

T-Axi (algorithmic)

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Syntax

The syntax in the algorithmic system is the same, but we will have far fewer annotations in terms

Judgements

Kind inference judgement: $\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma'$

Type conversion judgements: $\Gamma \vdash A \hat{=} B \Rightarrow \text{Type}_r \dashv \Gamma'$,
 $\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma'$

Type checking judgement: $\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$

Type inference judgement: $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

Term conversion judgements: $\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow A \dashv \Gamma'$,
 $\Gamma \vdash e_1 \hat{\sim} e_2 \Rightarrow A \dashv \Gamma'$, $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$,
 $\Gamma \vdash e_1 \sim e_2 \Rightarrow A \dashv \Gamma'$

Proposition checking judgement: $\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma'$

Proposition conversion judgements: $\Gamma \vdash P \hat{=} Q \dashv \Gamma'$,
 $\Gamma \vdash P \equiv Q \dashv \Gamma'$, $\Gamma \vdash P \sim Q \dashv \Gamma'$

Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\overline{* \leq_{\text{dec}} +}$$

$$\overline{0 \leq_{\text{dec}} 1}$$

$$\overline{0 \leq_{\text{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how “big” they are.

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	?	+	*
0	*	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
?	$(*, ?)$?	?	$(0, ?)$	$(0, ?)$
+	$(*, +)$	+	$(*, 1)$	+	$(*, 1)$
*	$(*, *)$	*	*	*	*

Decrement variable in context

• $-x = \mathbf{undefined}$

$$(\Gamma, rx : A) - x = \Gamma, (r-1)x : A$$

$$(\Gamma, ry : A) - x = \Gamma - x, ry : A$$

$$(\Gamma, rx : A := e) - x = \Gamma, (r-1)x : A := e$$

$$(\Gamma, ry : A := e) - x = \Gamma - x, ry : A := e$$

$$(\Gamma, h : P) - x = \Gamma - x, h : P$$

$$(\Gamma, a : \text{Type}_r) - x = \Gamma - x, a : \text{Type}_r$$

Context division with remainder

$$\overline{\cdot/r} = \cdot$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A)/q = ((\Gamma_1, r_1 x : A), (\Gamma_2, r_2 x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A := e)/q = ((\Gamma_1, r_1 x : A := e), (\Gamma_2, r_2 x : A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \text{Type}_r)/q = ((\Gamma_1, a : \text{Type}_r), (\Gamma_2, a : \text{Type}_r))}$$

Context clean-up

$\Gamma, rx : A \vdash_i e \Leftarrow A \dashv \Gamma', 0x : A$ is a shorthand for
 $\Gamma, rx : A \vdash_i e \Leftarrow A \dashv \Gamma', r'x : A$ with the additional condition
 $r' \sqsubseteq 0$ when $i = c$

$\Gamma, rx : A \vdash_i e \Rightarrow A \dashv \Gamma', 0x : A$ is a shorthand for
 $\Gamma, rx : A \vdash_i e \Rightarrow A \dashv \Gamma', r'x : A$ with the additional condition
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Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \triangleq B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2} \text{ANNOT}$$

Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_c x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{nc} x \Rightarrow A \dashv \Gamma}$$

Functions

$$\frac{\Gamma, r_A^i x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0 x : A}{\Gamma \vdash_i \lambda x. e \Leftarrow r A \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 / r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f a \Rightarrow B \dashv \Gamma_3 + r \Gamma_4}$$

Box

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \text{box } e \Leftarrow !_r A \dashv \Gamma_2 + r \Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0 x : A}{\Gamma \vdash_i \text{let box } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash_i e \Leftarrow \text{Empty} \dashv \Gamma'}{\Gamma \vdash_i \text{Empty-elim } e \Leftarrow A \dashv \Gamma'}$$

Unit

$$\overline{\Gamma \vdash_i \text{unit} \Leftarrow \text{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash_i \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1_A^i x : A, 1_B^i y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0x : A, 0y : B}{\Gamma \vdash_i \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \vdash \Gamma'}{\Gamma \vdash_i \text{inl } e \Leftarrow A \oplus B \dashv \vdash \Gamma'} \quad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \vdash \Gamma'}{\Gamma \vdash_i \text{inr } e \Leftarrow A \oplus B \dashv \vdash \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Leftarrow 1A \rightarrow C \dashv \vdash \Gamma_2 \\ \Gamma_1 \vdash_i g \Leftarrow 1B \rightarrow C \dashv \vdash \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Leftarrow C \dashv \vdash \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash_i f \Rightarrow 1A \rightarrow C \dashv \vdash \Gamma_2 \\ \Gamma_1 \vdash_i g \Rightarrow 1B \rightarrow C \dashv \vdash \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Rightarrow C \dashv \vdash \Gamma_2 \sqcup \Gamma_3}$$

Let

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r \Gamma_3), r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash_i \text{let}_r x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_4}$$

Polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda a. e \Leftarrow \forall a : \text{Type}_r. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e A \Rightarrow B[a := A] \dashv \Gamma_2}$$

Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda \{a\}. e \Leftarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e @A \Rightarrow B [a := A] \dashv \Gamma_2}$$

Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash P \triangleq P' \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow P \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash P \Leftarrow_{\text{prop}} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \text{proving } P \text{ by } p \Rightarrow P \dashv \Gamma_2} \text{ANNOT}$$

Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \quad \frac{(h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \mathbf{assume} \ h \ \mathbf{in} \ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{apply} \ q \ p \Rightarrow Q \dashv \Gamma_2}$$

Propositional logic

$$\frac{}{\Gamma \vdash \mathbf{trivial} \Leftarrow \top \dashv \Gamma} \quad \frac{\Gamma \vdash p \Leftarrow \perp \dashv \Gamma'}{\Gamma \vdash \mathbf{absurd} \ p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \mathbf{both} \ p \ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \mathbf{and-left} \ pq \Rightarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \mathbf{and-right} \ pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \mathbf{or-left} \ p \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \mathbf{or-right} \ q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \mathbf{cases} \ pq \ r_1 \ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \mathbf{destruct} \, pq \, \mathbf{as} \, h_1 \, h_2 \, \mathbf{in} \, r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

Utilities

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2 \quad \Gamma_2, h : P \vdash q \Leftarrow Q \dashv \Gamma_3, h : P}{\Gamma \vdash \text{lemma } h : P \text{ by } p \text{ in } q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P \dashv \Gamma_3}{\Gamma \vdash \text{suffices } P \text{ by } q \text{ in } p \Leftarrow Q \dashv \Gamma_2}$$

Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P[x := y] \dashv \Gamma'}{\Gamma \vdash \mathbf{pick-any} \ y \ \mathbf{in} \ p \Leftarrow \forall x : A. P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A. P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \mathbf{instantiate} \ p \ \mathbf{with} \ e \Rightarrow P[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P[x := e] \dashv \Gamma_2}{\Gamma \vdash \mathbf{witness} \ e \ \mathbf{such \ that} \ p \Leftarrow \exists x : A. P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \mathbf{pick-witness} \ y \ h \ \mathbf{for} \ p \ \mathbf{in} \ q \Leftarrow Q \dashv \Gamma_2}$$

Equality

$$\frac{\Gamma \vdash e_1 \triangleq e_2 \Leftarrow A \vdash \Gamma'}{\Gamma \vdash \mathbf{refl} \Leftarrow e_1 =_A e_2 \vdash \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1, x : A \vdash P \Leftarrow \text{prop} \vdash \Gamma_2, x : A \\ \Gamma_2 \vdash p \Leftarrow P[x := e_2] \vdash \Gamma_3 \end{array}}{\Gamma \vdash \mathbf{rewrite} \, q \, \text{at} \, x.P \, \text{in} \, p \Rightarrow P[x := e_1] \vdash \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \vdash \Gamma', x : A}{\Gamma \vdash \mathbf{funext} \, x \, \text{in} \, p \Leftarrow f =_{rA \rightarrow B} g \vdash \Gamma'}$$

Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma', h : \neg P}{\Gamma \vdash \text{by-contradiction } h \text{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} \text{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash \text{choose-spec } p \Rightarrow P[x := \text{choose } p] \dashv \Gamma'}$$

Classical logic

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \text{choose } p, h$$

$$\Gamma \vdash \text{choose-witness } y \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2$$

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P[x := y] \vdash_{\text{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \text{choose } p, h$$

$$\Gamma \vdash_{\text{nc}} \text{choose-witness } y \text{ for } p \text{ in } e \Rightarrow B \dashv \Gamma_2$$

Conversion of types

$$\overline{\Gamma \vdash \text{Unit} \equiv \text{Unit} \Rightarrow \text{Type} \vdash \Gamma}$$

$$\overline{\Gamma \vdash \text{Empty} \equiv \text{Empty} \Rightarrow \text{Type} \vdash \Gamma}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \Rightarrow \text{Type} \vdash \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 \Rightarrow \text{Type} \vdash \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \Rightarrow \text{Type} \vdash \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 \Rightarrow \text{Type} \vdash \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \Rightarrow \text{Type} \vdash \Gamma_2}{\Gamma \vdash r A_1 \rightarrow B_1 \equiv r A_2 \rightarrow B_2 \Rightarrow \text{Type} \vdash \Gamma_2}$$

$$\Gamma \vdash A_1 \triangleq A_2 \Rightarrow \text{Type} \vdash \Gamma'$$

Conversion of propositions

$$\overline{\Gamma \vdash \top \equiv \top \dashv \Gamma} \quad \overline{\Gamma \vdash \perp \equiv \perp \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2 \dashv \Gamma_2}$$

Conversion of propositions

$$\frac{\Gamma \vdash A_1 \cong A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \cong P_2 [x_2 := x_1] \vdash \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1. P_1 \equiv \forall x_2 : A_2. P_2 \vdash \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \cong A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \cong P_2 [x_2 := x_1] \vdash \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1. P_1 \equiv \exists x_2 : A_2. P_2 \vdash \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \cong A_2 \Rightarrow \text{Type} \vdash \Gamma_1 \quad \Gamma_1 \vdash e_1 \cong e_2 \Leftarrow A_1 \vdash \Gamma_2 \quad \Gamma_2 \vdash e'_1 \cong e'_2 \Leftarrow A_1}{\Gamma \vdash e_1 =_{A_1} e'_1 \equiv e_2 =_{A_2} e'_2 \vdash \Gamma_3}$$

Conversion of propositions

$$\frac{\Gamma \vdash N_1 \sim N_2 \dashv \vdash \Gamma'}{\Gamma \vdash N_1 \equiv N_2 \dashv \vdash \Gamma'}$$

Conversion of terms

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \hat{\sim} x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma, x : A \vdash f \ x \hat{=} g \ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \equiv g \Leftarrow r A \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash n_1 \sim n_2 \Rightarrow r A \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash u_1 \hat{=} u_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1 \ u_1 \hat{\sim} n_2 \ u_2 \Rightarrow B \dashv \Gamma_2}$$