T-Axi

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Quantities

Quantities:
$r ::= 0 \mid 1 \mid \omega$

ω is the default quantity, so when there's nothing to indicate quantity, it means it's ω.

Quantities are ordered:

$$\omega \leq r$$
$$0 \leq 0$$
$$1 \leq 1$$

Operations on quantities

Addition of quantities:

$$
\begin{aligned}
0 + r &= r \\
\omega + r &= \omega \\
1 + 0 &= 1 \\
1 + 1 &= \omega \\
1 + \omega &= \omega
\end{aligned}
$$

Multiplication:

$$
\begin{aligned}
0 \cdot r &= 0 \\
1 \cdot r &= r \\
\omega \cdot 0 &= 0 \\
\omega \cdot 1 &= \omega \\
\omega \cdot \omega &= \omega
\end{aligned}
$$

Types

Types:

$$A, B ::= r\ A \to B \mid !_r A \mid A \otimes B \mid A \oplus B \mid \mathbf{1} \mid \mathbf{0}$$

Terms

Terms:

e ::=

$x \mid \lambda_r\, x : A.\, e \mid e_1\ e_2 \mid$

$\texttt{box}_r\ e \mid \texttt{letbox}\ x = e_1\ \texttt{in}\ e_2$

$\texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \mid$

$\texttt{choose}\ p$

$\texttt{choose}\ p$ is a noncomputable terms, whereas all others are computable.

Propositions

Propositions:

$P, Q ::=$

$\top \mid \bot \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid$

$\forall x : A.\, P \mid \exists x : A.\, P \mid$

$e_1 =_A e_2$

Notations:

$\neg P$ stands for $P \Rightarrow \bot$

$P \Leftrightarrow Q$ stands for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

h | **trivial** | **absurd** p
assume $h : P$ **in** q | **apply** $p_1\ p_2$ |
both $p_1\ p_2$ | **and-left** p | **and-right** p |
or-left p | **or-right** p | **cases** $p_1\ p_2\ p_3$ |
lemma $h : P$ **by** p **in** q | **proving** P **by** p |
pick-any $x : A$ **in** e | **instantiate** p **with** e |
witness e **such that** p | **pick-witness** $x\ h$ **for** p_1 **in** p_2 |
refl e | **rewrite** p_1 **in** P_2 | **funext** $x : A$ **in** p
by-contradiction $h : P$ **in** q | **choose-spec** p

Contexts

Contexts:

$\Gamma ::= \cdot \mid \Gamma, r\,x : A \mid \Gamma, r\,x : A := e \mid \Gamma, h : P$

Operations on contexts:

$\Gamma_1 + \Gamma_2$

$r\,\Gamma$

$\Gamma_1 \leq \Gamma_2$

$|\Gamma|$

Context addition

$$
\begin{gathered}
\cdot + \cdot = \cdot \\
(\Gamma_1, r_1\, x : A) + (\Gamma_2, r_2\, x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : A \\
(\Gamma_1, r_1\, x : A := e) + (\Gamma_2, r_2\, x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : \\
A := e \\
(\Gamma_1, p : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P
\end{gathered}
$$

Context scaling

$$
\begin{aligned}
s \cdot &= \cdot \\
s\,(\Gamma, r\,x : A) &= s\,\Gamma, (s \cdot r)\,x : A \\
s\,(\Gamma, r\,x : A := e) &= s\,\Gamma, (s \cdot r)\,x : A := e \\
s\,(\Gamma, h : P) &= s\,\Gamma, h : P
\end{aligned}
$$

Context ordering

$$\frac{}{\cdot \leq \cdot}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A \leq \Gamma_2, r_2\, x : A}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A := e \leq \Gamma_2, r_2\, x : A := e}$$

$$\frac{\Gamma_1 \leq \Gamma_2}{\Gamma_1, h : P \leq \Gamma_2, h : P}$$

Cartesianization

Cartesianization turns a context into a traditional context that doesn't mention any quantities.

$$
\begin{aligned}
|\cdot| &= \cdot \\
|\Gamma, r\,x : A| &= |\Gamma|, x : A \\
|\Gamma, r\,x : A := e| &= |\Gamma|, x : A := e \\
|\Gamma, h : P| &= |\Gamma|, h : P
\end{aligned}
$$

Judgements

Valid type judgement: $\Gamma \vdash A \; \mathtt{type}$
Typing judgement: $\Gamma \vdash e : A$

Well-formed proposition judgement: $\Gamma \vdash P \; \mathtt{prop}$
Proof judgement: $\Gamma \vdash p : P$

Functions

$$\frac{\Gamma, r\,x : A \vdash e : B}{\Gamma \vdash \lambda_r\,x : A.\,e : r\ A \to B}$$

$$\frac{\Gamma \leq \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash e_1 : r\ A \to B \quad \Gamma_2 \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

Box

$$\frac{\Gamma \leq r\,\Gamma' \quad \Gamma' \vdash e : A}{\Gamma \vdash \texttt{box}_r\ e : !_r\,A}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : !_r\,A \quad \Gamma_2, r\,x : A \vdash e_2 : B}{\Gamma \vdash \texttt{letbox}\ x = e_1\ \texttt{in}\ e_2 : B}$$

Let

$$\frac{\Gamma \le \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash e_1 : A \quad \Gamma_2, r\,x : A \vdash e_2 : B}{\Gamma \vdash \mathtt{let}_r\ x = e_1\ \mathtt{in}\ e_2 : B}$$

Well-formed propositions

$$\frac{\Gamma, x : A \vdash P \ \texttt{prop}}{\Gamma \vdash \forall x : A.\, P \ \texttt{prop}} \qquad \frac{\Gamma, x : A \vdash P \ \texttt{prop}}{\Gamma \vdash \exists x : A.\, P \ \texttt{prop}}$$

$$\frac{\Gamma \vdash A \ \texttt{type} \quad \Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 =_A e_2 \ \texttt{prop}}$$

Substitution

The notation is $P\,[x := e]$ for substitution in propositions.

Proofs

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \textbf{trivial} : \top}$$

$$\frac{\Gamma\ \texttt{cartesian} \quad (h : P) \in \Gamma}{\Gamma \vdash \textbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{assume}\ h : P\ \textbf{in}\ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{apply}\ q\ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{lemma}\ h : P\ \textbf{by}\ p\ \textbf{in}\ q : Q}$$

Proofs

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \textbf{pick-any } x : A \textbf{ in } p : \forall x : A.\, P}$$

$$\frac{\Gamma \vdash p : \forall x : A.\, P \quad \Gamma \vdash e : A}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e : P\,[x := e]}$$

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash p : P\,[x := e]}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p : \exists x : A.\, P}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q \texttt{ prop} \quad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \textbf{pick-witness } x\ h \textbf{ for } p \textbf{ in } q : Q}$$

Axiom of Choice

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P}{\Gamma \vdash \texttt{choose}\ p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P}{\Gamma \vdash \textbf{choose-spec}\ p : P\,[x := \texttt{choose}\ p]}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q\ \texttt{prop} \quad \Gamma, x : A := \texttt{choose}\ p, h : P \vdash q : Q}{\Gamma \vdash \textbf{choose-witness}\ x\ h\ \textbf{for}\ p\ \textbf{in}\ q : Q}$$

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P \quad |\Gamma| \vdash B\ \texttt{type} \quad \Gamma, x : A := \texttt{choose}\ p, h : P \vdash e : B}{\Gamma \vdash \texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e : B}$$