T-Axi

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Quantities

Quantities:
$r ::= 0 \mid 1 \mid \omega$

ω is the default quantity, so when there's nothing to indicate quantity, it means it's ω.

Quantities are ordered:

$$\omega \le r$$
$$0 \le 0$$
$$1 \le 1$$

Operations on quantities

Addition of quantities:

$$\begin{aligned} 0 + r &= r \\ \omega + r &= \omega \\ 1 + 0 &= 1 \\ 1 + 1 &= \omega \\ 1 + \omega &= \omega \end{aligned}$$

Multiplication:

$$\begin{aligned} 0 \cdot r &= 0 \\ 1 \cdot r &= r \\ \omega \cdot 0 &= 0 \\ \omega \cdot 1 &= \omega \\ \omega \cdot \omega &= \omega \end{aligned}$$

Types

Types:

$$A, B ::= r\ A \to B \mid\ !_r A \mid A \otimes B \mid A \oplus B \mid \mathbf{1} \mid \mathbf{0}$$

Terms

Terms:

$$
\begin{aligned}
e ::= \;& \\
& x \mid \lambda_r\, x : A.\, e \mid e_1\ e_2 \mid \\
& \texttt{box}_r\ e \mid \texttt{letbox}\ x = e_1\ \texttt{in}\ e_2 \\
& \texttt{let}_r\ x = e_1\ \texttt{in}\ e_2 \mid \\
& \texttt{choose}\ p \mid \texttt{choose-witness}\ x\ h\ \texttt{for}\ p\ \texttt{in}\ e
\end{aligned}
$$

$\texttt{choose}\ p$ is a noncomputable terms, whereas all others are computable.

Propositions

Propositions:

$$
\begin{aligned}
P, Q ::= &\\
&\top \mid \bot \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid\\
&\forall x : A.\, P \mid \exists x : A.\, P \mid\\
&e_1 =_A e_2
\end{aligned}
$$

Notations:

$\neg P$ stands for $P \Rightarrow \bot$

$P \Leftrightarrow Q$ stands for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

h | **assumption** | **trivial** | **absurd** p
assume $h : P$ **in** q | **apply** p_1 p_2 |
both p_1 p_2 | **and-left** p | **and-right** p |
or-left p | **or-right** p | **cases** p_1 p_2 p_3 |
lemma $h : P$ **by** p **in** q | **proving** P **by** p |
suffices P **by** q **in** p |
pick-any $x : A$ **in** e | **instantiate** p **with** e |
witness e **such that** p | **pick-witness** x h **for** p_1 **in** p_2 |
refl e | **rewrite** p_1 **in** p_2 | **funext** $x : A$ **in** p
by-contradiction $h : \neg P$ **in** q |
choose-spec p | **choose-witness** x h **for** p **in** q

Contexts

Contexts:
$\Gamma ::= \cdot \mid \Gamma, r\, x : A \mid \Gamma, r\, x : A := e \mid \Gamma, h : P$

Operations on contexts:
$\Gamma_1 + \Gamma_2$ – context addition
$r\, \Gamma$ – context scaling
$\Gamma_1 \leq \Gamma_2$ – context subusaging/ordering
$|\Gamma|$ – cartesianization

Context addition

$$\cdot + \cdot = \cdot$$

$$(\Gamma_1, r_1\, x : A) + (\Gamma_2, r_2\, x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : A$$

$$(\Gamma_1, r_1\, x : A := e) + (\Gamma_2, r_2\, x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2)\, x : A := e$$

$$(\Gamma_1, p : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P$$

Context scaling

$$
\begin{aligned}
s \cdot &= \cdot \\
s\,(\Gamma, r\,x : A) &= s\,\Gamma, (s \cdot r)\,x : A \\
s\,(\Gamma, r\,x : A := e) &= s\,\Gamma, (s \cdot r)\,x : A := e \\
s\,(\Gamma, h : P) &= s\,\Gamma, h : P
\end{aligned}
$$

Context ordering

$$\frac{}{\cdot \leq \cdot}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A \leq \Gamma_2, r_2\, x : A}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1\, x : A := e \leq \Gamma_2, r_2\, x : A := e}$$

$$\frac{\Gamma_1 \leq \Gamma_2}{\Gamma_1, h : P \leq \Gamma_2, h : P}$$

Cartesianization

Cartesianization turns a context into a traditional context that doesn't mention any quantities.

$$
\begin{aligned}
|\cdot| &= \cdot \\
|\Gamma, r\,x : A| &= |\Gamma|, x : A \\
|\Gamma, r\,x : A := e| &= |\Gamma|, x : A := e \\
|\Gamma, h : P| &= |\Gamma|, h : P
\end{aligned}
$$

Judgements

Well-formed context judgement: $\Gamma\ \mathtt{ctx}$
Cartesian context judgement: $\Gamma\ \mathtt{cartesian}$

Well-formed type judgement: $\Gamma \vdash A\ \mathtt{type}$
Typing judgement: $\Gamma \vdash e : A$

Well-formed proposition judgement: $\Gamma \vdash P\ \mathtt{prop}$
Proof judgement: $\Gamma \vdash p : P$

Sanity checks

We'll set up the system so that:

- If $\Gamma \vdash e : A$, then $|\Gamma| \vdash A\ \mathtt{type}$.
- If $\Gamma \vdash p : P$, then $\Gamma \vdash P\ \mathtt{prop}$.
- If $\Gamma \vdash P\ \mathtt{prop}$, then $\Gamma\ \mathtt{cartesian}$.
- If $\Gamma \vdash A\ \mathtt{type}$, then $\Gamma\ \mathtt{cartesian}$.
- If $\Gamma\ \mathtt{cartesian}$, then $\Gamma\ \mathtt{ctx}$.

Well-formed contexts

$$\frac{}{\cdot\ \mathtt{ctx}}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad |\Gamma| \vdash A\ \mathtt{type} \quad x \notin \Gamma}{\Gamma, r\,x : A\ \mathtt{ctx}}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad \Gamma \vdash e : A \quad x \notin \Gamma}{\Gamma, r\,x : A := e\ \mathtt{ctx}}\,\text{DUBIOUS}$$

$$\frac{\Gamma\ \mathtt{ctx} \quad |\Gamma| \vdash P\ \mathtt{prop} \quad h \notin \Gamma}{\Gamma, h : P\ \mathtt{ctx}}$$

Cartesian contexts

$$\frac{\Gamma\ \mathtt{ctx} \qquad \Gamma = |\Gamma|}{\Gamma\ \mathtt{cartesian}}$$

Well-formed types

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \mathbf{1}\ \texttt{type}} \qquad \frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \mathbf{0}\ \texttt{type}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \qquad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash r\ A \to B\ \texttt{type}} \qquad \frac{\Gamma \vdash A\ \texttt{type}}{\Gamma \vdash\ !_r\, A\ \texttt{type}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \qquad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash A \otimes B\ \texttt{type}} \qquad \frac{\Gamma \vdash A\ \texttt{type} \qquad \Gamma \vdash B\ \texttt{type}}{\Gamma \vdash A \oplus B\ \texttt{type}}$$

Functions

$$\frac{\Gamma, r\,x : A \vdash e : B}{\Gamma \vdash \lambda_r x : A.\, e : r\ A \to B}$$

$$\frac{\Gamma \leq \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash e_1 : r\ A \to B \quad \Gamma_2 \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

Box

$$\frac{\Gamma \leq r\,\Gamma' \quad \Gamma' \vdash e : A}{\Gamma \vdash \texttt{box}_r\ e : \,!_r\, A}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : \,!_r\, A \quad \Gamma_2, r\,x : A \vdash e_2 : B}{\Gamma \vdash \texttt{letbox}\ x = e_1\ \texttt{in}\ e_2 : B}$$

Let

$$\frac{\Gamma \leq \Gamma_1 + r\,\Gamma_2 \quad \Gamma_1 \vdash e_1 : A \quad \Gamma_2, r\,x : A \vdash e_2 : B}{\Gamma \vdash \mathtt{let}_r\ x = e_1\ \mathtt{in}\ e_2 : B}$$

Well-formed propositions

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \top\ \texttt{prop}} \qquad \frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \bot\ \texttt{prop}}$$

$$\frac{\Gamma \vdash P\ \texttt{prop} \quad \Gamma \vdash Q\ \texttt{prop}}{\Gamma \vdash P \Rightarrow Q\ \texttt{prop}}$$

$$\frac{\Gamma \vdash P\ \texttt{prop} \quad \Gamma \vdash Q\ \texttt{prop}}{\Gamma \vdash P \wedge Q\ \texttt{prop}} \qquad \frac{\Gamma \vdash P\ \texttt{prop} \quad \Gamma \vdash Q\ \texttt{prop}}{\Gamma \vdash P \vee Q\ \texttt{prop}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma, x : A \vdash P\ \texttt{prop}}{\Gamma \vdash \forall x : A.\, P\ \texttt{prop}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma, x : A \vdash P\ \texttt{prop}}{\Gamma \vdash \exists x : A.\, P\ \texttt{prop}}$$

$$\frac{\Gamma \vdash A\ \texttt{type} \quad \Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 =_A e_2\ \texttt{prop}}$$

Substitution

The notation is $P[x := e]$ for substitution in propositions.

Assumptions and implication

$$\frac{\Gamma \vdash P\ \mathtt{prop} \quad (h : P) \in \Gamma}{\Gamma \vdash h : P} \qquad \frac{\Gamma \vdash P\ \mathtt{prop} \quad (h : P) \in \Gamma}{\Gamma \vdash \textbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{assume}\ h : P\ \textbf{in}\ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{apply}\ q\ p : Q}$$

Propositional logic

$$\frac{\Gamma\ \texttt{cartesian}}{\Gamma \vdash \textbf{trivial} : \top} \qquad \frac{\Gamma \vdash Q\ \texttt{prop} \quad \Gamma \vdash p : \bot}{\Gamma \vdash \textbf{absurd}\ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{both}\ p\ q : P \wedge Q}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-left}\ pq : P} \qquad \frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \textbf{and-right}\ pq : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash Q\ \texttt{prop}}{\Gamma \vdash \textbf{or-left}\ p : P \vee Q} \qquad \frac{\Gamma \vdash P\ \texttt{prop} \quad \Gamma \vdash q : Q}{\Gamma \vdash \textbf{or-right}\ q : P \vee Q}$$

$$\frac{\Gamma \vdash pq : P \vee Q \quad \Gamma \vdash r_1 : P \Rightarrow R \quad \Gamma \vdash r_2 : Q \Rightarrow R}{\Gamma \vdash \textbf{cases}\ pq\ r_1\ r_2 : R}$$

Utilities

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \textbf{lemma } h : P \textbf{ by } p \textbf{ in } q : Q}$$

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \textbf{proving } P \textbf{ by } p : P}$$

$$\frac{\Gamma \vdash pq : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \textbf{suffices } P \textbf{ by } pq \textbf{ in } p : Q}$$

Quantifiers

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \textbf{pick-any } x : A \textbf{ in } p : \forall x : A.\, P}$$

$$\frac{\Gamma \vdash p : \forall x : A.\, P \qquad \Gamma \vdash e : A}{\Gamma \vdash \textbf{instantiate } p \textbf{ with } e : P\,[x := e]}$$

$$\frac{\Gamma \vdash e : A \qquad \Gamma, x : A \vdash P \ \mathtt{prop} \qquad \Gamma \vdash p : P\,[x := e]}{\Gamma \vdash \textbf{witness } e \textbf{ such that } p : \exists x : A.\, P}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \qquad \Gamma \vdash Q \ \mathtt{prop} \qquad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \textbf{pick-witness } x \ h \textbf{ for } p \textbf{ in } q : Q}$$

Equality

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \textbf{refl}\ e : e =_A e}$$

$$\frac{\Gamma \vdash q : e_1 =_A e_2 \quad \Gamma, x : A \vdash P\ \mathtt{prop} \quad \Gamma \vdash p : P\,[x := e_2]}{\Gamma \vdash \textbf{rewrite}\ q\ \textbf{in}\ p : P\,[x := e_1]}$$

$$\frac{\Gamma, x : A \vdash p : f\ x =_B g\ x}{\Gamma \vdash \textbf{funext}\ x : A\ \textbf{in}\ p : f =_{r\ A \to B} g}$$

Classical Logic

$$\frac{\Gamma, h : \neg P \vdash q : \bot}{\Gamma \vdash \textbf{by-contradiction } h : \neg P \textbf{ in } q : P}$$

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P}{\Gamma \vdash \texttt{choose } p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P}{\Gamma \vdash \textbf{choose-spec } p : P\,[x := \texttt{choose } p]}$$

$$\frac{\Gamma \vdash p : \exists x : A.\, P \quad \Gamma \vdash Q \texttt{ prop} \quad \Gamma, x : A := \texttt{choose } p, h : P \vdash q : Q}{\Gamma \vdash \textbf{choose-witness } x\ h \textbf{ for } p \textbf{ in } q : Q}$$

$$\frac{|\Gamma| \vdash p : \exists x : A.\, P \quad |\Gamma| \vdash B \texttt{ type} \quad \Gamma, x : A := \texttt{choose } p, h : P \vdash e : B}{\Gamma \vdash \texttt{choose-witness } x\ h \texttt{ for } p \texttt{ in } e : B}$$