

# Poor Man's Axi: Algorithmic Version

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# Syntax

# Values (cbv)

$$\overline{\lambda x. e \text{ value}}$$

$$\frac{v_1 \text{ value} \quad v_2 \text{ value}}{(v_1, v_2) \text{ value}}$$

$$\frac{v \text{ value}}{\text{inl } v \text{ value}} \quad \frac{v \text{ value}}{\text{inr } v \text{ value}}$$

$$\overline{\text{unit value}}$$

Values are the final results of computation. Note that a function is a value whether or not its body is. Other values are pairs of values, values injected into a sum on the left or right, and **unit**.

# Big-step semantics (cbv)

$$\frac{}{\lambda x. e \Downarrow \lambda x. e} \quad \frac{e_1 \Downarrow \lambda x. e \quad e_2 \Downarrow v \quad e[x := v] \Downarrow v'}{e_1 \ e_2 \Downarrow v'}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \quad \frac{e \Downarrow (v_1, v_2)}{\text{outl } e \Downarrow v_1} \quad \frac{e \Downarrow (v_1, v_2)}{\text{outr } e \Downarrow v_2}$$

$$\frac{e \Downarrow v}{\text{inl } e \Downarrow \text{inl } v} \quad \frac{e \Downarrow v}{\text{inr } e \Downarrow \text{inr } v}$$

$$\frac{e \Downarrow \text{inl } v \quad f \ v \Downarrow v'}{\text{case } e \text{ of } (f, g) \Downarrow v'} \quad \frac{e \Downarrow \text{inr } v \quad g \ v \Downarrow v'}{\text{case } e \text{ of } (f, g) \Downarrow v'}$$

$$\frac{}{\text{unit} \Downarrow \text{unit}}$$

# Small-step semantics (cbv) – basic rules

$$\frac{v \text{ value}}{(\lambda x. e) \ v \longrightarrow e[x := v]}$$

$$\frac{v_1 \text{ value} \quad v_2 \text{ value}}{\text{outl } (v_1, v_2) \longrightarrow v_1} \quad \frac{v_1 \text{ value} \quad v_2 \text{ value}}{\text{outr } (v_1, v_2) \longrightarrow v_2}$$

$$\frac{v \text{ value}}{\text{case } (\text{inl } v) \text{ of } (f, g) \longrightarrow f \ v}$$

$$\frac{v \text{ value}}{\text{case } (\text{inr } v) \text{ of } (f, g) \longrightarrow g \ v}$$

# Small-step semantics (cbv) – boring rules

$$\frac{e_1 \longrightarrow e'_1}{e_1 \ e_2 \longrightarrow e'_1 \ e_2} \qquad \frac{v_1 \text{ value} \quad e_2 \longrightarrow e'_2}{v_1 \ e_2 \longrightarrow v_1 \ e'_2}$$

$$\frac{e_1 \longrightarrow e'_1}{(e_1, e_2) \longrightarrow (e'_1, e_2)} \qquad \frac{v_1 \text{ value} \quad e_2 \longrightarrow e'_2}{(v_1, e_2) \longrightarrow (v_1, e'_2)}$$

$$\frac{e \longrightarrow e'}{\text{outl } e \longrightarrow \text{outl } e'} \qquad \frac{e \longrightarrow e'}{\text{outr } e \longrightarrow \text{outr } e'}$$

$$\frac{e \longrightarrow e'}{\text{inl } e \longrightarrow \text{inl } e'} \qquad \frac{e \longrightarrow e'}{\text{inr } e \longrightarrow \text{inr } e'}$$

$$\frac{e \longrightarrow e'}{\text{case } e \text{ of } (f, g) \longrightarrow \text{case } e' \text{ of } (f, g)}$$

# Algorithmic typing

# Weak head normal forms

$$\overline{n \text{ whnf}}$$

$$\overline{\lambda x. e \text{ whnf}}$$

$$\overline{(e_1, e_2) \text{ whnf}}$$

$$\overline{\text{inl } e \text{ whnf}}$$

$$\overline{\text{inr } e \text{ whnf}}$$

$$\overline{\text{unit whnf}}$$

$$\overline{\text{elim}_0 e \text{ whnf}}$$



# Weak head normal forms grammar

Weak head normal forms:

$e ::=$

$n \mid \lambda x. e \mid$   
 $(e_1, e_2) \mid$   
 $\text{inl } e \mid \text{inr } e \mid$   
 $\text{unit} \mid \text{elim}_0 e$

Neutral forms:

$n ::=$

$x \mid n e \mid$   
 $\text{outl } n \mid \text{outr } n \mid$   
 $\text{case } n \text{ of } (e_1, e_2) \mid$

# Whnf reduction – basic rules

$$\overline{(\lambda x. e_1) e_2 \longrightarrow_{\text{whnf}} e_1 [x := e_2]}$$

$$\overline{\text{outl } (e_1, e_2) \longrightarrow_{\text{whnf}} e_1} \quad \overline{\text{outr } (e_1, e_2) \longrightarrow_{\text{whnf}} e_2}$$

$$\overline{\text{case } (\text{inl } e) \text{ of } (f, g) \longrightarrow_{\text{whnf}} f e}$$

$$\overline{\text{case } (\text{inr } e) \text{ of } (f, g) \longrightarrow_{\text{whnf}} g e}$$

# Whnf reduction – boring rules

$$\frac{e_1 \longrightarrow_{\text{whnf}} e'_1}{e_1 \ e_2 \longrightarrow_{\text{whnf}} e'_1 \ e_2}$$

$$\frac{e_1 \text{ whnf} \quad e_2 \longrightarrow_{\text{whnf}} e'_2}{e_1 \ e_2 \longrightarrow_{\text{whnf}} e_1 \ e'_2}$$

$$\frac{e \longrightarrow_{\text{whnf}} e'}{\text{outl } e \longrightarrow_{\text{whnf}} \text{outl } e'}$$

$$\frac{e \longrightarrow_{\text{whnf}} e'}{\text{outr } e \longrightarrow_{\text{whnf}} \text{outr } e'}$$

$$\frac{e \longrightarrow_{\text{whnf}} e'}{\text{case } e \text{ of } (f, g) \longrightarrow_{\text{whnf}} \text{case } e' \text{ of } (f, g)}$$

# Algorithmic computational equality 0

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \equiv x \Rightarrow A}^{\text{VAR}}$$

$$\frac{\Gamma \vdash e \equiv e' \Rightarrow B \quad A = B}{\Gamma \vdash e \equiv e' \Leftarrow A}^{\text{SUB}}$$

$$\frac{e_1 \longrightarrow_{\text{whnf}} e'_1 \quad e_2 \longrightarrow_{\text{whnf}} e'_2 \quad \Gamma \vdash e'_1 \equiv e'_2 \Leftarrow A}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow_{\text{whnf}} A}$$

$$\frac{e_1 \longrightarrow_{\text{whnf}} e'_1 \quad e_2 \longrightarrow_{\text{whnf}} e'_2 \quad \Gamma \vdash e'_1 \equiv e'_2 \Rightarrow A}{\Gamma \vdash e_1 \equiv e_2 \Rightarrow_{\text{whnf}} A}$$

# Algorithmic computational equality 1

$$\frac{\Gamma, x : A \vdash e_1 \equiv e_2 \leftarrow_{\text{whnf}} B}{\Gamma \vdash \lambda x. e_1 \equiv \lambda x. e_2 \leftarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \rightarrow B \quad \Gamma \vdash e_1 \equiv e_2 \leftarrow_{\text{whnf}} A}{\Gamma \vdash n_1 e_1 \equiv n_2 e_2 \Rightarrow B}$$

$$\frac{\Gamma \vdash a_1 \equiv a_2 \leftarrow_{\text{whnf}} A \quad \Gamma \vdash b_1 \equiv b_2 \leftarrow_{\text{whnf}} B}{\Gamma \vdash (a_1, b_1) \equiv (a_2, b_2) \leftarrow A \times B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \times B}{\Gamma \vdash \text{outl } n_1 \equiv \text{outl } n_2 \Rightarrow A}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A \times B}{\Gamma \vdash \text{outr } n_1 \equiv \text{outr } n_2 \Rightarrow B}$$

# Algorithmic computational equality 2

$$\frac{\Gamma \vdash e_1 \equiv e_2 \leftarrow_{\text{whnf}} A}{\Gamma \vdash \text{inl } e_1 \equiv \text{inl } e_2 \leftarrow A + B} \quad \frac{\Gamma \vdash e_1 \equiv e_2 \leftarrow_{\text{whnf}} B}{\Gamma \vdash \text{inr } e_1 \equiv \text{inr } e_2 \leftarrow A + B}$$

$$\frac{\Gamma \vdash n_1 \equiv n_2 \Rightarrow A + B \quad \begin{array}{l} \Gamma \vdash f_1 \equiv f_2 \leftarrow_{\text{whnf}} A \rightarrow C \\ \Gamma \vdash g_1 \equiv g_2 \leftarrow_{\text{whnf}} B \rightarrow C \end{array}}{\Gamma \vdash \text{case } n_1 \text{ of } (f_1, g_1) \equiv \text{case } n_2 \text{ of } (f_2, g_2) \Rightarrow C}$$

$$\frac{}{\Gamma \vdash \text{unit} \equiv \text{unit} \leftarrow \mathbf{1}} \quad \frac{\Gamma \vdash e_1 \equiv e_2 \leftarrow \mathbf{0}}{\Gamma \vdash \text{elim}_0 e_1 \equiv \text{elim}_0 e_2 \leftarrow A}$$

# Uniqueness rules (asymmetric, contraction-like)

$$\frac{\Gamma \vdash f \Leftarrow A \rightarrow B}{\Gamma \vdash f \equiv \lambda x. f \ x \Leftarrow A \rightarrow B} \text{FUN-UNIQ}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times B}{\Gamma \vdash e \equiv (\text{outl } e, \text{outr } e) \Leftarrow A \times B} \text{PROD-UNIQ}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{1}}{\Gamma \vdash e \equiv \text{unit} \Leftarrow \mathbf{1}} \text{UNIT-UNIQ}$$

# Uniqueness rules (symmetric, prop-like)

$$\frac{\Gamma, x : A \vdash f \ x \equiv g \ x \Leftarrow B}{\Gamma \vdash f \equiv g \Leftarrow A \rightarrow B} \text{FUN-UNIQ-ALT}$$

$$\frac{\begin{array}{l} \Gamma \vdash \text{outl } e_1 \equiv \text{outl } e_2 \Leftarrow A \\ \Gamma \vdash \text{outr } e_1 \equiv \text{outr } e_2 \Leftarrow B \end{array}}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \times B} \text{PROD-UNIQ-ALT}$$

$$\frac{\Gamma \vdash e_1 \Leftarrow \mathbf{1} \quad \Gamma \vdash e_2 \Leftarrow \mathbf{1}}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \mathbf{1}} \text{UNIT-UNIQ-ALT}$$

$$\frac{\Gamma \vdash e_1 \Leftarrow \mathbf{0} \quad \Gamma \vdash e_2 \Leftarrow \mathbf{0}}{\Gamma \vdash e_1 \equiv e_2 \Leftarrow \mathbf{0}} \text{EMPTY-UNIQ-ALT}$$



# Subsumption

$$\frac{\Gamma \vdash e \Rightarrow P' \quad P = P'}{\Gamma \vdash e \Leftarrow P} \text{SUB}$$

# Annotations

$$\frac{\Gamma \vdash e \Leftarrow P}{\Gamma \vdash \mathbf{have} \, P \, \mathbf{from} \, e \Rightarrow P} \text{PROOF-ANNOT}$$

# Assumptions

$$\frac{\Gamma \text{ ctx} \quad P \in \Gamma}{\Gamma \vdash P \Rightarrow P} \text{Ass}$$

# True and False

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbf{true} \Rightarrow \top} \text{TRUE-INTRO}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash e \Leftarrow \perp}{\Gamma \vdash \mathbf{exfalse} \ e \Leftarrow P} \text{FALSE-ELIM}$$

# Implication

$$\frac{\Gamma, P \vdash e \Leftarrow Q}{\Gamma \vdash \text{assume } P \text{ in } e \Leftarrow P \Rightarrow Q} \text{IMPL-INTRO-CHECK}$$

$$\frac{\Gamma, P \vdash e \Rightarrow Q}{\Gamma \vdash \text{assume } P \text{ in } e \Rightarrow P \Rightarrow Q} \text{IMPL-INTRO-INFER}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow P \Rightarrow Q \quad \Gamma \vdash e_2 \Leftarrow P}{\Gamma \vdash \text{modus-ponens } e_1 \ e_2 \Rightarrow Q} \text{IMPL-ELIM}$$

# Negation

$$\frac{\Gamma, P \vdash e \Leftarrow \perp}{\Gamma \vdash \text{suppose-absurd } P \text{ in } e \Rightarrow \neg P} \text{NOT-INTRO}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \neg P \quad \Gamma \vdash e_2 \Leftarrow P}{\Gamma \vdash \text{absurd } e_1 \ e_2 \Rightarrow \perp} \text{NOT-ELIM}$$

# Conjunction

$$\frac{\Gamma \vdash e_1 \Leftarrow P \quad \Gamma \vdash e_2 \Leftarrow Q}{\Gamma \vdash \mathbf{both} \ e_1 \ e_2 \Leftarrow P \wedge Q} \text{AND-INTRO}$$

$$\frac{\Gamma \vdash e \Rightarrow P \wedge Q}{\Gamma \vdash \mathbf{left-and} \ e \Rightarrow P} \text{AND-ELIM-L}$$

$$\frac{\Gamma \vdash e \Rightarrow P \wedge Q}{\Gamma \vdash \mathbf{right-and} \ e \Rightarrow Q} \text{AND-ELIM-R}$$

# Biconditional

$$\frac{\Gamma \vdash e_1 \Leftarrow P \Rightarrow Q \quad \Gamma \vdash e_2 \Leftarrow Q \Rightarrow P}{\Gamma \vdash \mathbf{equivalence} \ e_1 \ e_2 \Leftarrow P \Leftrightarrow Q} \text{IFF-INTRO}$$

$$\frac{\Gamma \vdash e \Rightarrow P \Leftrightarrow Q}{\Gamma \vdash \mathbf{left-iff} \ e \Rightarrow P \Rightarrow Q} \text{IFF-ELIM-L}$$

$$\frac{\Gamma \vdash e \Rightarrow P \Leftrightarrow Q}{\Gamma \vdash \mathbf{right-iff} \ e \Rightarrow Q \Rightarrow P} \text{IFF-ELIM-R}$$



# Disjunction – checking

$$\frac{\Gamma \vdash Q \text{ prop} \quad \Gamma \vdash e \Leftarrow P}{\Gamma \vdash \text{left-either } Q \ e \Leftarrow P \vee Q} \text{OR-INTRO-L}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash e \Leftarrow Q}{\Gamma \vdash \text{right-either } P \ e \Leftarrow P \vee Q} \text{OR-INTRO-R}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow P \vee Q \quad \Gamma \vdash e_2 \Leftarrow P \Rightarrow R \quad \Gamma \vdash e_3 \Leftarrow Q \Rightarrow R}{\Gamma \vdash \text{constructive-dilemma } e_1 \ e_2 \ e_3 \Leftarrow R} \text{OR-ELIM}$$

# Disjunction – inference

$$\frac{\Gamma \vdash Q \text{ prop} \quad \Gamma \vdash e \Rightarrow P}{\Gamma \vdash \text{left-either } Q \ e \Rightarrow P \vee Q} \text{OR-INTRO-L-INFER}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash e \Rightarrow Q}{\Gamma \vdash \text{right-either } P \ e \Rightarrow P \vee Q} \text{OR-INTRO-R-INFER}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow P \vee Q \quad \Gamma \vdash e_2 \Rightarrow P \Rightarrow R \quad \Gamma \vdash e_3 \Rightarrow Q \Rightarrow R}{\Gamma \vdash \text{constructive-dilemma } e_1 \ e_2 \ e_3 \Rightarrow R} \text{OR-ELIM}$$

# Double Negation Elimination

$$\frac{\Gamma \vdash e \Leftarrow \neg\neg P}{\Gamma \vdash \text{double-negation } e \Leftarrow P} \text{CLASSIC-CHECK}$$

$$\frac{\Gamma \vdash e \Rightarrow \neg\neg P}{\Gamma \vdash \text{double-negation } e \Rightarrow P} \text{CLASSIC-INFER}$$

# Cut rule

$$\frac{\Gamma \vdash e_1 \Rightarrow P \quad \Gamma, P \vdash e_2 \Leftarrow Q}{\Gamma \vdash e_1; e_2 \Leftarrow Q} \text{CUT}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow P \quad \Gamma, P \vdash e_2 \Rightarrow Q}{\Gamma \vdash e_1; e_2 \Rightarrow Q} \text{CUT-INFERR}$$

# Universal quantifier

$$\frac{\Gamma, y : A \vdash e \Leftarrow P[x := y]}{\Gamma \vdash \text{pick-any } y \text{ in } e \Leftarrow \forall x : A. P} \text{FORALL-INTRO}$$

$$\frac{\Gamma, x : A \vdash e \Rightarrow P}{\Gamma \vdash \text{pick-any } x : A \text{ in } e \Rightarrow \forall x : A. P} \text{FORALL-INTRO-INFER}$$

$$\frac{\Gamma \vdash e \Rightarrow \forall x : A. P \quad \Gamma \vdash t \Leftarrow A}{\Gamma \vdash \text{specialize } e \text{ with } t \Rightarrow P[x := t]} \text{FORALL-ELIM}$$

# Existential quantifier

$$\frac{\Gamma \vdash t \Leftarrow A \quad \Gamma \vdash e \Leftarrow P[x := t]}{\Gamma \vdash \text{exists } t \text{ such that } e \Leftarrow \exists x : A. P} \text{EXISTS-INTRO}$$

$$\frac{\Gamma \vdash R \text{ prop} \quad \Gamma \vdash e_1 \Rightarrow \exists x : A. P \quad \Gamma, y : A, P[x := y] \vdash e_2 \Rightarrow R}{\Gamma \vdash \text{pick-witness } y \text{ for } e_1 \text{ in } e_2 \Rightarrow R} \text{EXISTS-ELIM}$$

$$\frac{\Gamma \vdash R \text{ prop} \quad \Gamma \vdash e_1 \Rightarrow \exists x : A. P \quad \Gamma, y : A, P[x := y] \vdash e_2 \Leftarrow R}{\Gamma \vdash \text{pick-witness } y \text{ for } e_1 \text{ in } e_2 \Leftarrow R} \text{EXISTS-ELIM}$$

# Conversion rule – TODO

$$\frac{\Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash t_1 \equiv t_2 : A \quad \Gamma \vdash e : P[x := t_1]}{\Gamma \vdash e : P[x := t_2]} \text{CONV}$$

# Equality introduction and elimination

$$\frac{\Gamma \vdash t \Leftarrow A}{\Gamma \vdash \mathbf{refl} \ t \Leftarrow t =_A t} \text{EQ-INTRO}$$

$$\frac{\Gamma \vdash t \Rightarrow A}{\Gamma \vdash \mathbf{refl} \ t \Rightarrow t =_A t} \text{EQ-INTRO-INFER}$$

$$\frac{\Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash e \Rightarrow t_1 =_A t_2 \quad \Gamma \vdash e' \Leftarrow P[x := t_1]}{\Gamma \vdash \mathbf{rewrite} \ e \text{ in } e' \Rightarrow P[x := t_2]} \text{EQ-ELIM}$$



# Experimenting with **refl**

$$\frac{\Gamma \vdash t_1 \equiv t_2 \Leftarrow A}{\Gamma \vdash \mathbf{refl} \Leftarrow t_1 =_A t_2} \text{EQ-INTRO-ALT}$$

# Function extensionality

$$\frac{\Gamma \vdash e \Leftarrow \forall x : A. f \ x =_B g \ x}{\Gamma \vdash \mathbf{funext} \ e \Leftarrow f =_{A \rightarrow B} g} \text{FUNEXT}$$

# Reasoning by cases (for sums)

$$\frac{\Gamma, x : A + B \vdash P \text{ prop} \quad \Gamma \vdash t \Rightarrow A + B \quad \begin{array}{l} \Gamma, a : A \vdash e_1 \Leftarrow P[x := \text{inl } a] \\ \Gamma, b : B \vdash e_2 \Leftarrow P[x := \text{inr } b] \end{array}}{\Gamma \vdash \text{case } t \text{ of } (\text{inl } a \rightarrow e_1, \text{inr } b \rightarrow e_2) \Rightarrow P[x := t]}$$