Quantities

Quantities:
$r, s, q ::= 0 \mid 1 \mid ? \mid + \mid *$

Quantities – explanation

- 0 means a resource has been used up and is no longer available.
- 1 means a resource must be used exactly once.
- ? (pronounced "few") means a resource must be used at most once.
- $+$ (pronounced "many") means a resource must be used at least once.
- $*$ (pronounced "any") means no restrictions on usage.

Subusage ordering

$r \sqsubseteq s$ means that a resource with quantity r may be used when quantity s is expected. This ordering is called sub-usaging. The definition below is just the skeleton, the full ordering is the reflexive-transitive closure of it.

$$* \sqsubseteq ?$$
$$* \sqsubseteq +$$
$$+ \sqsubseteq 1$$
$$? \sqsubseteq 1$$
$$? \sqsubseteq 0$$

We will denote the infimum in this order with $r \sqcap s$ and the supremum (if it exists) with $r \sqcup s$.

Addition

When we have two quantities of the same resource, we can sum the quantities.

$$0 + r = r$$
$$r + 0 = r$$
$$?+? = *$$
$$? + * = *$$
$$*+? = *$$
$$* + * = *$$
$$\_ + \_ = +$$

Multiplication

When we have a quantity r of resource A that contains quantitty s of resource B, then we in fact have quantity $r \cdot s$ of resource B.

$$0 \cdot r = 0$$
$$r \cdot 0 = 0$$
$$1 \cdot r = r$$
$$r \cdot 1 = r$$
$$? \cdot ? = ?$$
$$+ \cdot + = +$$
$$\_ \cdot \_ = *$$

The algebra of quantities

Quantities $\mathcal{Q}$ form a positive ordered commutative semiring with no zero divisors, i.e.:

- $(\mathcal{Q}, +, 0)$ is a commutative monoid.
- $(\mathcal{Q}, \cdot, 1)$ is a commutative monoid.
- 0 annihilates multiplication.
- Multiplication distributes over addition.
- Addition and multiplication preserve the subusage ordering in both arguments.
- If $r + s = 0$, then $r = 0$ and $s = 0$.
- If $r \cdot s = 0$, then $r = 0$ or $s = 0$.

Infimum

We can explicitly calculate the infimum as follows.

$$0 \sqcap 1 = ?$$
$$0 \sqcap + = *$$
$$0 \sqcap s = s$$
$$1 \sqcap 0 = ?$$
$$+ \sqcap 0 = *$$
$$r \sqcap 0 = r$$
$$r \sqcap s = r \cdot s$$

Complement

Complement of r is the quantity on the opposite side of the diamond. The complement of 0 is 0.

$$0^{\text{op}} = 0$$
$$1^{\text{op}} = *$$
$$?^{\text{op}} = +$$
$$+^{\text{op}} = ?$$
$$*^{\text{op}} = 1$$

Supremum

We can explicitly calculate the supremum as follows.

$$0 \sqcup 1 = \texttt{undefined}$$
$$0 \sqcup + = \texttt{undefined}$$
$$1 \sqcup 0 = \texttt{undefined}$$
$$1 \sqcup + = \texttt{undefined}$$
$$r \sqcup s = (r^{\mathrm{op}} \cdot s^{\mathrm{op}})^{\mathrm{op}}$$

Subtraction

$r - s = \inf\{q \in \mathcal{Q} \mid r \sqsubseteq q + s\}$, where the inf is taken according to the subusage ordering. Explicitly:

$r - s$	0	1	?	$+$	$*$
0	0				
1	1	0			
?	?	0	0		
$+$	$+$	$*$	$+$	$*$	$+$
$*$	$*$	$*$	$*$	$*$	$*$

Subtraction order

$r \leq_{\mathrm{sub}} s$ holds when $s - r$ is defined.

Explicitly: $0 \leq_{\mathrm{sub}} 1 \leq_{\mathrm{sub}} ? \leq_{\mathrm{sub}} + \leq_{\mathrm{sub}} * \leq_{\mathrm{sub}} +$

Decrementation order

$r \leq_{\mathrm{decr}} s$ holds when $r = s - 1$.

$$* \leq_{\mathrm{decr}} *$$
$$* \leq_{\mathrm{decr}} +$$
$$0 \leq_{\mathrm{decr}} 1$$
$$0 \leq_{\mathrm{decr}}?$$

Division with remainder

$r/s = \sup_{a \in \mathcal{Q}} \inf_{b \in \mathcal{Q}} \{(a, b) \mid r = a \cdot s + b\}$, where the sup and inf are taken according to the subtraction order. Note that $r/s = a$ means that $b = 0$. Explicitly:

r/s	0	1	?	$+$	$*$
0	$*$	0	0	0	0
1	$(*, 1)$	1	$(0, 1)$	$(0, 1)$	$(0, 1)$
?	$(*, ?)$	?	?	$(0, ?)$	$(0, ?)$
$+$	$(*, +)$	$+$	$(*, 1)$	$+$	$(*, 1)$
$*$	$(*, *)$	$*$	$*$	$*$	$*$

Trait ordering

The trait ordering is similar to the subusage ordering, but 0 is above 1. Explicitly:

$$* \leq_{\text{trait}} ?$$
$$* \leq_{\text{trait}} +$$
$$? \leq_{\text{trait}} 1$$
$$+ \leq_{\text{trait}} 1$$
$$1 \leq_{\text{trait}} 0$$

Trait division

$r/s = \sup\{q \in \mathcal{Q} \mid q \cdot s \sqsubseteq r\}$, where the sup is taken according to the trait ordering. Explicitly:

r/s	0	1	?	$+$	$*$
0	0	0	0	0	0
1		1	1	1	1
?		?	1	?	1
$+$		$+$	$+$	1	1
$*$		$*$	$+$	?	1

Arithmetic order

The arithmetic order on quantities is
$0 \leq_{\texttt{arith}} 1 \leq_{\texttt{arith}} ? \leq_{\texttt{arith}} + \leq_{\texttt{arith}} *$. The idea is to compare the quantities by how "big" they are.