

Quantities

Quantities:

$r ::= 0 \mid 1 \mid ? \mid + \mid *$

* is the default quantity, so when there's nothing to indicate quantity, it means it's *.

Quantities – explanation

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced “few”) means a resource must be used at most once.
- + (pronounced “many”) means a resource must be used at least once.
- * (pronounced “any”) means no restrictions on usage.

Subusage ordering

$r_1 \sqsubseteq r_2$ means that a resource with quantity r_1 may be used when quantity r_2 is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$* \sqsubseteq ?$$

$$* \sqsubseteq +$$

$$+ \sqsubseteq 1$$

$$? \sqsubseteq 1$$

$$? \sqsubseteq 0$$

We will denote the greatest lower bound in this order with $r_1 \sqcap r_2$ and the least upper bound (if it exists) with $r_1 \sqcup r_2$.

Addition of quantities

When we have two quantities of the same resource, we can sum the quantities.

$$0 + r = r$$

$$r + 0 = r$$

$$? + ? = *$$

$$? + * = *$$

$$* + ? = *$$

$$* + * = *$$

$$- + - = +$$

Multiplication of quantities

When we have a quantity r_1 of resource A that contains quantity r_2 of resource B , then we in fact have quantity $r_1 \cdot r_2$ of resource B .

$$0 \cdot r = 0$$

$$r \cdot 0 = 0$$

$$1 \cdot r = r$$

$$r \cdot 1 = r$$

$$?.? = ?$$

$$+ \cdot + = +$$

$$- \cdot - = *$$

The algebra of quantities

Quantities \mathcal{Q} form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$ is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$ is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If $r_1 + r_2 = 0$, then $r_1 = 0$ and $r_2 = 0$.
- If $r_1 \cdot r_2 = 0$, then $r_1 = 0$ or $r_2 = 0$.

Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\overline{* \leq_{\text{dec}} *}$$

$$\overline{* \leq_{\text{dec}} +}$$

$$\overline{0 \leq_{\text{dec}} 1}$$

$$\overline{0 \leq_{\text{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how “big” they are.

Trait-checking Division

$$r_1/r_2 = \sup\{s \in \mathcal{Q} \mid s \cdot r_2 \sqsubseteq r_1\}$$

r_1/r_2	0	1	?	+	*
0	0	0	0	0	0
1		1	1	1	1
?		?	1	?	1
+		+	+	1	1
*		*	+	?	1

Division with remainder

$r_1/r_2 = (a, b)$ when $r_1 = a \cdot r_2 + b$, with a as large as possible and b being as small as possible according to the subtraction order.

Note that $r_1/r_2 = a$ means that $b = 0$.

r_1/r_2	0	1	?	+	*
0	*	0	0	0	0
1	(*, 1)	1	(0, 1)	(0, 1)	(0, 1)
?	(*, ?)	?	?	(0, ?)	(0, ?)
+	(*, +)	+	(*, 1)	+	(*, 1)
*	(*, *)	*	*	*	*