Hint-based unification for STLC

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Terms, contexts and judgements

Terms:

$e ::=$

$x \mid (e : H) \mid$

$\lambda x.\, e \mid e_1\ e_2 \mid$

$(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid$

$\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid$

$\texttt{unit} \mid \mathbf{0}\texttt{-elim}\ e$

Contexts:

$\Gamma ::= \cdot \mid \Gamma, x : H$

Judgements:

$\Gamma \vdash e \Leftarrow H \Rightarrow H' \dashv \Gamma'$ – in context Γ, term e checks with hint H and infers hint H' in output context Γ'

Output contexts – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

$$\frac{\Gamma \vdash e \Leftarrow A \sqcup B \Rightarrow C \dashv \Gamma'}{\Gamma \vdash (e : A) \Leftarrow B \Rightarrow C \dashv \Gamma'}\textsc{Annot}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathtt{hint}(e) \Rightarrow A \dashv \Gamma' \quad e\ \mathtt{constructor}}{\Gamma \vdash e \Leftarrow \mathbf{?} \Rightarrow A \dashv \Gamma'}\textsc{Hole}$$

Output contexts – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Leftarrow \mathbf{?} \to B \Rightarrow A \to B' \dashv \Gamma' \quad \Gamma' \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f\ a \Leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \Leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \Leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times \mathbf{?} \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \mathtt{outl}\ e \Leftarrow A \Rightarrow A' \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{?} \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \mathtt{outr}\ e \Leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Output contexts – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow A \Rightarrow A' \dashv \Gamma'}{\Gamma \vdash \mathtt{inl}\ e \Leftarrow A + B \Rightarrow A' + B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma'}{\Gamma \vdash \mathtt{inr}\ e \Leftarrow A + B \Rightarrow A + B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \dashv \Gamma_2 \\ \Gamma_2 \vdash g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathtt{case}\ e\ \mathtt{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3 \dashv \Gamma_3}$$

$$\frac{}{\Gamma \vdash \mathtt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1} \dashv \Gamma} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0} \dashv \Gamma'}{\Gamma \vdash \mathbf{0}\text{-}\mathtt{elim}\ e \Leftarrow A \Rightarrow A \dashv \Gamma'}$$

Order on contexts

$$\frac{}{\cdot \sqsubseteq \cdot}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad A \sqsubseteq B}{\Gamma_1, x : A \sqsubseteq \Gamma_2, x : B}$$

Order on contexts – properties

$\sqsubseteq$ is a partial order, whose structure is inherited from the order on hints.

- Reflexivity: $\Gamma \sqsubseteq \Gamma$
- Transitivity: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_3 \implies \Gamma_1 \sqsubseteq \Gamma_3$
- Weak antisymmetry: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_1 \implies \Gamma_1 = \Gamma_2$

Metatheory

If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, then:

- (Information increase) $\Gamma \sqsubseteq \Gamma'$ (proof: induction)
- (Information increase) $A \sqsubseteq B$ (proof: induction)
- (Context squeeze) If $\Gamma \sqsubseteq \Delta \sqsubseteq \Gamma'$, then $\Delta \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: induction)
- (Hint squeeze) If $A \sqsubseteq A'$ and $A' \sqsubseteq B$, then $\Gamma \vdash e \Leftarrow A' \Rightarrow B \dashv \Gamma'$ (proof: induction)

Metatheory – decidability and determinism

- (Decidability) For Γ, e, A it is decidable whether there exist B and Γ' such that $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: the rules are literally the algorithm)
- (Determinism) If $\Gamma \vdash e \Leftarrow A \Rightarrow B_1 \dashv \Gamma_1$ and $\Gamma \vdash e \Leftarrow A \Rightarrow B_2 \dashv \Gamma_2$, then $B_1 = B_2$ and $\Gamma_1 = \Gamma_2$ (proof: induction)

Metatheory – soundness

(Soundness) If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, B is a type and Γ' only contains types, then $\Gamma' \vdash e : B$.

Abridged rules for output contexts

The rules for STLC with output contexts can get quite lengthy. We try to remedy this by introducing an abridged form of the rules, in which contexts are written explicitly only if some operation (like context extension) is performed on them. If contexts are only passed around, we keep them implicit.

Output contexts – abridged basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

$$\frac{e \Leftarrow A \sqcup B \Rightarrow C}{(e : A) \Leftarrow B \Rightarrow C}\textsc{Annot}$$

$$\frac{e \Leftarrow \mathtt{hint}(e) \Rightarrow A \qquad e\ \mathtt{constructor}}{e \Leftarrow \mathbf{?} \Rightarrow A}\textsc{Hole}$$

Output contexts – abridged type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{f \Leftarrow \textbf{?} \to B \Rightarrow A \to B' \qquad a \Leftarrow A \Rightarrow A'}{f\ \ a \Leftarrow B \Rightarrow B'}$$

$$\frac{a \Leftarrow A \Rightarrow A' \qquad b \Leftarrow B \Rightarrow B'}{(a, b) \Leftarrow A \times B \Rightarrow A' \times B'}$$

$$\frac{e \Leftarrow A \times \textbf{?} \Rightarrow A' \times B}{\texttt{outl}\ \ e \Leftarrow A \Rightarrow A'} \qquad \frac{e \Leftarrow \textbf{?} \times B \Rightarrow A \times B'}{\texttt{outr}\ \ e \Leftarrow B \Rightarrow B'}$$

Output contexts – abridged type-directed rules

$$\frac{e \Leftarrow A \Rightarrow A'}{\texttt{inl}\ e \Leftarrow A + B \Rightarrow A' + B}$$

$$\frac{e \Leftarrow B \Rightarrow B'}{\texttt{inr}\ e \Leftarrow A + B \Rightarrow A + B'}$$

$$\frac{e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \qquad \begin{array}{c} f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \\ g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \end{array}}{\texttt{case}\ e\ \texttt{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3}$$

$$\frac{}{\texttt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}} \qquad \frac{e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\mathbf{0}\texttt{-elim}\ e \Leftarrow A \Rightarrow A}$$

Types, holes, terms and contexts

Holes:

$H ::= \alpha \mid H_1 \to H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$

Terms:

$$
\begin{aligned}
e ::= \\
&x \mid (e : H) \mid \\
&\lambda x.\, e \mid e_1\ e_2 \mid \\
&(e_1, e_2) \mid \texttt{outl}\ e \mid \texttt{outr}\ e \mid \\
&\texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}\ e\ \texttt{of}\ (e_1, e_2) \mid \\
&\texttt{unit} \mid \mathbf{0}\texttt{-elim}\ e
\end{aligned}
$$

Note: the terms are the same as in STLC with Hints.

Contexts and judgements

Contexts:
$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha \mid \Gamma, \alpha := H$

We introduce unification variables, denoted with Greek letter $(\alpha, \beta, \gamma, \ldots)$. When extending the context with a unification variable, it is set to some hint, which at the beginning will be just **?**.

Judgements:
$\Gamma \vdash e \Leftarrow H \Rightarrow H' \dashv \Gamma'$ – in context Γ, term e checks with hint H and infers hint H' in output context Γ'

Unification – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2}\textsc{Var}$$

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Unification – type-directed rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x.\, e \Leftarrow A \to B \Rightarrow A' \to B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Leftarrow \mathbf{?} \to B \Rightarrow A \to B' \dashv \Gamma' \quad \Gamma' \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f\ a \Leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \Leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \Leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \Leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \Leftarrow A \times \mathbf{?} \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \texttt{outl}\ e \Leftarrow A \Rightarrow A' \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{?} \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \texttt{outr}\ e \Leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Unification – type-directed rules

$$\frac{\Gamma \vdash e \Leftarrow A \Rightarrow A' \dashv \Gamma'}{\Gamma \vdash \texttt{inl}\ e \Leftarrow A + B \Rightarrow A' + B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow B \Rightarrow B' \dashv \Gamma'}{\Gamma \vdash \texttt{inr}\ e \Leftarrow A + B \Rightarrow A + B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Leftarrow \mathbf{?} + \mathbf{?} \Rightarrow A + B \dashv \Gamma_1 \qquad \begin{array}{c} \Gamma_1 \vdash f \Leftarrow A \to C_1 \Rightarrow A' \to C_2 \dashv \Gamma_2 \\ \Gamma_2 \vdash g \Leftarrow B \to C_2 \Rightarrow B' \to C_3 \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C_1 \Rightarrow C_3 \dashv \Gamma_3}$$

$$\frac{}{\Gamma \vdash \texttt{unit} \Leftarrow \mathbf{1} \Rightarrow \mathbf{1} \dashv \Gamma} \qquad \frac{\Gamma \vdash e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0} \dashv \Gamma'}{\Gamma \vdash \mathbf{0}\texttt{-elim}\ e \Leftarrow A \Rightarrow A \dashv \Gamma'}$$