

# Judgements

- $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$  – in context  $\Gamma$ , term  $e$  infers type  $A$ , with output context  $\Gamma'$
- $\Gamma \vdash A \rightsquigarrow B \dashv \Gamma'$  – in context  $\Gamma$ , type  $A$  matches (output) type  $B$ , with output context  $\Gamma'$
- $\Gamma \vdash r \Rightarrow A <: B \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ ,  $s$  copies of subtype  $A$  are needed to produce  $r$  copies of supertype  $B$ , with output context  $\Gamma'$ .
- $\Gamma \vdash r \Rightarrow \hat{a} <: A \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ , instantiate the existential variable  $\hat{a}$  to a subtype of  $!_r A$ , lacking  $s$ , returning context  $\Gamma'$
- $\Gamma \vdash r \Rightarrow A <: \hat{a} \Rightarrow s \dashv \Gamma'$  – in context  $\Gamma$ , instantiate the existential variable  $\hat{a}$ , so that  $A <: !_r \hat{a}$ , lacking  $s$ , returning context  $\Gamma$

# Variables and annotations

$$\frac{\Gamma(x) = A \quad 0 \Gamma + x \vdash \langle 0 \Gamma + x \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{\text{c}} x \Rightarrow B \dashv \Gamma' + \Gamma}$$

$$\frac{\Gamma(x) = A \quad \Gamma \vdash \langle \Gamma \rangle A \rightsquigarrow B \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} x \Rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \begin{array}{l} 0 \Gamma_1, \square : A \vdash_i e \Rightarrow A' \dashv \Gamma_2 \\ \Gamma_2 \vdash \langle \Gamma_2 \rangle A' \rightsquigarrow B \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i (e : A) \Rightarrow B \dashv \Gamma_3 + \Gamma_1}$$

# Stationary rules

$$\frac{\Gamma, \Delta, a, \Box : A \vdash_i e \Rightarrow A' \dashv \Gamma', a, \Theta}{\Gamma, \Box : \forall\{a\}. A, \Delta \vdash_i e \Rightarrow \forall\{a\}. A \dashv \Gamma'}$$

$$\frac{0\Gamma, 0\Delta, \Box : A \vdash_i e \Rightarrow A' \dashv \Gamma'}{\Gamma, \Box : !_r A, \Delta \vdash_i e \Rightarrow !_r A \dashv r\Gamma' + \Gamma, \Delta}$$

# Application

$$\frac{\Gamma, \Box e_2 \vdash_i e_1 \Rightarrow A \rightarrow B \dashv \Gamma'}{\Gamma \vdash_i e_1 e_2 \Rightarrow B \dashv \Gamma'}$$

# Functions

$$\frac{\Gamma, \Delta, 0x : A, \square : B \vdash_i e \Rightarrow B' \dashv \Gamma_1, rx : A, \Theta \quad \Gamma_1 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma, \square : A \rightarrow B, \Delta \vdash_i \lambda x. e \Rightarrow A \rightarrow B \dashv \Gamma_2}$$

$$\frac{\begin{array}{l} \Gamma \vdash_i e' \Rightarrow A \dashv \Gamma_1 \quad \Gamma_1, \Delta, 0x : A \vdash_i e \Rightarrow B \dashv \Gamma_2, rx : A, \Theta \\ \Gamma_2 \vdash A \Leftarrow \text{Type}_r \dashv \Gamma_3 \end{array}}{\Gamma, \square e', \Delta \vdash_i \lambda x. e \Rightarrow A \rightarrow B \dashv \Gamma_3}$$

$$\frac{\Gamma [\hat{a}_1, \hat{a}_2, \hat{a} = \hat{a}_1 \rightarrow \hat{a}_2], \Delta, 0x : \hat{a}_1, \square : \hat{a}_2 \vdash_i e \Rightarrow A \dashv \Gamma_1, rx : \hat{a}_1, \Theta \quad \Gamma_1 \vdash \hat{a}_1 \Leftarrow \text{Type}_r \dashv \Gamma_2}{\Gamma [\hat{a}], \square : \hat{a}, \Delta \vdash_i \lambda x. e \Rightarrow \hat{a} \dashv \Gamma_2}$$

$$\frac{\begin{array}{l} \Gamma, \blacktriangleright_{\hat{a}}, \hat{a}, 0x : \hat{a}, \square \vdash_i e \Rightarrow A \dashv \Gamma_1, rx : \hat{a}, \Gamma_4 \\ \Gamma_1 \vdash \hat{a} \Leftarrow \text{Type}_r \dashv \Gamma_2, \blacktriangleright_{\hat{a}}, \Gamma_3 \\ B = \forall \{ \text{unsolved}(\Gamma_3) \}. \langle \Gamma_3 \rangle (\hat{a} \rightarrow \forall \{ \text{unsolved}(\Gamma_4) \}. \langle \Gamma_4 \rangle A) \end{array}}{\Gamma, \square, \Delta \vdash_i \lambda x. e \Rightarrow B \dashv \Gamma_2}$$

# Matching

$$\overline{\Gamma, \square, \Delta \vdash A \rightsquigarrow A \dashv \Gamma, \Delta}$$

$$\frac{\Gamma, \Delta \vdash 1 \Rightarrow A <: B \Rightarrow r \dashv \Gamma'}{\Gamma, \square : B, \Delta \vdash A \rightsquigarrow B \dashv r \Gamma'}$$

$$\frac{\Gamma, \square : A \vdash_i e \Rightarrow A' \dashv \Gamma_1 \quad \Gamma_1, \Delta \vdash B \rightsquigarrow B' \dashv \Gamma_2}{\Gamma, \square e, \Delta \vdash A \rightarrow B \rightsquigarrow A \rightarrow B' \dashv \Gamma_2}$$

# Matching

$$\frac{\Gamma, \Box e, \Delta \vdash A \rightsquigarrow B \dashv \vdash \Gamma'}{\Gamma, \Box e, \Delta \vdash !_r A \rightsquigarrow B \dashv \vdash \Gamma'}$$

$$\frac{\Gamma, \Delta, \Box e, \hat{a} \vdash A[a := \hat{a}] \rightsquigarrow B \dashv \vdash \Gamma'}{\Gamma, \Box e, \Delta \vdash \forall\{a\}. A \rightsquigarrow B \dashv \vdash \Gamma'}$$

$$\frac{\Gamma \vdash B \Rightarrow \text{Type}_r \dashv \vdash \Gamma_1 \quad \Gamma_1, \Delta \vdash A[a := B] \rightsquigarrow C \dashv \vdash \Gamma'}{\Gamma, \Box B, \Delta \vdash \forall a. A \rightsquigarrow C \dashv \vdash \Gamma'}$$

# Subtyping

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Unit} <: \text{Unit} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{}{\Gamma \vdash r \Rightarrow \text{Empty} <: \text{Empty} \Rightarrow 1 \dashv \Gamma}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow B_1 <: B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \otimes B_1 <: A_2 \otimes B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash r \Rightarrow A_1 <: A_2 \Rightarrow s_A \dashv \Gamma_1 \quad \Gamma_1 \vdash r \Rightarrow B_1 <: B_2 \Rightarrow s_B \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \oplus B_1 <: A_2 \oplus B_2 \Rightarrow s_A \sqcap s_B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash 1 \Rightarrow A_2 <: A_1 \Rightarrow 1 \dashv \Gamma_1 \quad \Gamma_1 \vdash 1 \Rightarrow B_1 <: B_2 \Rightarrow 1 \dashv \Gamma_2}{\Gamma \vdash r \Rightarrow A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 \Rightarrow r \dashv \Gamma_2}$$



# Subtyping

$$\overline{\Gamma [\hat{a}] \vdash r \Rightarrow \hat{a} <: \hat{a} \Rightarrow r \dashv \Gamma [\hat{a}]}$$

$$\overline{\Gamma [a : \text{Type}_s] \vdash r \Rightarrow a <: a \Rightarrow r/s \dashv \Gamma [a : \text{Type}_s]}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \text{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow A <: \forall \{a : \text{Type}_s\}. B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, \blacktriangleright_{\hat{a}}, \hat{a} \vdash r \Rightarrow A[a := !_s \hat{a}] <: B \Rightarrow t \dashv \Gamma', \blacktriangleright_{\hat{a}}, \Theta}{\Gamma \vdash r \Rightarrow \forall \{a : \text{Type}_s\}. A <: B \Rightarrow t \dashv \Gamma'}$$

$$\frac{\Gamma, a : \text{Type}_s \vdash r \Rightarrow A <: B \Rightarrow t \dashv \Gamma', a : \text{Type}_s, \Theta}{\Gamma \vdash r \Rightarrow \forall a : \text{Type}_s. A <: \forall a : \text{Type}_s. B \Rightarrow t \dashv \Gamma'}$$

# Subtyping

$$\frac{\Gamma \vdash r \cdot s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow A <: !_s B \Rightarrow t \dashv \Gamma'}$$

$$\frac{B \neq !_s B' \quad \Gamma \vdash r/s \Rightarrow A <: B \Rightarrow t \dashv \Gamma'}{\Gamma \vdash r \Rightarrow !_s A <: B \Rightarrow t \dashv \Gamma'}$$