

Zipper Contexts

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Intro

What we do here is explore an interesting idea. In some algorithms for bidirectional typechecking of polymorphic lambda calculus, there appears a special thing called a mark. The idea is that the mark designates the place in the context up until which type variables can be generalized when forming universally quantified types.

Since such a mark is just another way of extending the context, besides $\Gamma, x : A$ (which corresponds to $\lambda x : A. e$) and Γ, α which corresponds to $\Lambda \alpha. e$, it might be worthwhile to explore what would happen if we had more ways of extending the context that correspond to terms?

The answer is interesting: contexts would turn into zippers!

Contexts as zippers

Contexts:

$\Gamma ::=$

$\cdot \mid$
 $\Gamma, \lambda x. \circ, x : A \mid \Gamma, \circ e_2 \mid \Gamma, e_1 \circ \mid$
 $\Gamma, (\circ, e_2) \mid \Gamma, (e_1, \circ) \mid \Gamma, \text{outl } \circ \mid \Gamma, \text{outr } \circ \mid$
 $\Gamma, \text{inl } \circ \mid \Gamma, \text{inr } \circ \mid \Gamma, \text{case } \circ \text{ of } (f, g) \mid$
 $\Gamma, \text{case } e \text{ of } (x. \circ, g), x : A \mid \Gamma, \text{case } e \text{ of } (f, x. \circ), x : A \mid$
 $\Gamma, \text{elim}_0 \circ$

Contexts

To be clear, $(x : A) \in \Gamma$ if and only if either
 $\Gamma = \Gamma_L, \lambda x. \circ, x : A, \Gamma_R$, or $\Gamma = \Gamma_L, \lambda x. \circ, x : A := e, \Gamma_R$, or
 $\Gamma = \Gamma, \text{case } e \text{ of } (x. \circ, y. e_2), x : A$

Terms

Terms:

$e ::=$

$x \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{elim}_0 e$

Declarative typing – basics

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

Declarative typing – type-directed rules

$$\frac{\Gamma, \lambda x. \circ, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma, \circ a \vdash f : A \rightarrow B \quad \Gamma, f \circ \vdash a : A}{\Gamma \vdash f a : B}$$

$$\frac{\Gamma, (\circ, b) \vdash a : A \quad \Gamma, (a, \circ) \vdash b : B}{\Gamma \vdash (a, b) : A \times B}$$

$$\frac{\Gamma, \text{outl } \circ \vdash e : A \times B}{\Gamma \vdash \text{outl } e : A} \quad \frac{\Gamma, \text{outr } \circ \vdash e : A \times B}{\Gamma \vdash \text{outr } e : B}$$

$$\frac{\Gamma, \text{inl } \circ \vdash e : A}{\Gamma \vdash \text{inl } e : A + B} \quad \frac{\Gamma, \text{inr } \circ \vdash e : B}{\Gamma \vdash \text{inr } e : A + B}$$

Declarative typing – type-directed rules

$$\frac{\begin{array}{l} \Gamma, \text{case } \circ \text{ of } (x. e_1, y. e_2) \vdash e : A + B \\ \Gamma, \text{case } e \text{ of } (x. \circ, y. e_2), x : A \vdash e_1 : C \\ \Gamma, \text{case } e \text{ of } (x. e_1, y. \circ), y : B \vdash e_2 : C \end{array}}{\Gamma \vdash \text{case } e \text{ of } (x. e_1, y. e_2) : C}$$

$$\frac{}{\Gamma \vdash \text{unit} : \mathbf{1}} \quad \frac{\Gamma, \text{elim}_0 \circ \vdash e : \mathbf{0}}{\Gamma \vdash \text{elim}_0 e : A}$$

What's the point?

What is the point of it? By now I don't really remember, but I think it was that we add definitions to the context, i.e. $\Gamma, x : A := e$, and then we turn the term-related context extensions into smart constructors, which can turn ordinary variable bindings into definitions.

$$\Gamma, \circ e, \lambda x. \circ, x : A \rightsquigarrow \Gamma, \circ e, \lambda x. \circ, x : A := e$$

$$\begin{aligned} \Gamma, \text{case } (\text{inl } e) \text{ of } (x. \circ, y. e_2), x : A &\rightsquigarrow \\ \Gamma, \text{case } (\text{inl } e) \text{ of } (x. \circ, y. e_2), x : A &:= e \end{aligned}$$

$$\begin{aligned} \Gamma, \text{case } (\text{inr } e) \text{ of } (x. e_1, y. \circ), y : B &\rightsquigarrow \\ \Gamma, \text{case } (\text{inr } e) \text{ of } (x. e_1, y. \circ), y : B &:= e \end{aligned}$$