Poor Man's Axi: Algorithmic Version

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Values (cbv)

$$\frac{}{\lambda x.e \; \mathtt{value}}$$

$$\frac{v_1 \; \mathtt{value} \quad v_2 \; \mathtt{value}}{(v_1, v_2) \; \mathtt{value}}$$

$$\frac{v \; \mathtt{value}}{\mathtt{inl} \; v \; \mathtt{value}} \qquad \frac{v \; \mathtt{value}}{\mathtt{inr} \; v \; \mathtt{value}}$$

$$\frac{}{\mathtt{unit} \; \mathtt{value}}$$

Values are the final results of computation. Note that a function is a value whether or not its body is. Other values are pairs of values, values injected into a sum on the left or right, and unit.

Big-step semantics (cbv)

$$\frac{}{\lambda x.e \Downarrow \lambda x.e} \qquad \frac{e_1 \Downarrow \lambda x.e \quad e_2 \Downarrow v \quad e\,[x := v] \Downarrow v'}{e_1 \;\; e_2 \Downarrow v'}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{outl}\ e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{outr}\ e \Downarrow v_2}$$

$$\frac{e \Downarrow v}{\texttt{inl}\ e \Downarrow \texttt{inl}\ v} \qquad \frac{e \Downarrow v}{\texttt{inr}\ e \Downarrow \texttt{inr}\ v}$$

$$\frac{e \Downarrow \texttt{inl}\ v \quad f\ v \Downarrow v'}{\texttt{case}\ e\ \texttt{of}\ (f, g) \Downarrow v'} \qquad \frac{e \Downarrow \texttt{inr}\ v \quad g\ v \Downarrow v'}{\texttt{case}\ e\ \texttt{of}\ (f, g) \Downarrow v'}$$

$$\frac{}{\texttt{unit} \Downarrow \texttt{unit}}$$

Small-step semantics (cbv) – basic rules

$$\frac{v \ \texttt{value}}{(\lambda x.e)\ v \longrightarrow e\,[x := v]}$$

$$\frac{v_1 \ \texttt{value} \quad v_2 \ \texttt{value}}{\texttt{outl}\ (v_1, v_2) \longrightarrow v_1} \qquad \frac{v_1 \ \texttt{value} \quad v_2 \ \texttt{value}}{\texttt{outr}\ (v_1, v_2) \longrightarrow v_2}$$

$$\frac{v \ \texttt{value}}{\texttt{case}\ (\texttt{inl}\ v)\ \texttt{of}\ (f, g) \longrightarrow f\ v}$$

$$\frac{v \ \texttt{value}}{\texttt{case}\ (\texttt{inr}\ v)\ \texttt{of}\ (f, g) \longrightarrow g\ v}$$

Small-step semantics (cbv) – boring rules

$$\frac{e_1 \longrightarrow e_1'}{e_1\ e_2 \longrightarrow e_1'\ e_2} \qquad \frac{v_1\ \texttt{value} \quad e_2 \longrightarrow e_2'}{v_1\ e_2 \longrightarrow v_1\ e_2'}$$

$$\frac{e_1 \longrightarrow e_1'}{(e_1, e_2) \longrightarrow (e_1', e_2)} \qquad \frac{v_1\ \texttt{value} \quad e_2 \longrightarrow e_2'}{(v_1, e_2) \longrightarrow (v_1, e_2')}$$

$$\frac{e \longrightarrow e'}{\texttt{outl}\ e \longrightarrow \texttt{outl}\ e'} \qquad \frac{e \longrightarrow e'}{\texttt{outr}\ e \longrightarrow \texttt{outr}\ e'}$$

$$\frac{e \longrightarrow e'}{\texttt{inl}\ e \longrightarrow \texttt{inl}\ e'} \qquad \frac{e \longrightarrow e'}{\texttt{inr}\ e \longrightarrow \texttt{inr}\ e'}$$

$$\frac{e \longrightarrow e'}{\texttt{case}\ e\ \texttt{of}\ (f, g) \longrightarrow \texttt{case}\ e'\ \texttt{of}\ (f, g)}$$

Example semantics – explanations

$e \Downarrow v$ should be read "term e evalutes to value v". It describes computation in a coarse-grained manner, telling us what is the result of evaluating each term. If $e \Downarrow v$, then v `value`.

$e \longrightarrow e'$ should be read "term e reduces to term e'". It describes computation in a fine-grained manner, telling us what can happen in a single computation step. To actually describe computation fully in this style, we need to take the transitive closure of this relation, written $e \longrightarrow^* v$.

Both presented semantics are call-by-value, i.e. we evaluate a function's argument before performing a substitution. They are equivalent, i.e. $e \Downarrow v$ if and only if $e \longrightarrow^* v$

Weak head normal forms

$$\frac{}{x \ \texttt{whnf}}$$

$$\frac{}{\lambda x.e \ \texttt{whnf}}$$

$$\frac{}{(e_1, e_2) \ \texttt{whnf}}$$

$$\frac{}{\texttt{inl}\ e \ \texttt{whnf}} \qquad \frac{}{\texttt{inr}\ e \ \texttt{whnf}}$$

$$\frac{}{\texttt{unit} \ \texttt{whnf}}$$

Weak head normal forms grammar

Weak head normal forms:

$e ::=$

$n \mid \lambda x.e \mid$
$(e_1, e_2) \mid$
$\texttt{inl}\ e \mid \texttt{inr}\ e \mid$
$\texttt{unit} \mid \texttt{elim}_{\texttt{Empty}}\ e$

Neutral forms:

$n ::=$

$x \mid n\ e \mid$
$\texttt{outl}\ n \mid \texttt{outr}\ n \mid$
$\texttt{case}\ n\ \texttt{of}\ (e_1, e_2) \mid$

Whnf reduction – basic rules

$$\frac{}{(\lambda x.e_1)\ e_2 \longrightarrow_{\mathtt{whnf}} e_1\,[x := e_2]}$$

$$\frac{}{\mathtt{outl}\ (e_1, e_2) \longrightarrow_{\mathtt{whnf}} e_1} \qquad \frac{}{\mathtt{outr}\ (e_1, e_2) \longrightarrow_{\mathtt{whnf}} e_2}$$

$$\frac{}{\mathtt{case}\ (\mathtt{inl}\ e)\ \mathtt{of}\ (f, g) \longrightarrow_{\mathtt{whnf}} f\ e}$$

$$\frac{}{\mathtt{case}\ (\mathtt{inr}\ e)\ \mathtt{of}\ (f, g) \longrightarrow_{\mathtt{whnf}} g\ e}$$

Whnf reduction – boring rules

$$\frac{e_1 \longrightarrow_{\texttt{whnf}} e_1'}{e_1 \; e_2 \longrightarrow_{\texttt{whnf}} e_1' \; e_2} \qquad \frac{e_1 \; \texttt{whnf} \qquad e_2 \longrightarrow_{\texttt{whnf}} e_2'}{e_1 \; e_2 \longrightarrow_{\texttt{whnf}} e_1 \; e_2'}$$

$$\frac{e \longrightarrow_{\texttt{whnf}} e'}{\texttt{outl}\; e \longrightarrow_{\texttt{whnf}} \texttt{outl}\; e'} \qquad \frac{e \longrightarrow_{\texttt{whnf}} e'}{\texttt{outr}\; e \longrightarrow_{\texttt{whnf}} \texttt{outr}\; e'}$$

$$\frac{e \longrightarrow_{\texttt{whnf}} e'}{\texttt{case}\; e \;\texttt{of}\; (f, g) \longrightarrow_{\texttt{whnf}} \texttt{case}\; e' \;\texttt{of}\; (f, g)}$$

Algorithmic computational equality 0

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \equiv x \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \equiv e' \Rightarrow B \qquad A = B}{\Gamma \vdash e \equiv e' \Leftarrow A}\textsc{Sub}$$

$$\frac{e_1 \longrightarrow_{\texttt{whnf}} e_1' \qquad e_2 \longrightarrow_{\texttt{whnf}} e_2' \qquad \Gamma \vdash e_1' \equiv e_2' \Leftarrow A}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\texttt{whnf}} A}$$

$$\frac{e_1 \longrightarrow_{\texttt{whnf}} e_1' \qquad e_2 \longrightarrow_{\texttt{whnf}} e_2' \qquad \Gamma \vdash e_1' \equiv e_2' \Rightarrow A}{\Gamma \vdash e_1 \equiv e_2 \Rightarrow_{\texttt{whnf}} A}$$

Algorithmic computational equality 1

$$\frac{\Gamma, x : A \vdash e \equiv e' \Leftarrow_{\texttt{whnf}} B}{\Gamma \vdash \lambda x.e \equiv \lambda x.e' \Leftarrow A \to B}$$

$$\frac{\Gamma \vdash f \equiv f' \Rightarrow A \to B \quad \Gamma \vdash a \equiv a' \Leftarrow_{\texttt{whnf}} A}{\Gamma \vdash f\ a \equiv f'\ a' \Rightarrow B}$$

$$\frac{\Gamma \vdash a \equiv a' \Leftarrow_{\texttt{whnf}} A \quad \Gamma \vdash b \equiv b' \Leftarrow_{\texttt{whnf}} B}{\Gamma \vdash (a, b) \equiv (a', b') \Leftarrow A \times B}$$

$$\frac{\Gamma \vdash e \equiv e' \Rightarrow A \times B}{\Gamma \vdash \texttt{outl}\ e \equiv \texttt{outl}\ e' \Rightarrow A} \qquad \frac{\Gamma \vdash e \equiv e' \Rightarrow A \times B}{\Gamma \vdash \texttt{outr}\ e \equiv \texttt{outr}\ e' \Rightarrow B}$$

Algorithmic computational equality 2

$$\frac{\Gamma \vdash e \equiv e' \Leftarrow_{\mathtt{whnf}} A}{\Gamma \vdash \mathtt{inl}\ e \equiv \mathtt{inl}\ e' \Leftarrow A + B} \qquad \frac{\Gamma \vdash e \equiv e' \Leftarrow_{\mathtt{whnf}} B}{\Gamma \vdash \mathtt{inr}\ e \equiv \mathtt{inr}\ e' \Leftarrow A + B}$$

$$\frac{\Gamma \vdash e \equiv e' \Rightarrow A + B \quad \Gamma \vdash f \equiv f' \Leftarrow_{\mathtt{whnf}} A \to C \quad \Gamma \vdash g \equiv g' \Leftarrow_{\mathtt{whnf}} B \to}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \equiv \mathsf{case}\ e'\ \mathsf{of}\ (f', g') \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \mathtt{unit} \equiv \mathtt{unit} \Leftarrow \mathtt{Unit}} \qquad \frac{\Gamma \vdash e \equiv e' \Leftarrow \mathtt{Empty}}{\Gamma \vdash \mathtt{elim}_{\mathtt{Empty}}\ e \equiv \mathtt{elim}_{\mathtt{Empty}}\ e' \Leftarrow A}$$

Uniqueness rules

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f \equiv \lambda x.f\ x \Leftarrow A \to B}\textsc{Fun-Uniq}$$

$$\frac{\Gamma \vdash e : A \times B}{\Gamma \vdash e \equiv (\mathtt{outl}\ e, \mathtt{outr}\ e) : A \times B}\textsc{Prod-Uniq}$$

$$\frac{\Gamma \vdash e_1 : \mathtt{Unit} \quad \Gamma \vdash e_2 : \mathtt{Unit}}{\Gamma \vdash e_1 \equiv e_2 \Rightarrow \mathtt{Unit}}\textsc{Unit-Uniq}$$

$$\frac{\Gamma \vdash e_1 : \mathtt{Empty} \quad \Gamma \vdash e_2 : \mathtt{Empty}}{\Gamma \vdash e_1 \equiv e_2 \Rightarrow \mathtt{Empty}}\textsc{Empty-Uniq}$$