Poor Man's Axi: Algorithmic Version

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Values (cbv)

$$\frac{}{\lambda x.\, e\ \texttt{value}}$$

$$\frac{v_1\ \texttt{value} \quad v_2\ \texttt{value}}{(v_1, v_2)\ \texttt{value}}$$

$$\frac{v\ \texttt{value}}{\texttt{inl}\ v\ \texttt{value}} \qquad \frac{v\ \texttt{value}}{\texttt{inr}\ v\ \texttt{value}}$$

$$\frac{}{\texttt{unit value}}$$

Values are the final results of computation. Note that a function is a value whether or not its body is. Other values are pairs of values, values injected into a sum on the left or right, and unit.

Big-step semantics (cbv)

$$\frac{}{\lambda x.\, e \Downarrow \lambda x.\, e} \qquad \frac{e_1 \Downarrow \lambda x.\, e \quad e_2 \Downarrow v \quad e\,[x := v] \Downarrow v'}{e_1\ e_2 \Downarrow v'}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \qquad \frac{e \Downarrow (v_1, v_2)}{\mathtt{outl}\ e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\mathtt{outr}\ e \Downarrow v_2}$$

$$\frac{e \Downarrow v}{\mathtt{inl}\ e \Downarrow \mathtt{inl}\ v} \qquad \frac{e \Downarrow v}{\mathtt{inr}\ e \Downarrow \mathtt{inr}\ v}$$

$$\frac{e \Downarrow \mathtt{inl}\ v \quad f\ v \Downarrow v'}{\mathtt{case}\ e\ \mathtt{of}\ (f, g) \Downarrow v'} \qquad \frac{e \Downarrow \mathtt{inr}\ v \quad g\ v \Downarrow v'}{\mathtt{case}\ e\ \mathtt{of}\ (f, g) \Downarrow v'}$$

$$\frac{}{\mathtt{unit} \Downarrow \mathtt{unit}}$$

Small-step semantics (cbv) – basic rules

$$\frac{v\ \texttt{value}}{(\lambda x.\, e)\ v \longrightarrow e\,[x := v]}$$

$$\frac{v_1\ \texttt{value} \quad v_2\ \texttt{value}}{\texttt{outl}\ (v_1, v_2) \longrightarrow v_1} \qquad \frac{v_1\ \texttt{value} \quad v_2\ \texttt{value}}{\texttt{outr}\ (v_1, v_2) \longrightarrow v_2}$$

$$\frac{v\ \texttt{value}}{\texttt{case}\ (\texttt{inl}\ v)\ \texttt{of}\ (f, g) \longrightarrow f\ v}$$

$$\frac{v\ \texttt{value}}{\texttt{case}\ (\texttt{inr}\ v)\ \texttt{of}\ (f, g) \longrightarrow g\ v}$$

Small-step semantics (cbv) – boring rules

$$\frac{e_1 \longrightarrow e_1'}{e_1\ e_2 \longrightarrow e_1'\ e_2} \qquad \frac{v_1\ \texttt{value} \quad e_2 \longrightarrow e_2'}{v_1\ e_2 \longrightarrow v_1\ e_2'}$$

$$\frac{e_1 \longrightarrow e_1'}{(e_1, e_2) \longrightarrow (e_1', e_2)} \qquad \frac{v_1\ \texttt{value} \quad e_2 \longrightarrow e_2'}{(v_1, e_2) \longrightarrow (v_1, e_2')}$$

$$\frac{e \longrightarrow e'}{\texttt{outl}\ e \longrightarrow \texttt{outl}\ e'} \qquad \frac{e \longrightarrow e'}{\texttt{outr}\ e \longrightarrow \texttt{outr}\ e'}$$

$$\frac{e \longrightarrow e'}{\texttt{inl}\ e \longrightarrow \texttt{inl}\ e'} \qquad \frac{e \longrightarrow e'}{\texttt{inr}\ e \longrightarrow \texttt{inr}\ e'}$$

$$\frac{e \longrightarrow e'}{\texttt{case}\ e\ \texttt{of}\ (f, g) \longrightarrow \texttt{case}\ e'\ \texttt{of}\ (f, g)}$$

Example semantics – explanations

$e \Downarrow v$ should be read "term e evalutes to value v". It describes computation in a coarse-grained manner, telling us what is the result of evaluating each term. If $e \Downarrow v$, then v `value`.

$e \longrightarrow e'$ should be read "term e reduces to term e'". It describes computation in a fine-grained manner, telling us what can happen in a single computation step. To actually describe computation fully in this style, we need to take the transitive closure of this relation, written $e \longrightarrow^* v$.

Both presented semantics are call-by-value, i.e. we evaluate a function's argument before performing a substitution. They are equivalent, i.e. $e \Downarrow v$ if and only if $e \longrightarrow^* v$

Weak head normal forms

$$\overline{x \ \mathtt{whnf}}$$

$$\overline{\lambda x.\, e \ \mathtt{whnf}}$$

$$\overline{(e_1, e_2) \ \mathtt{whnf}}$$

$$\overline{\mathtt{inl}\ e \ \mathtt{whnf}} \qquad \overline{\mathtt{inr}\ e \ \mathtt{whnf}}$$

$$\overline{\mathtt{unit} \ \mathtt{whnf}}$$

Weak head normal forms grammar

Weak head normal forms:

$e ::=$

$\quad n \mid \lambda x.\, e \mid$

$\quad (e_1, e_2) \mid$

$\quad \texttt{inl}\ e \mid \texttt{inr}\ e \mid$

$\quad \texttt{unit} \mid \texttt{elim}_{\mathbf{0}}\ e$

Neutral forms:

$n ::=$

$\quad x \mid n\ e \mid$

$\quad \texttt{outl}\ n \mid \texttt{outr}\ n \mid$

$\quad \texttt{case}\ n\ \texttt{of}\ (e_1, e_2) \mid$

Whnf reduction – basic rules

$$\frac{}{(\lambda x.\, e_1)\ e_2 \longrightarrow_{\texttt{whnf}} e_1\,[x := e_2]}$$

$$\frac{}{\texttt{outl}\ (e_1, e_2) \longrightarrow_{\texttt{whnf}} e_1} \qquad \frac{}{\texttt{outr}\ (e_1, e_2) \longrightarrow_{\texttt{whnf}} e_2}$$

$$\frac{}{\texttt{case}\ (\texttt{inl}\ e)\ \texttt{of}\ (f, g) \longrightarrow_{\texttt{whnf}} f\ e}$$

$$\frac{}{\texttt{case}\ (\texttt{inr}\ e)\ \texttt{of}\ (f, g) \longrightarrow_{\texttt{whnf}} g\ e}$$

Whnf reduction – boring rules

$$\frac{e_1 \longrightarrow_{\mathtt{whnf}} e_1'}{e_1\ e_2 \longrightarrow_{\mathtt{whnf}} e_1'\ e_2} \qquad \frac{e_1\ \mathtt{whnf} \qquad e_2 \longrightarrow_{\mathtt{whnf}} e_2'}{e_1\ e_2 \longrightarrow_{\mathtt{whnf}} e_1\ e_2'}$$

$$\frac{e \longrightarrow_{\mathtt{whnf}} e'}{\mathtt{outl}\ e \longrightarrow_{\mathtt{whnf}} \mathtt{outl}\ e'} \qquad \frac{e \longrightarrow_{\mathtt{whnf}} e'}{\mathtt{outr}\ e \longrightarrow_{\mathtt{whnf}} \mathtt{outr}\ e'}$$

$$\frac{e \longrightarrow_{\mathtt{whnf}} e'}{\mathtt{case}\ e\ \mathtt{of}\ (f, g) \longrightarrow_{\mathtt{whnf}} \mathtt{case}\ e'\ \mathtt{of}\ (f, g)}$$

Algorithmic computational equality 0

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \equiv x \Rightarrow A}\textsc{Var}$$

$$\frac{\Gamma \vdash e \equiv e' \Rightarrow B \qquad A = B}{\Gamma \vdash e \equiv e' \Leftarrow A}\textsc{Sub}$$

$$\frac{e_1 \longrightarrow_{\mathtt{whnf}} e_1' \qquad e_2 \longrightarrow_{\mathtt{whnf}} e_2' \qquad \Gamma \vdash e_1' \equiv e_2' \Leftarrow A}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\mathtt{whnf}} A}$$

$$\frac{e_1 \longrightarrow_{\mathtt{whnf}} e_1' \qquad e_2 \longrightarrow_{\mathtt{whnf}} e_2' \qquad \Gamma \vdash e_1' \equiv e_2' \Rightarrow A}{\Gamma \vdash e_1 \equiv e_2 \Rightarrow_{\mathtt{whnf}} A}$$

Algorithmic computational equality 1

$$\frac{\Gamma, x : A \vdash e_1 \equiv e_2 \Leftarrow_{\texttt{whnf}} B}{\Gamma \vdash \lambda x.\, e_1 \equiv \lambda x.\, e_2 \Leftarrow A \to B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \to B \quad \Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\texttt{whnf}} A}{\Gamma \vdash n_1\ e_1 \equiv n_2\ e_2 \Rightarrow B}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 \Leftarrow_{\texttt{whnf}} A \quad \Gamma \vdash b_1 \equiv b_2 \Leftarrow_{\texttt{whnf}} B}{\Gamma \vdash (a_1, b_1) \equiv (a_2, b_2) \Leftarrow A \times B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \times B}{\Gamma \vdash \texttt{outl}\ n_1 \equiv \texttt{outl}\ n_2 \Rightarrow A} \qquad \frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \times B}{\Gamma \vdash \texttt{outr}\ n_1 \equiv \texttt{outr}\ n_2 \Rightarrow B}$$

Algorithmic computational equality 2

$$\frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\texttt{whnf}} A}{\Gamma \vdash \texttt{inl}\ e_1 \equiv \texttt{inl}\ e_2 \Leftarrow A + B} \qquad \frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\texttt{whnf}} B}{\Gamma \vdash \texttt{inr}\ e_1 \equiv \texttt{inr}\ e_2 \Leftarrow A + B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A + B \qquad \begin{array}{c}\Gamma \vdash f_1 \equiv f_2 \Leftarrow_{\texttt{whnf}} A \to C \\ \Gamma \vdash g_1 \equiv g_2 \Leftarrow_{\texttt{whnf}} B \to C\end{array}}{\Gamma \vdash \texttt{case}\ n_1\ \texttt{of}\ (f_1, g_1) \equiv \texttt{case}\ n_2\ \texttt{of}\ (f_2, g_2) \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \texttt{unit} \equiv \texttt{unit} \Leftarrow \mathbf{1}} \qquad \frac{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \mathbf{0}}{\Gamma \vdash \texttt{elim}_{\mathbf{0}}\ [e] \equiv \texttt{elim}_{\mathbf{0}}\ [e'] \Leftarrow A}$$

Uniqueness rules (asymmetric, contraction-like)

$$\frac{\Gamma \vdash f \Leftarrow A \to B}{\Gamma \vdash f \equiv \lambda x.\, f\ x \Leftarrow A \to B}\textsc{Fun-Uniq}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times B}{\Gamma \vdash e \equiv (\mathtt{outl}\ e, \mathtt{outr}\ e) \Leftarrow A \times B}\textsc{Prod-Uniq}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{1}}{\Gamma \vdash e \equiv \mathtt{unit} \Leftarrow \mathbf{1}}\textsc{Unit-Uniq}$$