $\cdot \overline{\text{ctx}_c}$ 

$$\frac{\Gamma \text{ ctx}_c \quad |\Gamma| \vdash A : \text{Type}_s \quad x \notin \Gamma}{\Gamma, r x : A \text{ ctx}_c}$$

$$\frac{\Gamma \text{ ctx}_c \quad |\Gamma| \vdash_{\text{nc}} e : A \quad x \notin \Gamma}{\Gamma, r x : A := e \text{ ctx}_c}$$

$$\frac{\Gamma \text{ ctx}_c \quad |\Gamma| \vdash P \text{ prop} \quad h \notin \Gamma}{\Gamma, h : P \text{ ctx}_c}$$

$$\frac{\Gamma \text{ ctx}_c \quad a \notin \Gamma \quad r \neq 0}{\Gamma, a : \text{Type}_r \text{ ctx}_c}$$

# Well-formed cartesian contexts

$$\frac{\Gamma \text{ ctx}_c \quad \Gamma = |\Gamma|}{\Gamma \text{ ctx}_{nc}}$$

# Well-formed types

$$\frac{\Gamma \text{ ctx}_{\text{nc}}}{\Gamma \vdash \text{Unit} : \text{Type}}$$

$$\frac{\Gamma \text{ ctx}_{\text{nc}}}{\Gamma \vdash \text{Empty} : \text{Type}}$$

$$\frac{\Gamma \vdash A : \text{Type}_{s_1} \quad \Gamma \vdash B : \text{Type}_{s_2}}{\Gamma \vdash rA \rightarrow B : \text{Type}_1}$$

$$\frac{\Gamma \vdash A : \text{Type}_s}{\Gamma \vdash !_0 A : \text{Type}}$$

$$\frac{\Gamma \vdash A : \text{Type}_s \quad r \neq 0}{\Gamma \vdash !_r A : \text{Type}_{r,s}}$$

$$\frac{\Gamma \vdash A : \text{Type}_{s_1} \quad \Gamma \vdash B : \text{Type}_{s_2}}{\Gamma \vdash A \otimes B : \text{Type}_{s_1 \sqcup s_2}}$$

$$\frac{\Gamma \vdash A : \text{Type}_{s_1} \quad \Gamma \vdash B : \text{Type}_{s_2}}{\Gamma \vdash A \oplus B : \text{Type}_{s_1 \sqcup s_2}}$$

# Well-formed types

$$\frac{\Gamma \text{ ctx}_{\text{nc}} \quad (a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a : \text{Type}_r}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B : \text{Type}_s}{\Gamma \vdash \forall a : \text{Type}_r. B : \text{Type}_s}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B : \text{Type}_s}{\Gamma \vdash \forall \{a : \text{Type}_r\}. B : \text{Type}_s}$$

# The inherent quantity of a type

$$\begin{aligned} \text{qty}(\text{Unit}) &= * \\ \text{qty}(\text{Empty}) &= * \\ \text{qty}(!_0 A) &= * \\ \text{qty}(!_r A) &= r \cdot \text{qty}(A) \\ \text{qty}(A \otimes B) &= \text{qty}(A) \sqcup \text{qty}(B) \\ \text{qty}(A \oplus B) &= \text{qty}(A) \sqcup \text{qty}(B) \\ \text{qty}(r A \rightarrow B) &= 1 \\ \text{qty}(\forall a : \text{Type}_r. B) &= \text{qty}(B) \\ \text{qty}(\forall \{a : \text{Type}_r\}. B) &= \text{qty}(B) \end{aligned}$$

# Total quantity added to context

$$\begin{aligned}r_A^{\text{nc}} &= * \\r_A^{\text{c}} &= r \cdot \text{qty}(A)\end{aligned}$$

# Using variables

$$\frac{\Gamma \text{ ctx}_i \quad (x : A) \in \Gamma \quad \Gamma \sqsubseteq |\Gamma|_x}{\Gamma \vdash_i x : A}$$

# Functions

$$\frac{\Gamma, r_A^i x : A \vdash_i e : B}{\Gamma \vdash_i \lambda_r x : A. e : r A \rightarrow B}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + r \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : r A \rightarrow B \quad \Gamma_2 \vdash_i e_2 : A}{\Gamma \vdash_i e_1 \ e_2 : B}$$

# Box

$$\frac{\Gamma \sqsubseteq r \Gamma' \quad \Gamma' \vdash_i e : A}{\Gamma \vdash_i \text{box}_r e : !_r A}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : !_r A \quad \Gamma_2, r_A^i x : A \vdash_i e_2 : B}{\Gamma \vdash_i \text{let}_B \text{ box } x = e_1 \text{ in } e_2 : B}$$

# Empty

$$\frac{|\Gamma| \vdash A : \text{Type}_r \quad \Gamma \vdash; e : \text{Empty}}{\Gamma \vdash; \text{Empty-elim}_A e : A}$$

# Unit

$$\frac{\Gamma \text{ ctx}_i \quad \Gamma \sqsubseteq 0\Gamma}{\Gamma \vdash_i \text{unit} : \text{Unit}}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : \text{Unit} \quad \Gamma_2 \vdash_i e_2 : A}{\Gamma \vdash_i \text{let}_A \text{ unit} = e_1 \text{ in } e_2 : A}$$

# Products

$$\frac{\Gamma \sqsubseteq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash; a : A \quad \Gamma_2 \vdash; b : B}{\Gamma \vdash; (a, b) : A \otimes B}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash; e_1 : A \otimes B \quad \Gamma_2, 1_A^i x : A, 1_B^i y : B \vdash; e_2 : C}{\Gamma \vdash; \text{let}_C (x, y) = e_1 \text{ in } e_2 : C}$$

# Sums

$$\frac{\Gamma \vdash; e : A \quad |\Gamma| \vdash B : \text{Type}_r}{\Gamma \vdash; \text{inl}_B e : A \oplus B} \quad \frac{|\Gamma| \vdash A : \text{Type}_r \quad \Gamma \vdash; e : B}{\Gamma \vdash; \text{inr}_A e : A \oplus B}$$

$$\frac{\Gamma \sqsubseteq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash; e : A \oplus B \quad \Gamma_2 \vdash; f : 1A \rightarrow C \quad \Gamma_2 \vdash; g : 1B \rightarrow C}{\Gamma \vdash; \text{case}_C e \text{ of } (f, g) : C}$$

Q: Do we want first-order representation of the branches?  
Probably yes.

# Let

$$\frac{\Gamma \sqsubseteq r \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash_i e_1 : A \quad \Gamma_2, r_A^i x : A := e_1 \vdash_i e_2 : B}{\Gamma \vdash_i \text{let}_r x : A = e_1 \text{ in } e_2 : B}$$

# Polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e : B}{\Gamma \vdash_i \Lambda a : \text{Type}_r. e : \forall a : \text{Type}_r. B}$$

$$\frac{\Gamma \vdash_i e : \forall a : \text{Type}_r. B \quad |\Gamma| \vdash A : \text{Type}_r}{\Gamma \vdash_i e A : B [a := A]}$$

# Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e : B}{\Gamma \vdash_i \Lambda \{a : \text{Type}_r\}. e : \forall \{a : \text{Type}_r\}. B}$$

$$\frac{\Gamma \vdash_i e : \forall \{a : \text{Type}_r\}. B \quad |\Gamma| \vdash A : \text{Type}_r}{\Gamma \vdash_i e @ A : B [a := A]}$$

# Well-formed propositions

$$\frac{}{\Gamma \vdash \top \text{ prop}} \quad \frac{}{\Gamma \vdash \perp \text{ prop}}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \Rightarrow Q \text{ prop}}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \wedge Q \text{ prop}} \quad \frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \vee Q \text{ prop}}$$

$$\frac{\Gamma \vdash A : \text{Type}, \quad \Gamma, x : A \vdash P \text{ prop}}{\Gamma \vdash \forall x : A. P \text{ prop}}$$

$$\frac{\Gamma \vdash A : \text{Type}, \quad \Gamma, x : A \vdash P \text{ prop}}{\Gamma \vdash \exists x : A. P \text{ prop}}$$

$$\frac{\Gamma \vdash A : \text{Type}, \quad \Gamma \vdash_{\text{nc}} e_1 : A \quad \Gamma \vdash_{\text{nc}} e_2 : A}{\Gamma \vdash e_1 =_A e_2 \text{ prop}}$$

# Substitution

The notation is  $P[x := e]$  for substitution in propositions.

# Assumptions and implication

$$\frac{\Gamma \text{ ctx}_{\text{nc}} \quad (h : P) \in \Gamma}{\Gamma \vdash h : P} \quad \frac{\Gamma \text{ ctx}_{\text{nc}} \quad (h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \mathbf{assume} \ h : P \ \mathbf{in} \ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \mathbf{apply} \ q \ p : Q}$$

# Propositional logic

$$\frac{\Gamma \text{ ctx}_{\text{nc}}}{\Gamma \vdash \mathbf{trivial} : \top}$$

$$\frac{\Gamma \vdash Q \text{ prop} \quad \Gamma \vdash p : \perp}{\Gamma \vdash \mathbf{absurd} \ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash \mathbf{both} \ p \ q : P \wedge Q}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \mathbf{and-left} \ pq : P}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \mathbf{and-right} \ pq : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash \mathbf{or-left} \ p : P \vee Q}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash q : Q}{\Gamma \vdash \mathbf{or-right} \ q : P \vee Q}$$

$$\frac{\Gamma \vdash pq : P \vee Q \quad \Gamma \vdash r_1 : P \Rightarrow R \quad \Gamma \vdash r_2 : Q \Rightarrow R}{\Gamma \vdash \mathbf{cases} \ pq \ r_1 \ r_2 : R}$$

# Utilities

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \mathbf{lemma} \ h : P \ \mathbf{by} \ p \ \mathbf{in} \ q : Q}$$

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \mathbf{proving} \ P \ \mathbf{by} \ p : P}$$

$$\frac{\Gamma \vdash pq : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \mathbf{suffices} \ P \ \mathbf{by} \ pq \ \mathbf{in} \ p : Q}$$

# Quantifiers

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \mathbf{pick-any} \ x : A \ \mathbf{in} \ p : \forall x : A. P}$$

$$\frac{\Gamma \vdash p : \forall x : A. P \quad \Gamma \vdash_{\text{nc}} e : A}{\Gamma \vdash \mathbf{stantiate} \ p \ \mathbf{with} \ e : P[x := e]}$$

$$\frac{\Gamma \vdash_{\text{nc}} e : A \quad \Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash p : P[x := e]}{\Gamma \vdash \mathbf{witness} \ e \ \mathbf{such \ that} \ p : \exists x : A. P}$$

$$\frac{\Gamma \vdash p : \exists x : A. P \quad \Gamma \vdash Q \text{ prop} \quad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \mathbf{pick-witness} \ x \ h \ \mathbf{for} \ p \ \mathbf{in} \ q : Q}$$

# Equality

$$\frac{\Gamma \vdash_{\text{nc}} e : A}{\Gamma \vdash \mathbf{refl} e : e =_A e}$$

$$\frac{\Gamma \vdash q : e_1 =_A e_2 \quad \Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash p : P [x := e_2]}{\Gamma \vdash \mathbf{rewrite} \ q \ \mathbf{in} \ p : P [x := e_1]}$$

$$\frac{\Gamma, x : A \vdash p : f \ x =_B g \ x}{\Gamma \vdash \mathbf{funext} \ x : A \ \mathbf{in} \ p : f =_{rA \rightarrow B} g}$$

# Classical Logic

$$\frac{\Gamma, h : \neg P \vdash q : \perp}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q : P}$$

$$\frac{\Gamma \vdash p : \exists x : A. P}{\Gamma \vdash_{\text{nc}} \text{choose } p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A. P}{\Gamma \vdash \text{choose-spec } p : P[x := \text{choose } p]}$$

$$\frac{\Gamma \vdash p : \exists x : A. P \quad \Gamma \vdash Q \text{ prop} \quad \Gamma, x : A := \text{choose } p, h : P \vdash q : Q}{\Gamma \vdash \text{choose-witness } x \text{ for } p \text{ in } q : Q}$$

$$\frac{\Gamma \vdash p : \exists x : A. P \quad \Gamma \vdash B : \text{Type}_r \quad \Gamma, x : A := \text{choose } p, h : P \vdash_{\text{nc}} e : B}{\Gamma \vdash_{\text{nc}} \text{choose-witness } x \text{ for } p \text{ in } e : B}$$

# Conversion rules

$$\frac{\Gamma \vdash_i e : A \quad |\Gamma| \vdash A \equiv B : \text{Type}_r}{\Gamma \vdash_i e : B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \quad \Gamma \vdash A \equiv B : \text{Type}_r}{\Gamma \vdash e_1 \equiv e_2 : B}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash P \equiv Q \text{ prop}}{\Gamma \vdash p : Q}$$

# Type conversion

$$\frac{\Gamma \vdash \text{ctx}_{\text{nc}}}{\Gamma \vdash \text{Unit} \equiv \text{Unit} : \text{Type}}$$
$$\frac{\Gamma \vdash \text{ctx}_{\text{nc}}}{\Gamma \vdash \text{Empty} \equiv \text{Empty} : \text{Type}}$$
$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_{s_1} \quad \Gamma \vdash B_1 \equiv B_2 : \text{Type}_{s_2}}{\Gamma \vdash r A_1 \rightarrow B_1 \equiv r A_2 \rightarrow B_2 : \text{Type}_1}$$
$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_s}{\Gamma \vdash !_0 A_1 \equiv !_0 A_2 : \text{Type}}$$
$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_s \quad r \neq 0}{\Gamma \vdash !_r A_1 \equiv !_r A_2 : \text{Type}_{r.s}}$$
$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_{s_1} \quad \Gamma \vdash B_1 \equiv B_2 : \text{Type}_{s_2}}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 : \text{Type}_{s_1 \sqcup s_2}}$$
$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_{s_1} \quad \Gamma \vdash B_1 \equiv B_2 : \text{Type}_{s_2}}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 : \text{Type}_{s_1 \sqcup s_2}}$$

# Type conversion

$$\frac{\Gamma \text{ ctx}_{\text{nc}} \quad (a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \equiv a : \text{Type}_r}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B_1 \equiv B_2 : \text{Type}_s}{\Gamma \vdash \forall a : \text{Type}_r. B_1 \equiv \forall a : \text{Type}_r. B_2 : \text{Type}_s}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B_1 \equiv B_2 : \text{Type}_s}{\Gamma \vdash \forall \{a : \text{Type}_r\}. B_1 \equiv \forall \{a : \text{Type}_r\}. B_2 : \text{Type}_s}$$

## Type conversion – properties

The rules above are the complete definition of type conversion. We don't need to take any closures – type conversion already is an equivalence relation.

- $\Gamma \vdash A \equiv A : \text{Type}$
- If  $\Gamma \vdash A \equiv B : \text{Type}$  then  $\Gamma \vdash B \equiv A : \text{Type}$
- If  $\Gamma \vdash A \equiv B : \text{Type}$  and  $\Gamma \vdash B \equiv C : \text{Type}$  then  
 $\Gamma \vdash A \equiv C : \text{Type}$

# Proposition conversion

$$\frac{\Gamma \vdash \text{ctx}_{\text{nc}}}{\Gamma \vdash \top \equiv \top \text{ prop}} \quad \frac{\Gamma \vdash \text{ctx}_{\text{nc}}}{\Gamma \vdash \perp \equiv \perp \text{ prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \text{ prop} \quad \Gamma \vdash Q_1 \equiv Q_2 \text{ prop}}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2 \text{ prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \text{ prop} \quad \Gamma \vdash Q_1 \equiv Q_2 \text{ prop}}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2 \text{ prop}}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \text{ prop} \quad \Gamma \vdash Q_1 \equiv Q_2 \text{ prop}}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2 \text{ prop}}$$

# Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_r \quad \Gamma, x : A_1 \vdash P_1 \equiv P_2 \text{ prop}}{\Gamma \vdash \forall x : A_1. P_1 \equiv \forall x : A_2. P_2 \text{ prop}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_r \quad \Gamma, x : A_1 \vdash P_1 \equiv P_2 \text{ prop}}{\Gamma \vdash \exists x : A_1. P_1 \equiv \exists x : A_2. P_2 \text{ prop}}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 : \text{Type}_r \quad \Gamma \vdash e_1 \equiv e_2 : A_1 \quad \Gamma \vdash e'_1 \equiv e'_2 : A_1}{\Gamma \vdash e_1 =_{A_1} e'_1 \equiv e_2 =_{A_2} e'_2 \text{ prop}}$$

Note that we don't worry about  $\alpha$ -conversion and assume the variable is the same on both sides.

# Term conversion – closure

$$\frac{\Gamma \vdash_i e : A}{\Gamma \vdash e \equiv e : A} \text{REFL}$$

$$\frac{\Gamma \vdash e_2 \equiv e_1 : A}{\Gamma \vdash e_1 \equiv e_2 : A} \text{SYM}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \quad \Gamma \vdash e_2 \equiv e_3 : A}{\Gamma \vdash e_1 \equiv e_3 : A} \text{TRANS}$$

# Term conversion – Empty

$$\frac{\Gamma \vdash_{\text{nc}} e_1 : \text{Empty} \quad \Gamma \vdash_{\text{nc}} e_2 : \text{Empty}}{\Gamma \vdash e_1 \equiv e_2 : \text{Empty}}$$

$$\frac{\Gamma \vdash A : \text{Type}_r \quad \Gamma \vdash e_1 \equiv e_2 : \text{Empty}}{\Gamma \vdash \text{Empty-elim}_A e_1 \equiv \text{Empty-elim}_A e_2 : A}$$

# Term conversion – Unit

$$\frac{\Gamma \vdash_{\text{nc}} a : A}{\Gamma \vdash \text{let unit} = \text{unit} \text{ in } a \equiv a : A}$$

$$\frac{\Gamma \vdash_{\text{nc}} u_1 : \text{Unit} \quad \Gamma \vdash_{\text{nc}} u_2 : \text{Unit}}{\Gamma \vdash u_1 \equiv u_2 : \text{Unit}}$$

$$\frac{\Gamma \vdash u_1 \equiv u_2 : \text{Unit} \quad \Gamma \vdash a_1 \equiv a_2 : A}{\Gamma \vdash \text{let}_A \text{unit} = u_1 \text{ in } a_1 \equiv \text{let}_A \text{unit} = u_2 \text{ in } a_2 : A}$$

# Term conversion – functions

$$\frac{\Gamma, x : A \vdash_{\text{nc}} b : B \quad \Gamma \vdash_{\text{nc}} a : A}{\Gamma \vdash (\lambda_r x : A. b) \ a \equiv b[x := a] : B}$$

$$\frac{\Gamma, x : A \vdash f \ x \equiv g \ x : B}{\Gamma \vdash f \equiv g : r A \rightarrow B}$$

$$\frac{\Gamma \vdash f_1 \equiv f_2 : r A \rightarrow B \quad \Gamma \vdash a_1 \equiv a_2 : A}{\Gamma \vdash f_1 \ a_1 \equiv f_2 \ a_2 : B}$$

# Term conversion – box

$$\frac{\Gamma \vdash_{\text{nc}} a : A \quad \Gamma, x : A \vdash_{\text{nc}} b : B}{\Gamma \vdash \text{let}_B \text{ box } x = \text{box}_r a \text{ in } b \equiv b[x := a] : B}$$

$$\frac{\Gamma \vdash \text{let}_A \text{ box } x = e_1 \text{ in } x \equiv \text{let}_A \text{ box } x = e_2 \text{ in } x : A}{\Gamma \vdash e_1 \equiv e_2 : !_r A}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : !_r A \quad \Gamma, x : A_1 \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \text{let}_B \text{ box } x = a_1 \text{ in } b_1 \equiv \text{let}_B \text{ box } x = a_2 \text{ in } b_2 : B}$$

# Term conversion – product

$$\frac{\Gamma \vdash_{\text{nc}} a : A \quad \Gamma \vdash_{\text{nc}} b : B \quad \Gamma, x : A, y : B \vdash_{\text{nc}} c : C}{\Gamma \vdash \text{let}_C (x, y) = (a, b) \text{ in } c \equiv c [x := a] [y := b] : C}$$

$$\frac{\begin{array}{c} \Gamma \vdash \text{let}_A (x, y) = e_1 \text{ in } x \equiv \text{let}_A (x, y) = e_2 \text{ in } x : A \\ \Gamma \vdash \text{let}_B (x, y) = e_1 \text{ in } y \equiv \text{let}_B (x, y) = e_2 \text{ in } y : B \end{array}}{\Gamma \vdash e_1 \equiv e_2 : A \otimes B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \otimes B \quad \Gamma, x : A, y : B \vdash c_1 \equiv c_2 : C}{\Gamma \vdash \text{let}_C (x, y) = e_1 \text{ in } c_1 \equiv \text{let}_C (x, y) = e_2 \text{ in } c_2 : C}$$

# Term conversion – sum

$$\frac{\Gamma \vdash_{\text{nc}} a : A \quad \Gamma \vdash_{\text{nc}} f : 1A \rightarrow C \quad \Gamma \vdash_{\text{nc}} g : 1B \rightarrow C}{\Gamma \vdash \text{case}_C \text{ inl } a \text{ of } (f, g) \equiv f \ a : C}$$

$$\frac{\Gamma \vdash_{\text{nc}} b : B \quad \Gamma \vdash_{\text{nc}} f : 1A \rightarrow C \quad \Gamma \vdash_{\text{nc}} g : 1B \rightarrow C}{\Gamma \vdash \text{case}_C \text{ inr } b \text{ of } (f, g) \equiv g \ b : C}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : A \quad \Gamma \vdash B : \text{Type}}{\Gamma \vdash \text{inl } a_1 \equiv \text{inl } a_2 : A \oplus B}$$

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \text{inr } b_1 \equiv \text{inr } b_2 : A \oplus B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : A \oplus B \quad \begin{array}{c} \Gamma \vdash f_1 \equiv f_2 : 1A \rightarrow C \\ \Gamma \vdash g_1 \equiv g_2 : 1B \rightarrow C \end{array}}{\Gamma \vdash \text{case}_C \ e_1 \text{ of } (f_1, g_1) \equiv \text{case}_C \ e_2 \text{ of } (f_2, g_2) : C}$$

# Term conversion – let

$$\frac{\Gamma \vdash_{\text{nc}} e_1 : A \quad \Gamma, x : A \vdash_{\text{nc}} e_2 : B}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \equiv e_2[x := e_1] : B}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 : A \quad \Gamma, x : A \vdash b_1 \equiv b_2 : B}{\Gamma \vdash \text{let } x = a_1 \text{ in } b_1 \equiv \text{let } x = a_2 \text{ in } b_2 : B}$$

Q: Should we have both rules, or just the first one?

# Term conversion – polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_{\text{nc}} e : B \quad |\Gamma| \vdash A : \text{Type}_r}{\Gamma \vdash (\Lambda a : \text{Type}_r. e) A \equiv e[a := A] : B[a := A]}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f \ a \equiv g \ a : B}{\Gamma \vdash f \equiv g : \forall a : \text{Type}_r. B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \forall a : \text{Type}_r. B \quad \Gamma \vdash A_1 \equiv A_2 : \text{Type}_r}{\Gamma \vdash e_1 \ A_1 \equiv e_2 \ A_2 : B[a := A_1]}$$

# Term conversion – polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_{\text{nc}} e : B \quad |\Gamma| \vdash A : \text{Type}_r}{\Gamma \vdash (\Lambda \{a : \text{Type}_r\}. e) @A \equiv e [a := A] : B [a := A]}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f @a \equiv g @a : B}{\Gamma \vdash f \equiv g : \forall \{a : \text{Type}_r\}. B}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : \forall \{a : \text{Type}_r\}. B \quad \Gamma \vdash A_1 \equiv A_2 : \text{Type}_r}{\Gamma \vdash e_1 @A_1 \equiv e_2 @A_2 : B [a := A_1]}$$

# Environments

Global environments:

$\Sigma ::=$

$\emptyset \mid \Sigma, h : P := p \mid \Sigma, x : A := e \mid$

$\Sigma, \text{partial } x : A := e \mid \Sigma, \text{totality } x \ p$

# Well-formed environments

 $\overline{\emptyset \text{ env}}$ 

$$\frac{\Sigma \text{ env} \quad h \notin \Sigma \quad \Sigma | \cdot \vdash p : P}{\Sigma, h : P := p \text{ env}}$$

$$\frac{\Sigma \text{ env} \quad x \notin \Sigma \quad \Sigma | \cdot \vdash_c e : A}{\Sigma, x : A := e \text{ env}}$$

$$\frac{\Sigma \text{ env} \quad x \notin \Sigma \quad \Sigma | \cdot \vdash_c e : A \quad e \text{ fails syntactic check}}{\Sigma, \text{partial } x : A := e \text{ env}}$$

$$\frac{\Sigma \text{ env} \quad \Sigma = \Sigma_1, \text{partial } x : A := e, \Sigma_2 \quad \Sigma | \cdot \vdash p : \exists r : A. e = ? = r}{\Sigma, \text{totality } x \ p \text{ env}}$$