

T-Axi (algorithmic)

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Algorithmic terms

Terms:

$e ::=$

$(e : A) \mid x \mid \lambda x. e \mid e_1 e_2 \mid$
 $\text{box } e \mid \text{let box } x = e_1 \text{ in } e_2 \mid$
 $(e_1, e_2) \mid \text{let } (x, y) = e_1 \text{ in } e_2 \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{let unit} = e_1 \text{ in } e_2 \mid$
 $\text{Empty-elim } e \mid$
 $\text{let}_r x = e_1 \text{ in } e_2 \mid$
 $\Lambda a. e \mid e A \mid$
 $\Lambda \{a\}. e \mid e @A \mid$
 $\text{choose } p \mid \text{choose-witness } x \text{ } h \text{ for } p \text{ in } e$

The algorithmic terms are almost the same to the declarative ones,
but there are far fewer annotations.

Recovering annotated terms

We can recover some of the original terms by combining algorithmic terms with annotations.

- $\text{let unit} = e_1 \text{ in } e_2 \equiv \text{let unit} = e_1 \text{ in } (e_2 : A)$
- $\text{Empty-elim } [A]e_1 \equiv (\text{Empty-elim } e_1 : A)$
- $\text{let}_r x : A = e_1 \text{ in } e_2 \equiv \text{let}_r x = (e_1 : A) \text{ in } e_2$

However, as long as annotations must be types (and not partial types, for example), we cannot recover more, like $\text{inl}_A e$, because we don't have partial annotations.

Algorithmic proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

$h \mid \text{assumption} \mid \text{trivial} \mid \text{absurd } p$
 $\text{assume } h \text{ in } q \mid \text{apply } p_1 \ p_2 \mid$
 $\text{both } p_1 \ p_2 \mid \text{and-left } p \mid \text{and-right } p \mid$
 $\text{or-left } p \mid \text{or-right } p \mid \text{cases } p_1 \ p_2 \ p_3 \mid$
 $\text{lemma } h : P \text{ by } p \text{ in } q \mid \text{proving } P \text{ by } p \mid$
 $\text{suffices } P \text{ by } q \text{ in } p \mid$
 $\text{pick-any } x \text{ in } e \mid \text{instantiate } p \text{ with } e \mid$
 $\text{witness } e \text{ such that } p \mid \text{pick-witness } x \ h \text{ for } p_1 \text{ in } p_2 \mid$
 $\text{refl} \mid \text{rewrite } p_1 \text{ at } x.P \text{ in } p_2 \mid \text{funext } x \text{ in } p$
 $\text{by-contradiction } h \text{ in } q \mid$
 $\text{choose-spec } p \mid \text{choose-witness } x \ h \text{ for } p \text{ in } q$

Recovering annotated proofterms

We can recover some of the original proofterms by combining algorithmic proofterms with annotations.

- **refl** $e : \equiv$ **proving** $e = e$ **by refl**
- **by-contradiction** $h : \neg P$ **in** $q : \equiv$
proving P **by-contradiction** h **in** q

Judgements

Kinding inference judgement: $\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma'$

Type conversion judgements: $\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma'$ (all types),
 $\Gamma \vdash A \hat{=} B \Rightarrow \text{Type}_r \dashv \Gamma'$ (types in whnf)

Type checking judgement: $\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$

Type inference judgement: $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

Term conversion judgements: $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$ (all terms),
 $\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow A \dashv \Gamma'$ (whnfs), $\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \dashv \Gamma'$ (neutral terms),
 $\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow A \dashv \Gamma'$ (neutral terms, type in whnf)

Proposition checking judgement: $\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma'$

Proposition conversion judgements: $\Gamma \vdash P \equiv Q \dashv \Gamma'$ (all propositions),
 $\Gamma \vdash P \hat{=} Q \dashv \Gamma'$ (whnfs), $\Gamma \vdash P \sim Q \dashv \Gamma'$ (whnfs)

Auxiliary judgements

Kind checking judgement: $\Gamma \vdash A \Leftarrow \text{Type}_r \dashv \Gamma'$

Type conversion (checking) judgement: $\Gamma \vdash A \equiv B \Leftarrow \text{Type}_r \dashv \Gamma'$

Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\overline{* \leq_{\text{dec}} +}$$

$$\overline{0 \leq_{\text{dec}} 1}$$

$$\overline{0 \leq_{\text{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how “big” they are.

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	?	+	*
0	*	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
?	$(*, ?)$?	?	$(0, ?)$	$(0, ?)$
+	$(*, +)$	+	$(*, 1)$	+	$(*, 1)$
*	$(*, *)$	*	*	*	*

Decrement variable in context

• $-x = \mathbf{undefined}$

$$(\Gamma, rx : A) - x = \Gamma, (r-1)x : A$$

$$(\Gamma, ry : A) - x = \Gamma - x, ry : A$$

$$(\Gamma, rx : A := e) - x = \Gamma, (r-1)x : A := e$$

$$(\Gamma, ry : A := e) - x = \Gamma - x, ry : A := e$$

$$(\Gamma, h : P) - x = \Gamma - x, h : P$$

$$(\Gamma, a : \text{Type}_r) - x = \Gamma - x, a : \text{Type}_r$$

Context division with remainder

$$\overline{\cdot/r} = \cdot$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A)/q = ((\Gamma_1, r_1 x : A), (\Gamma_2, r_2 x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A := e)/q = ((\Gamma_1, r_1 x : A := e), (\Gamma_2, r_2 x : A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \text{Type}_r)/q = ((\Gamma_1, a : \text{Type}_r), (\Gamma_2, a : \text{Type}_r))}$$

Kind inference

$$\overline{\Gamma \vdash \text{Unit} \Rightarrow \text{Type} \dashv \Gamma} \quad \overline{\Gamma \vdash \text{Empty} \Rightarrow \text{Type} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash r A \rightarrow B \Rightarrow \text{Type}_1 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma'}{\Gamma \vdash !_0 A \Rightarrow \text{Type} \dashv \Gamma'} \quad \frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma' \quad r \neq 0}{\Gamma \vdash !_r A \Rightarrow \text{Type}_{r.s} \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \otimes B \Rightarrow \text{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_{s_1} \dashv \Gamma_1 \quad \Gamma_1 \vdash B \Rightarrow \text{Type}_{s_2} \dashv \Gamma_2}{\Gamma \vdash A \oplus B \Rightarrow \text{Type}_{s_1 \sqcup s_2} \dashv \Gamma_2}$$

Kind inference

$$\frac{(a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \Rightarrow \text{Type}_r \dashv \Gamma}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall a : \text{Type}_r. B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash B \Rightarrow \text{Type}_s \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash \forall \{a : \text{Type}_r\}. B \Rightarrow \text{Type}_s \dashv \Gamma'}$$

Kind checking

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_s \dashv \Gamma' \quad r = s}{\Gamma \vdash A \Leftarrow \text{Type}_r \dashv \Gamma'}$$

Context clean-up

$\Gamma, rx : A \vdash_i e \Leftarrow A \dashv \Gamma', 0x : A$ is a shorthand for
 $\Gamma, rx : A \vdash_i e \Leftarrow A \dashv \Gamma', r'x : A$ with the additional condition
 $r' \sqsubseteq 0$ when $i = c$

$\Gamma, rx : A \vdash_i e \Rightarrow A \dashv \Gamma', 0x : A$ is a shorthand for
 $\Gamma, rx : A \vdash_i e \Rightarrow A \dashv \Gamma', r'x : A$ with the additional condition
 $r' \sqsubseteq 0$ when $i = c$

Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2} \text{ANNOT}$$

Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_c x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{nc} x \Rightarrow A \dashv \Gamma}$$

Functions

$$\frac{\Gamma, r_A^i x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0 x : A}{\Gamma \vdash_i \lambda x. e \Leftarrow r A \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 / r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f a \Rightarrow B \dashv \Gamma_3 + r \Gamma_4}$$

Box

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \text{box } e \Leftarrow !_r A \dashv \Gamma_2 + r \Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0 x : A}{\Gamma \vdash_i \text{let box } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash_i e \Leftarrow \text{Empty} \dashv \Gamma'}{\Gamma \vdash_i \text{Empty-elim } e \Leftarrow A \dashv \Gamma'}$$

Unit

$$\overline{\Gamma \vdash_i \text{unit} \Leftarrow \text{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash_i \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1_A^i x : A, 1_B^i y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0x : A, 0y}{\Gamma \vdash_i \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash_i \text{inl } e \Leftarrow A \oplus B \dashv \Gamma'} \quad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash_i \text{inr } e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Leftarrow 1A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Leftarrow 1B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Rightarrow 1A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Rightarrow 1B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

Let

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r \Gamma_3), r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash_i \text{let}_r x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_4}$$

Polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda a. e \Leftarrow \forall a : \text{Type}_r. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e A \Rightarrow B[a := A] \dashv \Gamma_2}$$

Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda \{a\}. e \Leftarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e @A \Rightarrow B[a := A] \dashv \Gamma_2}$$

Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash P \equiv Q \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow Q \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \text{proving } P \text{ by } p \Rightarrow P \dashv \Gamma_2} \text{ANNOT}$$

Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \quad \frac{(h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \mathbf{assume} \ h \ \mathbf{in} \ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{apply} \ q \ p \Rightarrow Q \dashv \Gamma_2}$$

Propositional logic

$$\frac{}{\Gamma \vdash \mathbf{trivial} \Leftarrow \top \dashv \Gamma} \quad \frac{\Gamma \vdash p \Leftarrow \perp \dashv \Gamma'}{\Gamma \vdash \mathbf{absurd} \ p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \mathbf{both} \ p \ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \mathbf{and-left} \ pq \Rightarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \mathbf{and-right} \ pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \mathbf{or-left} \ p \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \mathbf{or-right} \ q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \mathbf{cases} \ pq \ r_1 \ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \mathbf{destruct} \text{ } pq \text{ as } h_1 \ h_2 \text{ in } r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

Utilities

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2 \quad \Gamma_2, h : P \vdash q \Leftarrow Q \dashv \Gamma_3, h : P}{\Gamma \vdash \text{lemma } h : P \text{ by } p \text{ in } q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P \dashv \Gamma_3}{\Gamma \vdash \text{suffices } P \text{ by } q \text{ in } p \Leftarrow Q \dashv \Gamma_2}$$

Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P[x := y] \dashv \Gamma'}{\Gamma \vdash \text{pick-any } y \text{ in } p \Leftarrow \forall x : A. P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A. P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \text{instantiate } p \text{ with } e \Rightarrow P[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P[x := e] \dashv \Gamma_2}{\Gamma \vdash \text{witness } e \text{ such that } p \Leftarrow \exists x : A. P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \text{pick-witness } y \text{ h for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2}$$

Equality

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \vdash \Gamma'}{\Gamma \vdash \mathbf{refl} \Leftarrow e_1 =_A e_2 \vdash \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1, x : A \vdash P \Leftarrow \text{prop} \vdash \Gamma_2, x : A \\ \Gamma_2 \vdash p \Leftarrow P[x := e_2] \vdash \Gamma_3 \end{array}}{\Gamma \vdash \mathbf{rewrite } q \text{ at } x.P \text{ in } p \Rightarrow P[x := e_1] \vdash \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \vdash \Gamma', x : A}{\Gamma \vdash \mathbf{funext } x \text{ in } p \Leftarrow f =_{rA \rightarrow B} g \vdash \Gamma'}$$

Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma', h : \neg P}{\Gamma \vdash \text{by-contradiction } h \text{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} \text{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash \text{choose-spec } p \Rightarrow P[x := \text{choose } p] \dashv \Gamma'}$$

Classical logic

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \text{choose } p, h$$

$$\Gamma \vdash \text{choose-witness } y \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2$$

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P[x := y] \vdash_{\text{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \text{choose } p, h$$

$$\Gamma \vdash_{\text{nc}} \text{choose-witness } y \text{ for } p \text{ in } e \Rightarrow B \dashv \Gamma_2$$

Type conversion

$$\frac{\Gamma \vdash A \equiv B \Rightarrow \text{Type}_s \vdash \Gamma' \quad r = s}{\Gamma \vdash A \equiv B \Leftarrow \text{Type}_r \vdash \Gamma'}$$

$$\frac{\Gamma \vdash A \triangleq B \Rightarrow \text{Type}_r \vdash \Gamma'}{\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \vdash \Gamma'}$$

Type conversion

$$\overline{\Gamma \vdash \text{Unit} \hat{=} \text{Unit} \Rightarrow \text{Type} \vdash \Gamma}$$

$$\overline{\Gamma \vdash \text{Empty} \hat{=} \text{Empty} \Rightarrow \text{Type} \vdash \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \vdash \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \vdash \Gamma_2}{\Gamma \vdash r_1 A_1 \rightarrow B_1 \hat{=} r_2 A_2 \rightarrow B_2 \Rightarrow \text{Type}_1 \vdash \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_s \vdash \Gamma'}{\Gamma \vdash !_0 A_1 \hat{=} !_0 A_2 \Rightarrow \text{Type} \vdash \Gamma'}$$

$$\frac{r_1 = r_2 \quad r_1 \neq 0 \quad \Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_s \vdash \Gamma'}{\Gamma \vdash !_r A_1 \hat{=} !_r A_2 \Rightarrow \text{Type}_{r_1.s} \vdash \Gamma'}$$

Type conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \hat{=} A_2 \otimes B_2 \Rightarrow \text{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_{s_A} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \equiv B_2 \Rightarrow \text{Type}_{s_B} \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \hat{=} A_2 \oplus B_2 \Rightarrow \text{Type}_{s_A \sqcup s_B} \dashv \Gamma_2}$$

Type conversion

$$\frac{(a : \text{Type}_r) \in \Gamma}{\Gamma \vdash a \triangleq a \Rightarrow \text{Type}_r \dashv \Gamma}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \text{Type}_{r_1} \vdash B_1 \equiv B_2 [a_2 := a_1] \Rightarrow \text{Type}_s \dashv \Gamma', a_1 : \text{Type}_{r_1}}{\Gamma \vdash \forall a_1 : \text{Type}_{r_1}. B_1 \triangleq \forall a_2 : \text{Type}_{r_2}. B_2 \Rightarrow \text{Type}_s \dashv \Gamma'}$$

$$\frac{r_1 = r_2 \quad \Gamma, a_1 : \text{Type}_{r_1} \vdash B_1 \equiv B_2 [a_2 := a_1] \Rightarrow \text{Type}_s \dashv \Gamma', a_1 : \text{Type}_{r_1}}{\Gamma \vdash \forall \{a_1 : \text{Type}_{r_1}\}. B_1 \triangleq \forall \{a_2 : \text{Type}_{r_2}\}. B_2 \Rightarrow \text{Type}_s \dashv \Gamma'}$$

Proposition conversion

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma'}{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma'}$$

Proposition conversion

$$\overline{\Gamma \vdash \top \triangleq \top \dashv \Gamma} \quad \overline{\Gamma \vdash \perp \triangleq \perp \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \triangleq P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \triangleq P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \equiv Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \triangleq P_2 \vee Q_2 \dashv \Gamma_2}$$

Proposition conversion

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1. P_1 \hat{=} \forall x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \equiv P_2[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1. P_1 \hat{=} \exists x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \equiv A_2 \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash e_1 \equiv e_2 \leftarrow A_1 \dashv \Gamma_2 \\ \Gamma_2 \vdash e'_1 \equiv e'_2 \leftarrow A_1 \dashv \Gamma_3 \end{array}}{\Gamma \vdash e_1 =_{A_1} e'_1 \hat{=} e_2 =_{A_2} e'_2 \dashv \Gamma_3}$$

Whnfs and neutral terms

Whnfs:

$\hat{e} ::=$

$$n \mid \lambda_r x. e \mid \text{box}_r e \mid (e_1, e_2) \mid \text{inl } e \mid \text{inr } e \mid \text{unit} \\ \Lambda a : \text{Type}_r. e \mid \Lambda \{a : \text{Type}_r\}. e$$

Neutral forms:

$n ::=$

$$x \mid n e \mid n A \mid n @A \mid \\ \text{let unit} = n \text{ in } e \mid \text{let box } x = n \text{ in } e \mid \\ \text{let } (x, y) = n \text{ in } e \mid \text{case } n \text{ of } (e_1, e_2) \mid \\ \text{choose } p \mid \text{choose-witness } x \text{ } h \text{ for } p \text{ in } n$$

Note that `Empty-elim e` is neither whnf nor neutral.

Computation

$e_1 \longrightarrow e'_1$ – single step

$e_1 \longrightarrow^* e'_1$ – multi step

Term conversion (all terms)

$$\overline{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \text{Unit} \vdash \Gamma}$$

$$\overline{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \text{Empty} \vdash \Gamma}$$

$$\overline{\Gamma \vdash \text{Empty-elim } e_1 \equiv \text{Empty-elim } e_2 \Leftarrow A \vdash \Gamma'}$$

$$\frac{A \neq \text{Unit} \quad A \neq \text{Empty} \quad e_1 \longrightarrow^* e'_1 \quad e_2 \longrightarrow^* e'_2 \quad \Gamma \vdash e'_1 \hat{=} e'_2 \Leftarrow A \vdash \Gamma'}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \vdash \Gamma'}$$

Term conversion (whnfs)

$$\frac{\Gamma, x : A \vdash f \ x \equiv g \ x \Leftarrow B \vdash \Gamma', x : A}{\Gamma \vdash f \hat{=} g \Leftarrow r A \rightarrow B \vdash \Gamma'}$$

$$\frac{\Gamma \vdash \text{let box } x = e_1 \text{ in } x \equiv \text{let box } x = e_2 \text{ in } x \Leftarrow A \vdash \Gamma'}{\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow !_r A \vdash \Gamma'}$$

$$\frac{\begin{array}{l} \Gamma \vdash \text{let } (x, y) = e_1 \text{ in } x \equiv \text{let } (x, y) = e_2 \text{ in } x \Leftarrow A \vdash \Gamma_1 \\ \Gamma_1 \vdash \text{let } (x, y) = e_1 \text{ in } y \equiv \text{let } (x, y) = e_2 \text{ in } y \Leftarrow B \vdash \Gamma_2 \end{array}}{\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow A \otimes B \vdash \Gamma_2}$$

Term conversion (whnfs)

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \vdash \Gamma'}{\Gamma \vdash \text{inl } e_1 \hat{=} \text{inl } e_2 \Leftarrow A \oplus B \vdash \Gamma'}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow B \vdash \Gamma'}{\Gamma \vdash \text{inr } e_1 \hat{=} \text{inr } e_2 \Leftarrow A \oplus B \vdash \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f \ a \equiv g \ a \Leftarrow B \vdash \Gamma'}{\Gamma \vdash f \hat{=} g \Leftarrow \forall a : \text{Type}_r. B \vdash \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_r \vdash f \ @a \equiv g \ @a \Leftarrow B \vdash \Gamma'}{\Gamma \vdash f \hat{=} g \Leftarrow \forall \{a : \text{Type}_r\}. B \vdash \Gamma'}$$

Term conversion (neutral)

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \equiv x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1 e_1 \equiv n_2 e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 A_1 \equiv n_2 A_2 \Rightarrow B[a := A] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A_1 \equiv A_2 \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash n_1 @A_1 \equiv n_2 @A_2 \Rightarrow B[a := A] \dashv \Gamma_2}$$

Term conversion (neutral)

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \equiv e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash \text{let unit} = n_1 \text{ in } e_1 \equiv \text{let unit} = n_2 \text{ in } e_2 \Rightarrow A \dashv}$$

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, x : A \vdash e_1 \equiv e_2 \Rightarrow B \dashv \Gamma_2, x : A}{\Gamma \vdash \text{let box } x = n_1 \text{ in } e_1 \equiv \text{let box } x = n_2 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash n_1 \hat{=} n_2 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, x : A, y : B \vdash e_1 \equiv e_2 \Rightarrow C \dashv \Gamma_2, x : A, y : B}{\Gamma \vdash \text{let } (x, y) = n_1 \text{ in } e_1 \equiv \text{let } (x, y) = n_2 \text{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash f_1 \equiv f_2 \Rightarrow 1 A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash g_1 \equiv g_2 \Rightarrow 1 B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash \text{case } e_1 \text{ of } (f_1, g_1) \equiv \text{case } e_2 \text{ of } (f_2, g_2) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

Term conversion (choice)

$$\frac{\begin{array}{l} \Gamma \vdash p_1 \Rightarrow \exists x : A_1. P_1 \dashv \Gamma_1 \\ \Gamma_1 \vdash p_2 \Rightarrow \exists x : A_2. P_2 \dashv \Gamma_2 \end{array} \quad \Gamma_2 \vdash \exists x : A_1. P_1 \equiv \exists x : A_1. P_2 \dashv \Gamma_3}{\Gamma \vdash \text{choose } p_1 \equiv \text{choose } p_2 \Rightarrow A_1 \dashv \Gamma_3}$$