

# Hint-based typing for polymorphism

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# Types and judgements

Types:

$$A, B ::= \alpha \mid \forall \alpha. A \mid A \rightarrow B$$

Contexts:

$$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha$$

Judgements:

$$\Gamma \vdash e : A, \Gamma \vdash A \text{ type}$$

# Valid context judgement

$$\frac{}{\cdot \text{ ctx}} \text{CTX-EMPTY}$$

$$\frac{\Gamma \text{ ctx} \quad x \notin \Gamma \quad \Gamma \vdash A \text{ type}}{\Gamma, x : A \text{ ctx}} \text{CTX-EXTEND}$$

$$\frac{\Gamma \text{ ctx} \quad \alpha \notin \Gamma}{\Gamma, \alpha \text{ ctx}} \text{CTX-EXTENDTYPE}$$

# Valid type judgement

$$\frac{\Gamma \text{ ctx} \quad \alpha \in \Gamma}{\Gamma \vdash \alpha \text{ type}} \text{TYVAR}$$

$$\frac{\Gamma, \alpha \vdash A \text{ type}}{\Gamma \vdash \forall \alpha. A \text{ type}} \text{ALL}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \text{ type}} \text{FUN}$$

# Sanity checks

- If  $\Gamma \vdash A \text{ type}$ , then  $\Gamma \text{ ctx}$
- If  $\Gamma \text{ ctx}$  and  $(x : A) \in \Gamma$ , then  $\Gamma \vdash A \text{ type}$
- We will set up the system so that if  $\Gamma \vdash e : A$ , then  $\Gamma \vdash A \text{ type}$

# Terms

Terms:

$e ::=$

$x \mid$

$\lambda x : A. e \mid e_1 \ e_2 \mid$

$\Lambda\alpha. e \mid e @A$

# Declarative typing

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x : A. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. A}$$

$$\frac{\Gamma \vdash e : \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e @ B : A[\alpha := B]}$$

# Algorithmic typing

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma, x : A \vdash e \Rightarrow B}{\Gamma \vdash \lambda x : A. e \Rightarrow A \rightarrow B} \quad \frac{\Gamma \vdash f \Rightarrow A \rightarrow B \quad \Gamma \vdash a \Rightarrow A}{\Gamma \vdash f a \Rightarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Rightarrow A}{\Gamma \vdash \Lambda \alpha. e \Rightarrow \forall \alpha. A}$$

$$\frac{\Gamma \vdash e \Rightarrow \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e @ B \Rightarrow A[\alpha := B]}$$

# Terms

Terms:

$e ::=$

$x \mid$

$\lambda x. e \mid \text{app}_A e_1 e_2 \mid$

$e @ A$

# Declarative typing

$$\frac{\Gamma \text{ ctx} \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash \text{app}_A f a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha. A}$$

$$\frac{\Gamma \vdash e : \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e @B : A[\alpha := B]}$$

# Algorithmic typing

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma}{\Gamma \vdash x \Leftarrow A} \text{VAR}$$

$$\frac{\Gamma, x : A \vdash e \Leftarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \rightarrow B} \quad \frac{\Gamma \vdash f \Leftarrow A \rightarrow B \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash \text{app}_A f a \Leftarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Leftarrow A}{\Gamma \vdash e \Leftarrow \forall \alpha. A}$$

$$\frac{\Gamma \vdash e \Leftarrow \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e @ B \Leftarrow A[\alpha := B]}$$

# Terms

Terms:

$e ::=$

$$\begin{array}{l} x \mid \\ \lambda x. e \mid e_1 e_2 \mid \end{array}$$

# Declarative typing

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha. A}$$

$$\frac{\Gamma \vdash e : \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e : A[\alpha := B]}$$

# Hints

Hints:

$H ::= ? \mid H_1 \rightarrow H_2 \mid \forall \alpha. H \mid \alpha$

# Valid hint judgement

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash ? \text{ hint}} \text{HINT-HOLE}$$

$$\frac{\Gamma \text{ ctx} \quad \alpha \in \Gamma}{\Gamma \vdash \alpha \text{ hint}} \text{HINT-TYVAR}$$

$$\frac{\Gamma, \alpha \vdash H \text{ hint}}{\Gamma \vdash \forall \alpha. H \text{ hint}} \text{HINT-ALL}$$

$$\frac{\Gamma \vdash H_1 \text{ hint} \quad \Gamma \vdash H_2 \text{ hint}}{\Gamma \vdash H_1 \rightarrow H_2 \text{ hint}} \text{HINT-FUN}$$

# Order on hints

$$\frac{\Gamma \vdash H \text{ hint}}{\Gamma \vdash ? \sqsubseteq H}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \alpha \sqsubseteq \alpha}$$

$$\frac{\Gamma, \alpha \vdash H_1 \sqsubseteq H_2}{\Gamma \vdash \forall \alpha. H_1 \sqsubseteq \forall \alpha. H_2}$$

$$\frac{\Gamma \vdash H_1 \sqsubseteq H'_1 \quad \Gamma \vdash H_2 \sqsubseteq H'_2}{\Gamma \vdash H_1 \rightarrow H_2 \sqsubseteq H'_1 \rightarrow H'_2}$$

# Combining hints

$$\mathbf{?} \sqcup H = H$$

$$H \sqcup \mathbf{?} = H$$

$$(H_1 \rightarrow H_2) \sqcup (H'_1 \rightarrow H'_2) = (H_1 \sqcup H'_1) \rightarrow (H_2 \sqcup H'_2)$$

$$(\forall \alpha. H_1) \sqcup (\forall \alpha. H_2) = \forall \alpha. H_1 \sqcup H_2$$

$$\alpha \sqcup \alpha = \alpha$$

# Hints for term constructors

$$\text{hint}(\lambda x. e) = ? \rightarrow ?$$

# Terms

Terms:

$e ::=$

$$\begin{array}{c} x \mid (e : H) \mid \\ \lambda x. e \mid e_1 \ e_2 \mid \end{array}$$

# Declarative typing

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{VAR}$$

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash H \sqsubseteq A}{\Gamma \vdash (e : H) : A}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \rightarrow B} \quad \frac{\Gamma \vdash f : A \rightarrow B \quad \Gamma \vdash a : A}{\Gamma \vdash f\ a : B}$$

$$\frac{\Gamma, \alpha \vdash e : A}{\Gamma \vdash e : \forall \alpha. A}$$

$$\frac{\Gamma \vdash e : \forall \alpha. A \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash e : A[\alpha := B]}$$

# Algorithmic typing – basic rules

$$\frac{\Gamma \text{ ctx } (x : A) \in \Gamma \quad \Gamma \vdash H \sqsubseteq A}{\Gamma \vdash x \Leftarrow H \Rightarrow A} \text{VAR}$$

$$\frac{\Gamma \vdash e \Leftarrow H_1 \sqcup H_2 \Rightarrow A}{\Gamma \vdash (e : H_1) \Leftarrow H_2 \Rightarrow A} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \Leftarrow \text{hint}(e) \Rightarrow A \quad e \text{ constructor}}{\Gamma \vdash e \Leftarrow ? \Rightarrow A} \text{HOLE}$$

# Algorithmic typing – other rules

$$\frac{\Gamma, x : A \vdash e \Leftarrow H \Rightarrow B}{\Gamma \vdash \lambda x. e \Leftarrow A \rightarrow H \Rightarrow A \rightarrow B}$$

$$\frac{\Gamma \vdash a \Leftarrow ? \Rightarrow A \quad \Gamma \vdash f \Leftarrow A \rightarrow H \Rightarrow A \rightarrow B}{\Gamma \vdash f \ a \Leftarrow H \Rightarrow B}$$

$$\frac{\Gamma, \alpha \vdash e \Leftarrow H \Rightarrow A}{\Gamma \vdash e \Leftarrow \forall \alpha. H \Rightarrow \forall \alpha. A}$$

# Embedding

- $\lambda x : A. e \equiv (\lambda x. e : A \rightarrow ?)$
- $\text{app}_A f a \equiv (f : A \rightarrow ?) a$
- $\Lambda\alpha. e \equiv (e : \forall\alpha. ?)$
- $e @A \equiv \quad ???$