

# Quantities

Quantities:

$r ::= 0 \mid 1 \mid ? \mid + \mid *$

$*$  is the default quantity, so when there's nothing to indicate quantity, it means it's  $*$ .

## Quantities – explanation

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced “few”) means a resource must be used at most once.
- + (pronounced “many”) means a resource must be used at least once.
- \* (pronounced “any”) means no restrictions on usage.

# Subusage ordering

$r_1 \sqsubseteq r_2$  means that a resource with quantity  $r_1$  may be used when quantity  $r_2$  is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$* \sqsubseteq ?$$

$$* \sqsubseteq +$$

$$+ \sqsubseteq 1$$

$$? \sqsubseteq 1$$

$$? \sqsubseteq 0$$

We will denote the greatest lower bound in this order with  $r_1 \sqcap r_2$  and the least upper bound (if it exists) with  $r_1 \sqcup r_2$ .

# Addition of quantities

When we have two quantities of the same resource, we can sum the quantities.

$$0 + r = r$$

$$r + 0 = r$$

$$? + ? = *$$

$$? + * = *$$

$$* + ? = *$$

$$* + * = *$$

$$_ + _ = +$$

# Multiplication of quantities

When we have a quantity  $r_1$  of resource  $A$  that contains quantity  $r_2$  of resource  $B$ , then we in fact have quantity  $r_1 \cdot r_2$  of resource  $B$ .

$$0 \cdot r = 0$$

$$r \cdot 0 = 0$$

$$1 \cdot r = r$$

$$r \cdot 1 = r$$

$$?.? = ?$$

$$+ \cdot + = +$$

$$_ \cdot _ = *$$

# The algebra of quantities

Quantities  $\mathcal{Q}$  form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$  is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$  is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If  $r_1 + r_2 = 0$ , then  $r_1 = 0$  and  $r_2 = 0$ .
- If  $r_1 \cdot r_2 = 0$ , then  $r_1 = 0$  or  $r_2 = 0$ .

# Subtraction of quantities

$r_1 - r_2$  is the least  $r'$  such that  $r_1 \sqsubseteq r' + r_2$ .

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

# Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$  holds when  $r_2 - r_1$  is defined.

Explicitly:  $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$



# Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$  holds when  $r_2 - 1 = r_1$ .

$$\overline{* \leq_{\text{dec}} *}$$

$$\overline{* \leq_{\text{dec}} +}$$

$$\overline{0 \leq_{\text{dec}} 1}$$

$$\overline{0 \leq_{\text{dec}} ?}$$



# Trait-checking Division

$$r_1/r_2 = \sup\{s \in \mathcal{Q} \mid s \cdot r_2 \sqsubseteq r_1\}$$

$r_1/r_2$	0	1	?	+	*
0	0	0	0	0	0
1		1	1	1	1
?		?	1	?	1
+		+	+	1	1
*		*	+	?	1

# Division with remainder

$r_1/r_2 = (a, b)$  when  $r_1 = a \cdot r_2 + b$ , with  $a$  as large as possible and  $b$  being as small as possible according to the subtraction order.

Note that  $r_1/r_2 = a$  means that  $b = 0$ .

$r_1/r_2$	0	1	?	+	*
0	*	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
?	$(*, ?)$	?	?	$(0, ?)$	$(0, ?)$
+	$(*, +)$	+	$(*, 1)$	+	$(*, 1)$
*	$(*, *)$	*	*	*	*