T-Axi (declarative)

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TODO

- Kinding.
- Polymorphism.
- Higher-order quantification.
- Totality checker.
- Computation and conversion checking.
- Inductives and records.
- Algorithmic system.
- Distinguish noncomputable terms.
- Distinguish partial terms.

Quantities

Quantities:
$r ::= 0 \mid 1 \mid \omega$

ω is the default quantity, so when there's nothing to indicate quantity, it means it's ω.

Quantities are ordered:

$$\omega \leq r$$
$$0 \leq 0$$
$$1 \leq 1$$

Operations on quantities

Addition of quantities:

$$
\begin{aligned}
0 + r &= r \\
\omega + r &= \omega \\
1 + 0 &= 1 \\
1 + 1 &= \omega \\
1 + \omega &= \omega
\end{aligned}
$$

Multiplication:

$$
\begin{aligned}
0 \cdot r &= 0 \\
1 \cdot r &= r \\
\omega \cdot 0 &= 0 \\
\omega \cdot 1 &= \omega \\
\omega \cdot \omega &= \omega
\end{aligned}
$$

Types

Types:
$$A, B ::= r\,A \to B \mid \texttt{!}_r\,A \mid A \otimes B \mid A \oplus B \mid \texttt{Unit} \mid \texttt{Empty}$$

Terms

Terms:

$e ::=$

$x \mid \lambda_r\, x : A.\, e \mid e_1\ e_2 \mid$
$\mathtt{box}_r\ e \mid \mathtt{letbox}\ x : A = e_1\ \mathtt{in}\ e_2$
$(e_1, e_2) \mid \mathtt{letpair}\ (x, y) = e_1\ \mathtt{in}\ e_2 \mid$
$\mathtt{inl}_A\ e \mid \mathtt{inr}_A\ e \mid \mathtt{case}\ e\ \mathtt{of}\ (e_1, e_2) \mid$
$\mathtt{unit} \mid \mathtt{letunit}_A\ \mathtt{unit} = e_1\ \mathtt{in}\ e_2 \mid$
$\mathtt{Empty\text{-}elim}_A\ e \mid$
$\mathtt{let}_r\ x : A = e_1\ \mathtt{in}\ e_2 \mid$
$\mathtt{choose}_A\ p \mid \mathtt{choose\text{-}witness}\ x\ h\ \mathtt{for}\ p\ \mathtt{in}\ e$

$\mathtt{choose}_A\ p$ and $\mathtt{choose\text{-}witness}\ x\ h\ \mathtt{for}\ p\ \mathtt{in}\ e$ are noncomputable terms, whereas all others are computable.

Propositions

Propositions:

$P, Q ::=$

$\top \mid \bot \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid$

$\forall x : A.\, P \mid \exists x : A.\, P \mid$

$e_1 =_A e_2$

Notations:

$\neg P$ stands for $P \Rightarrow \bot$

$P \Leftrightarrow Q$ stands for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

h | **assumption** | **trivial** | **absurd** p
assume $h : P$ **in** q | **apply** p_1 p_2 |
both p_1 p_2 | **and-left** p | **and-right** p |
or-left p | **or-right** p | **cases** p_1 p_2 p_3 |
lemma $h : P$ **by** p **in** q | **proving** P **by** p |
suffices P **by** q **in** p |
pick-any $x : A$ **in** e | **instantiate** p **with** e |
witness e **such that** p | **pick-witness** x h **for** p_1 **in** p_2 |
refl e | **rewrite** p_1 **in** p_2 | **funext** $x : A$ **in** p
by-contradiction $h : \neg P$ **in** q |
choose-spec p | **choose-witness** x h **for** p **in** q

Contexts

Contexts:

$\Gamma ::= \cdot \mid \Gamma, r\, x : A \mid \Gamma, r\, x : A := e \mid \Gamma, h : P$

Operations on contexts:

$\Gamma_1 + \Gamma_2$ – context addition

$r\,\Gamma$ – context scaling

$\Gamma_1 \leq \Gamma_2$ – context subusaging/ordering

$|\Gamma|$ – cartesianization

Q: Definitions in context look fishy when combined with quantities!

Context addition

$$
\begin{gathered}
\cdot + \cdot = \cdot \\
(\Gamma_1, r_1\, x : A) + (\Gamma_2, r_2\, x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : A \\
(\Gamma_1, r_1\, x : A := e) + (\Gamma_2, r_2\, x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : \\
A := e \\
(\Gamma_1, p : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P
\end{gathered}
$$

Context scaling

$$
\begin{aligned}
s\,\cdot &= \cdot \\
s\,(\Gamma, r\,x : A) &= s\,\Gamma, (s \cdot r)\,x : A \\
s\,(\Gamma, r\,x : A := e) &= s\,\Gamma, (s \cdot r)\,x : A := e \\
s\,(\Gamma, h : P) &= s\,\Gamma, h : P
\end{aligned}
$$

Context ordering

$$\frac{}{\cdot \leq \cdot}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A \leq \Gamma_2, r_2\, x : A}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A := e \leq \Gamma_2, r_2\, x : A := e}$$

$$\frac{\Gamma_1 \leq \Gamma_2}{\Gamma_1, h : P \leq \Gamma_2, h : P}$$

Cartesianization

Cartesianization turns a context into a traditional context that doesn't mention any quantities.

$$
\begin{aligned}
|\cdot| &= \cdot \\
|\Gamma, r\, x : A| &= |\Gamma|, x : A \\
|\Gamma, r\, x : A := e| &= |\Gamma|, x : A := e \\
|\Gamma, h : P| &= |\Gamma|, h : P
\end{aligned}
$$

Judgements

Well-formed context judgement: $\Gamma\ \texttt{ctx}$
Cartesian context judgement: $\Gamma\ \texttt{cartesian}$

Well-formed type judgement: $\Gamma \vdash A\ \texttt{type}$
Typing judgement: $\Gamma \vdash e : A$

Well-formed proposition judgement: $\Gamma \vdash P\ \texttt{prop}$
Proof judgement: $\Gamma \vdash p : P$

Sanity checks

We'll set up the system so that:

- If $\Gamma \vdash e : A$, then $|\Gamma| \vdash A \ \mathtt{type}$.
- If $\Gamma \vdash p : P$, then $\Gamma \vdash P \ \mathtt{prop}$.
- If $\Gamma \vdash P \ \mathtt{prop}$, then $\Gamma \ \mathtt{cartesian}$.
- If $\Gamma \vdash A \ \mathtt{type}$, then $\Gamma \ \mathtt{cartesian}$.
- If $\Gamma \ \mathtt{cartesian}$, then $\Gamma \ \mathtt{ctx}$.

Well-formed contexts

$$\frac{}{\cdot\ \mathtt{ctx}}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad |\Gamma| \vdash A\ \mathtt{type} \quad x \notin \Gamma}{\Gamma, r\,x : A\ \mathtt{ctx}}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad \Gamma \vdash e : A \quad x \notin \Gamma}{\Gamma, r\,x : A := e\ \mathtt{ctx}}\mathrm{DUBIOUS}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad |\Gamma| \vdash P\ \mathtt{prop} \quad h \notin \Gamma}{\Gamma, h : P\ \mathtt{ctx}}$$

Q: Do we want to disallow shadowing?

Q: Do we really need to check type/proposition well-formedness in a cartesianized context?

Cartesian contexts

$$\frac{\Gamma\ \mathtt{ctx} \quad \Gamma = |\Gamma|}{\Gamma\ \mathtt{cartesian}}$$

Well-formed types

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \texttt{Unit type}} \qquad \frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \texttt{Empty type}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash r\, A \to B\ \texttt{type}} \qquad \frac{\Gamma \vdash A\ \texttt{type}}{\Gamma \vdash\ !_r\, A\ \texttt{type}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash A \otimes B\ \texttt{type}} \qquad \frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash A \oplus B\ \texttt{type}}$$

Using variables

$$\frac{\Gamma\ \texttt{ctx} \quad \Gamma = \Gamma_1, r\,x : A, \Gamma_2 \quad r \leq 1 \quad \Gamma_1 \leq 0\,\Gamma_1 \quad \Gamma_2 \leq 0\,\Gamma_2}{\Gamma \vdash x : A}$$

$$\frac{\Gamma\ \texttt{ctx} \quad \Gamma = \Gamma_1, r\,x : A := e, \Gamma_2 \quad r \leq 1 \quad \Gamma_1 \leq 0\,\Gamma_1 \quad \Gamma_2 \leq 0\,\Gamma_2}{\Gamma \vdash x : A}$$

Q: Is there a better notation?

Functions

$$\frac{\Gamma, r\,x : A \vdash e : B}{\Gamma \vdash \lambda_r\,x : A.\,e : r\,A \to B}$$

$$\frac{\Gamma \leq \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash e_1 : r\,A \to B \quad \Gamma_2 \vdash e_2 : A}{\Gamma \vdash e_1\;e_2 : B}$$

Q: Where exactly do we need quantity and type annotations?

Box

$$\frac{\Gamma \leq r\,\Gamma' \quad \Gamma' \vdash e : A}{\Gamma \vdash \texttt{box}_r\ e : \,!_r\, A}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : \,!_r\, A \quad \Gamma_2, r\,x : A \vdash e_2 : B}{\Gamma \vdash \texttt{letbox}\ x : A = e_1\ \texttt{in}\ e_2 : B}$$

Q: Notation for box intro and elim?

Empty

$$\frac{|\Gamma| \vdash A\ \mathtt{type} \quad \Gamma \vdash e : \mathtt{Empty}}{\Gamma \vdash \mathtt{Empty\text{-}elim}_A\ e : A}$$

Unit

$$\frac{\Gamma\ \mathtt{ctx} \qquad \Gamma \leq 0\,\Gamma}{\Gamma \vdash \mathtt{unit} : \mathtt{Unit}}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \qquad \Gamma_1 \vdash e_1 : \mathtt{Unit} \qquad \Gamma_2 \vdash e_2 : A}{\Gamma \vdash \mathtt{letunit}_A\ \mathtt{unit} = e_1\ \mathtt{in}\ e_2 : A}$$

Products

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash a : A \quad \Gamma_2 \vdash b : B}{\Gamma \vdash (a, b) : A \otimes B}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : A \otimes B \quad \Gamma_2, 1\, x : A, 1\, y : B \vdash e_2 : C}{\Gamma \vdash \texttt{letpair}\ (x, y) = e_1\ \texttt{in}\ e_2 : C}$$

Sums

$$\frac{\Gamma \vdash e : A \quad |\Gamma| \vdash B \;\texttt{type}}{\Gamma \vdash \texttt{inl}_B\; e : A \oplus B} \qquad \frac{|\Gamma| \vdash A \;\texttt{type} \quad \Gamma \vdash e : B}{\Gamma \vdash \texttt{inl}_A\; e : A \oplus B}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e : A \oplus B \quad \Gamma_2 \vdash f : 1\, A \to C \quad \Gamma_2 \vdash g : 1\, B \to C}{\Gamma \vdash \texttt{case}\; e \;\texttt{of}\; (f, g) : C}$$

Q: Do we want first-order representation of the branches?

Let

$$\frac{\Gamma \leq r\,\Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : A \quad \Gamma_2, r\,x : A := e_1 \vdash e_2 : B}{\Gamma \vdash \mathtt{let}_r\ x : A = e_1\ \mathtt{in}\ e_2 : B}$$

Well-formed propositions

$$\frac{\Gamma\ \mathtt{cartesian}}{\Gamma \vdash \top\ \mathtt{prop}} \qquad \frac{\Gamma\ \mathtt{cartesian}}{\Gamma \vdash \bot\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \Rightarrow Q\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \wedge Q\ \mathtt{prop}} \qquad \frac{\Gamma \vdash P\ \mathtt{prop} \quad \Gamma \vdash Q\ \mathtt{prop}}{\Gamma \vdash P \vee Q\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A\ \mathtt{type} \quad \Gamma, x : A \vdash P\ \mathtt{prop}}{\Gamma \vdash \forall x : A.\ P\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A\ \mathtt{type} \quad \Gamma, x : A \vdash P\ \mathtt{prop}}{\Gamma \vdash \exists x : A.\ P\ \mathtt{prop}}$$

$$\frac{\Gamma \vdash A\ \mathtt{type} \quad \Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 =_A e_2\ \mathtt{prop}}$$

Substitution

The notation is $P\,[x := e]$ for substitution in propositions.

Q: Do we need to check type well-formedness when checking proposition well-formedness? It should follow from the other sanity checks.

Assumptions and implication

$$\frac{\Gamma\ \texttt{cartesian} \quad (h : P) \in \Gamma}{\Gamma \vdash h : P} \qquad \frac{\Gamma\ \texttt{cartesian} \quad (h : P) \in \Gamma}{\Gamma \vdash \textbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{assume}\ h : P\ \textbf{in}\ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{apply}\ q\ p : Q}$$

Propositional logic

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \textbf{trivial} : \top} \qquad \frac{\Gamma \vdash Q\ \texttt{prop} \quad \Gamma \vdash p : \bot}{\Gamma \vdash \textbf{absurd}\ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{both}\ p\ q : P \wedge Q}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-left}\ pq : P} \qquad \frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-right}\ pq : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash Q\ \texttt{prop}}{\Gamma \vdash \textbf{or-left}\ p : P \vee Q} \qquad \frac{\Gamma \vdash P\ \texttt{prop} \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{or-right}\ q : P \vee Q}$$

$$\frac{\Gamma \vdash pq : P \vee Q \quad \Gamma \vdash r_1 : P \Rightarrow R \quad \Gamma \vdash r_2 : Q \Rightarrow R}{\Gamma \vdash \textbf{cases}\ pq\ r_1\ r_2 : R}$$

Utilities

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{lemma}\ h : P\ \textbf{by}\ p\ \textbf{in}\ q : Q}$$

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \textbf{proving}\ P\ \textbf{by}\ p : P}$$

$$\frac{\Gamma \vdash pq : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{suffices}\ P\ \textbf{by}\ pq\ \textbf{in}\ p : Q}$$

Quantifiers

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \textbf{pick-any } x : A \textbf{ in } p : \forall x : A.\, P}$$

$$\frac{\Gamma \vdash p : \forall x : A.\, P \quad \Gamma \vdash e : A}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e : P\,[x := e]}$$

$$\frac{\Gamma \vdash e : A \quad \Gamma, x : A \vdash P \ \texttt{prop} \quad \Gamma \vdash p : P\,[x := e]}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p : \exists x : A.\, P}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q \ \texttt{prop} \quad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \textbf{pick-witness } x\ h \textbf{ for } p \textbf{ in } q : Q}$$

Equality

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \textbf{refl } e : e =_A e}$$

$$\frac{\Gamma \vdash q : e_1 =_A e_2 \quad \Gamma, x : A \vdash P \ \mathtt{prop} \quad \Gamma \vdash p : P\,[x := e_2]}{\Gamma \vdash \textbf{rewrite } q \textbf{ in } p : P\,[x := e_1]}$$

$$\frac{\Gamma, x : A \vdash p : f\ x =_B g\ x}{\Gamma \vdash \textbf{funext } x : A \textbf{ in } p : f =_{r\,A \to B} g}$$

Classical Logic

$$\frac{\Gamma, h : \neg P \vdash q : \bot}{\Gamma \vdash \textbf{by-contradiction}\ h : \neg P\ \textbf{in}\ q : P}$$

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P}{\Gamma \vdash \texttt{choose}\ p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P}{\Gamma \vdash \textbf{choose-spec}\ p : P\,[x := \texttt{choose}\ p]}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q\ \texttt{prop} \quad \Gamma, x : A := \texttt{choose}\ p, h : P \vdash q : Q}{\Gamma \vdash \textbf{choose-witness}\ x\ h\ \textbf{for}\ p\ \textbf{in}\ q : Q}$$

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P \quad |\Gamma| \vdash B\ \texttt{type} \quad \Gamma, x : A := \texttt{choose}\ p, h : P \vdash e : B}{\Gamma \vdash \texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e : B}$$

Environments

Global environments:

$$\Sigma ::=$$
$$\emptyset \mid \Sigma, h : P := p \mid \Sigma, x : A := e \mid$$
$$\Sigma, \texttt{partial}\ x : A := e \mid \Sigma, \texttt{totality}\ x\ p$$

Well-formed environments

$$\frac{}{\emptyset \ \texttt{env}}$$

$$\frac{\Sigma \ \texttt{env} \quad h \notin \Sigma \quad \Sigma \mid \cdot \vdash p : P}{\Sigma, h : P := p \ \texttt{env}}$$

$$\frac{\Sigma \ \texttt{env} \quad x \notin \Sigma \quad \Sigma \mid \cdot \vdash e : A}{\Sigma, x : A := e \ \texttt{env}}$$

$$\frac{\Sigma \ \texttt{env} \quad x \notin \Sigma \quad \Sigma \mid \cdot \vdash e : A \quad e \textbf{ fails syntactic check}}{\Sigma, \texttt{partial} \ x : A := e \ \texttt{env}}$$

$$\frac{\Sigma \ \texttt{env} \quad \Sigma = \Sigma_1, \texttt{partial} \ x : A := e, \Sigma_2 \quad \Sigma \mid \cdot \vdash p : \exists r : A.\, e =?= r}{\Sigma, \texttt{totality} \ x \ p \ \texttt{env}}$$