Hint-based unification for STLC

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Terms, contexts and judgements

Terms:

$e ::=$

$x \mid (e : H) \mid$

$\lambda x.\, e \mid e_1\ e_2 \mid$

$(e_1, e_2) \mid \mathtt{outl}\ e \mid \mathtt{outr}\ e \mid$

$\mathtt{inl}\ e \mid \mathtt{inr}\ e \mid \mathtt{case}\ e\ \mathtt{of}\ (e_1, e_2) \mid$

$\mathtt{unit} \mid \mathtt{elim}_{\mathbf{0}}\ e$

Contexts:

$\Gamma ::= \cdot \mid \Gamma, x : H$

Judgements:

$\Gamma \vdash e \Leftarrow H \Rightarrow H' \dashv \Gamma'$ – in context Γ, term e checks with hint H and infers hint H' in output context Γ'

Output contexts – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow A \sqcup B \Rightarrow C \dashv \Gamma'}{\Gamma \vdash (e : A) \Leftarrow B \Rightarrow C \dashv \Gamma'}\textsc{Annot}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathtt{hint}(e) \Rightarrow A \dashv \Gamma' \quad e\ \mathtt{constructor}}{\Gamma \vdash e \Leftarrow \mathbf{?} \Rightarrow A \dashv \Gamma'}\textsc{Hole}$$

Output contexts – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Leftarrow \mathbf{?} \to B \Rightarrow A \to B' \dashv \Gamma' \quad \Gamma' \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f\ a \Leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \Leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \Leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times \mathbf{?} \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \mathtt{outl}\ e \Leftarrow A \Rightarrow A' \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{?} \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \mathtt{outr}\ e \Leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Output contexts – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow A \Rightarrow A' \dashv \Gamma'}{\Gamma \vdash \mathtt{inl}\ e \Leftarrow A + B \Rightarrow A' + B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma'}{\Gamma \vdash \mathtt{inr}\ e \Leftarrow A + B \Rightarrow A + B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \dashv \Gamma_2 \\ \Gamma_2 \vdash g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3 \dashv \Gamma_3}$$

$$\frac{}{\Gamma \vdash \mathtt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1} \dashv \Gamma} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0} \dashv \Gamma'}{\Gamma \vdash \mathtt{elim}_{\mathbf{0}}\ e \Leftarrow A \Rightarrow A \dashv \Gamma'}$$

Order on contexts

$$\frac{}{\cdot \sqsubseteq \cdot}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad A \sqsubseteq B}{\Gamma_1, x : A \sqsubseteq \Gamma_2, x : B}$$

Order on contexts – properties

$\sqsubseteq$ is a partial order, whose structure is inherited from the order on hints.

- Reflexivity: $\Gamma \sqsubseteq \Gamma$
- Transitivity: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_3 \implies \Gamma_1 \sqsubseteq \Gamma_3$
- Weak antisymmetry: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_1 \implies \Gamma_1 = \Gamma_2$

Metatheory

If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, then:

- (Information increase) $\Gamma \sqsubseteq \Gamma'$ (proof: induction)
- (Information increase) $A \sqsubseteq B$ (proof: induction)
- (Context squeeze) If $\Gamma \sqsubseteq \Delta \sqsubseteq \Gamma'$, then $\Delta \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: induction)
- (Hint squeeze) If $A \sqsubseteq A'$ and $A' \sqsubseteq B$, then $\Gamma \vdash e \Leftarrow A' \Rightarrow B \dashv \Gamma'$ (proof: induction)

Metatheory – decidability and determinism

- (Decidability) For Γ, e, A it is decidable whether there exist B and Γ' such that $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: the rules are literally the algorithm)
- (Determinism) If $\Gamma \vdash e \Leftarrow A \Rightarrow B_1 \dashv \Gamma_1$ and $\Gamma \vdash e \Leftarrow A \Rightarrow B_2 \dashv \Gamma_2$, then $B_1 = B_2$ and $\Gamma_1 = \Gamma_2$ (proof: induction)

Metatheory – soundness

(Soundness) If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, B is a type and Γ' only contains types, then $\Gamma' \vdash e : B$.

Output contexts – abridged basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

$$\frac{e \Leftarrow A \sqcup B \Rightarrow C}{(e : A) \Leftarrow B \Rightarrow C}\textsc{Annot}$$

$$\frac{e \Leftarrow \texttt{hint}(e) \Rightarrow A \qquad e\ \texttt{constructor}}{e \Leftarrow \textbf{?} \Rightarrow A}\textsc{Hole}$$

Output contexts – abridged type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{f \Leftarrow \mathbf{?} \to B \Rightarrow A \to B' \quad a \Leftarrow A \Rightarrow A'}{f\ a \Leftarrow B \Rightarrow B'}$$

$$\frac{a \Leftarrow A \Rightarrow A' \quad b \Leftarrow B \Rightarrow B'}{(a, b) \Leftarrow A \times B \Rightarrow A' \times B'}$$

$$\frac{e \Leftarrow A \times \mathbf{?} \Rightarrow A' \times B}{\texttt{outl}\ e \Leftarrow A \Rightarrow A'} \qquad \frac{e \Leftarrow \mathbf{?} \times B \Rightarrow A \times B'}{\texttt{outr}\ e \Leftarrow B \Rightarrow B'}$$

Output contexts – abridged type-directed rules

$$\frac{e \Leftarrow A \Rightarrow A'}{\texttt{inl}\ e \Leftarrow A + B \Rightarrow A' + B}$$

$$\frac{e \Leftarrow B \Rightarrow B'}{\texttt{inr}\ e \Leftarrow A + B \Rightarrow A + B'}$$

$$\frac{e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \qquad \begin{array}{c} f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \\ g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \end{array}}{\texttt{case}\ e\ \texttt{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3}$$

$$\frac{}{\texttt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \qquad \frac{e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\texttt{elim}_{\mathbf{0}}\ e \Leftarrow A \Rightarrow A}$$

Types, holes, terms and contexts

Holes:

$$H ::= \alpha \mid H_1 \to H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$$

Terms:

$$
\begin{aligned}
e ::= &\\
&x \mid (e : H) \mid \\
&\lambda x.\, e \mid e_1\ e_2 \mid \\
&(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid \\
&\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid \\
&\texttt{unit} \mid \texttt{elim}_{\mathbf{0}}\ e
\end{aligned}
$$

Note: the terms are the same as in STLC with Hints.

Contexts and judgements

Contexts:
$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha \mid \Gamma, \alpha := H$

We introduce unification variables, denoted with Greek letter $(\alpha, \beta, \gamma, \dots)$. When extending the context with a unification variable, it is set to some hint, which at the beginning will be just **?**.

Judgements:
$\Gamma \vdash e \Leftarrow H \Rightarrow H' \dashv \Gamma'$ – in context Γ, term e checks with hint H and infers hint H' in output context Γ'

Unification – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

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Unification – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Leftarrow \mathbf{?} \to B \Rightarrow A \to B' \dashv \Gamma' \quad \Gamma' \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f\ a \Leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \Leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \Leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times \mathbf{?} \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \mathtt{outl}\ e \Leftarrow A \Rightarrow A' \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{?} \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \mathtt{outr}\ e \Leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Unification – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow A \Rightarrow A' \dashv \Gamma'}{\Gamma \vdash \mathtt{inl}\ e \Leftarrow A + B \Rightarrow A' + B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma'}{\Gamma \vdash \mathtt{inr}\ e \Leftarrow A + B \Rightarrow A + B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \dashv \Gamma_2 \\ \Gamma_2 \vdash g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3 \dashv \Gamma_3}$$

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