T-Axi (algorithmic)

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Subtraction of quantities

$r_1 - r_2$ is the least r' such that $r_1 \sqsubseteq r' + r_2$.

$r_1 - r_2$	0	1	?	$+$	$*$
0	0				
1	1	0			
?	?	0	0		
$+$	$+$	$*$	$+$	$*$	$+$
$*$	$*$	$*$	$*$	$*$	$*$

Subtraction order on quantities

$r_1 \leq_{\mathtt{sub}} r_2$ holds when $r_2 - r_1$ is defined.

Explicitly: $0 \leq_{\mathtt{sub}} 1 \leq_{\mathtt{sub}} ? \leq_{\mathtt{sub}} + \leq_{\mathtt{sub}} * \leq_{\mathtt{sub}} +$

Decrementation order on quantities

$r_1 \leq_{\texttt{dec}} r_2$ holds when $r_2 - 1 = r_1$.

$$\frac{}{* \leq_{\texttt{dec}} +}$$

$$\frac{}{0 \leq_{\texttt{dec}} 1}$$

$$\frac{}{0 \leq_{\texttt{dec}} ?}$$

Arithmetic order on quantities

The arithmetic order on quantities is $0 \leq 1 \leq ? \leq + \leq *$. The idea is to compare the quantities by how "big" they are.

Division with remainder

$a/b = (q, r)$ when $a = b \cdot q + r$, with q as large as possible and r being as small as possible according to the arithmetic order. Note that $a/b = q$ means that $r = 0$.

r_1/r_2	0	1	$?$	$+$	$*$
0	$*$	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
$?$	$(*, ?)$	$?$	$?$	$(0, ?)$	$(0, ?)$
$+$	$(*, +)$	$+$	$(*, 1)$	$+$	$(*, 1)$
$*$	$(*, *)$	$*$	$*$	$*$	$*$

Decrement variable in context

$$
\begin{aligned}
\cdot - x &= \textbf{undefined} \\
(\Gamma, r\,x : A) - x &= \Gamma, (r-1)\,x : A \\
(\Gamma, r\,y : A) - x &= \Gamma - x, r\,y : A \\
(\Gamma, r\,x : A := e) - x &= \Gamma, (r-1)\,x : A := e \\
(\Gamma, r\,y : A := e) - x &= \Gamma - x, r\,y : A := e \\
(\Gamma, h : P) - x &= \Gamma - x, h : P
\end{aligned}
$$

Context division with remainder

$$\frac{}{\cdot/r = \cdot}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\,x : A)/q = ((\Gamma_1, r_1\,x : A), (\Gamma_2, r_2\,x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r\,x : A := e)/q = ((\Gamma_1, r_1\,x : A := e), (\Gamma_2, r_2\,x : A := e))}$$

$$\frac{\Gamma/q = \Gamma'}{(\Gamma, h : P)/q = ((\Gamma', h : P), (\Gamma', h : P))}$$

Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \Rightarrow A \dashv \Gamma - x}$$

Functions

$$\frac{\Gamma, r\,x : A \vdash e \Leftarrow B \dashv \Gamma', r'\,x : A \quad r' \sqsubseteq 0}{\Gamma \vdash \lambda\,x.\,e \Leftarrow r\,A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Rightarrow r\,A \to B \dashv \Gamma_1 \quad \Gamma_1/r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash f\ a \Rightarrow B \dashv \Gamma_3 + r\,\Gamma_4}$$

Useful shorthand (when not enough space):

$$\frac{\Gamma, r\,x : A \vdash e \Leftarrow B \dashv \Gamma', 0\,x : A}{\Gamma \vdash \lambda\,x.\,e \Leftarrow r\,A \to B \dashv \Gamma'}$$

Box

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash \texttt{box}\ e \Leftarrow\ !_r\, A \dashv \Gamma_2 + r\,\Gamma_3}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow\ !_r\, A \dashv \Gamma_1 \quad \Gamma_1, r\,x : A \vdash e_2 \Rightarrow B \dashv \Gamma_2, r'\,x : A \quad r' \sqsubseteq 0}{\Gamma \vdash \texttt{let}\ \ \texttt{box}\ x = e_1\ \texttt{in}\ e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash e \Leftarrow \texttt{Empty} \dashv \Gamma'}{\Gamma \vdash \texttt{Empty-elim}\ e \Leftarrow A \dashv \Gamma'}$$

Unit

$$\frac{}{\Gamma \vdash \texttt{unit} \color{blue}{\Leftarrow} \texttt{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash e_1 \color{red}{\Rightarrow} \texttt{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash e_2 \color{red}{\Rightarrow} A \dashv \Gamma_2}{\Gamma \vdash \texttt{let unit} = e_1 \texttt{ in } e_2 \color{red}{\Rightarrow} A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash a \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash b \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash (a, b) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1\, x : A, 1\, y : B \vdash e_2 \Rightarrow C \dashv \Gamma_2, 0\, x : A, 0\, y : B}{\Gamma \vdash \mathtt{let}\ (x, y) = e_1\ \mathtt{in}\ e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \texttt{inl}\ e \Leftarrow A \oplus B \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash \texttt{inr}\ e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Rightarrow A \oplus B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Leftarrow 1\, A \to C \dashv \Gamma_2 \\ \Gamma_1 \vdash g \Leftarrow 1\, B \to C \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash e \Rightarrow A \oplus B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Rightarrow 1\, A \to C \dashv \Gamma_2 \\ \Gamma_1 \vdash g \Rightarrow 1\, B \to C \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

Let

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash e_1 \Rightarrow A \dashv \Gamma_3 \quad \Gamma_2 + r\,\Gamma_3, r\,x : A \vdash e_2 \Rightarrow B \dashv \Gamma_4, 0\,x}{\Gamma \vdash \texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \Rightarrow B \dashv \Gamma_4}$$

Conversion of types

$$\frac{}{\Gamma \vdash \texttt{Unit} \equiv \texttt{Unit} \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \texttt{Empty} \equiv \texttt{Empty} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash r\, A_1 \to B_1 \equiv r\, A_2 \to B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma'}{\Gamma \vdash !_r\, A_1 \equiv !_r\, A_2 \dashv \Gamma'}$$

Conversion of propositions

$$\frac{}{\Gamma \vdash \top \equiv \top \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \bot \equiv \bot \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \triangleq P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \triangleq Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2 \dashv \Gamma_2}$$

Conversion of propositions

$$\frac{\Gamma \vdash A_1 \mathrel{\hat{\equiv}} A_2 \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \mathrel{\hat{\equiv}} P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1.\, P_1 \equiv \forall x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \mathrel{\hat{\equiv}} A_2 \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \mathrel{\hat{\equiv}} P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1.\, P_1 \equiv \exists x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \mathrel{\hat{\equiv}} A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \mathrel{\hat{\equiv}} e_2 \Leftarrow A_1 \dashv \Gamma_2 \quad \Gamma_2 \vdash e_1' \mathrel{\hat{\equiv}} e_2' \Leftarrow A_1 \dashv \Gamma_3}{\Gamma \vdash e_1 =_{A_1} e_1' \equiv e_2 =_{A_2} e_2' \dashv \Gamma_3}$$

Conversion of propositions

$$\frac{\Gamma \vdash N_1 \sim N_2 \dashv \Gamma'}{\Gamma \vdash N_1 \equiv N_2 \dashv \Gamma'}$$

Conversion of terms

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \mathrel{\hat{\sim}} x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma, x : A \vdash f\ x \mathrel{\hat{\equiv}} g\ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f\ \ \equiv g \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash n_1 \sim n_2 \Rightarrow r\, A \to B \dashv \Gamma_1 \quad \Gamma_1 \vdash u_1 \mathrel{\hat{\equiv}} u_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1\ u_1 \mathrel{\hat{\sim}} n_2\ u_2 \Rightarrow B \dashv \Gamma_2}$$