

# T-Axi (algorithmic)

Wojciech Kołowski  
Mateusz Pyzik

# Syntax

The syntax in the algorithmic system is the same, but we will have far fewer annotations in terms

# Judgements

Kind inference judgement:  $\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma'$

Type conversion judgements:  $\Gamma \vdash A \hat{=} B \Rightarrow \text{Type}_r \dashv \Gamma'$ ,  
 $\Gamma \vdash A \equiv B \Rightarrow \text{Type}_r \dashv \Gamma'$

Type checking judgement:  $\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'$

Type inference judgement:  $\Gamma \vdash_i e \Rightarrow A \dashv \Gamma'$

Term conversion judgements:  $\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow A \dashv \Gamma'$ ,  
 $\Gamma \vdash e_1 \hat{\sim} e_2 \Rightarrow A \dashv \Gamma'$ ,  $\Gamma \vdash e_1 \equiv e_2 \Leftarrow A \dashv \Gamma'$ ,  
 $\Gamma \vdash e_1 \sim e_2 \Rightarrow A \dashv \Gamma'$

Proposition checking judgement:  $\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma'$

Proposition conversion judgements:  $\Gamma \vdash P \hat{=} Q \dashv \Gamma'$ ,  
 $\Gamma \vdash P \equiv Q \dashv \Gamma'$ ,  $\Gamma \vdash P \sim Q \dashv \Gamma'$

# Subtraction of quantities

$r_1 - r_2$  is the least  $r'$  such that  $r_1 \sqsubseteq r' + r_2$ .

$r_1 - r_2$	0	1	?	+	*
0	0				
1	1	0			
?	?	0	0		
+	+	*	+	*	+
*	*	*	*	*	*

# Subtraction order on quantities

$r_1 \leq_{\text{sub}} r_2$  holds when  $r_2 - r_1$  is defined.

Explicitly:  $0 \leq_{\text{sub}} 1 \leq_{\text{sub}} ? \leq_{\text{sub}} + \leq_{\text{sub}} * \leq_{\text{sub}} +$

# Decrementation order on quantities

$r_1 \leq_{\text{dec}} r_2$  holds when  $r_2 - 1 = r_1$ .

$$\overline{* \leq_{\text{dec}} +}$$

$$\overline{0 \leq_{\text{dec}} 1}$$

$$\overline{0 \leq_{\text{dec}} ?}$$

# Arithmetic order on quantities

The arithmetic order on quantities is  $0 \leq 1 \leq ? \leq + \leq *$ . The idea is to compare the quantities by how “big” they are.

# Division with remainder

$a/b = (q, r)$  when  $a = b \cdot q + r$ , with  $q$  as large as possible and  $r$  being as small as possible according to the arithmetic order. Note that  $a/b = q$  means that  $r = 0$ .

$r_1/r_2$	0	1	?	+	*
0	*	0	0	0	0
1	(*, 1)	1	(0, 1)	(0, 1)	(0, 1)
?	(*, ?)	?	?	(0, ?)	(0, ?)
+	(*, +)	+	(*, 1)	+	(*, 1)
*	(*, *)	*	*	*	*

# Decrement variable in context

$\cdot - x = \mathbf{undefined}$

$$(\Gamma, r x : A) - x = \Gamma, (r - 1) x : A$$

$$(\Gamma, r y : A) - x = \Gamma - x, r y : A$$

$$(\Gamma, r x : A := e) - x = \Gamma, (r - 1) x : A := e$$

$$(\Gamma, r y : A := e) - x = \Gamma - x, r y : A := e$$

$$(\Gamma, h : P) - x = \Gamma - x, h : P$$

$$(\Gamma, a : \text{Type}_r) - x = \Gamma - x, a : \text{Type}_r$$

# Context division with remainder

$$\overline{\cdot / r = \cdot}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A)/q = ((\Gamma_1, r_1 x : A), (\Gamma_2, r_2 x : A))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2) \quad r/q = (r_1, r_2)}{(\Gamma, r x : A := e)/q = ((\Gamma_1, r_1 x : A := e), (\Gamma_2, r_2 x : A := e))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, h : P)/q = ((\Gamma_1, h : P), (\Gamma_2, h : P))}$$

$$\frac{\Gamma/q = (\Gamma_1, \Gamma_2)}{(\Gamma, a : \text{Type}_r)/q = ((\Gamma_1, a : \text{Type}_r), (\Gamma_2, a : \text{Type}_{\text{Type}_r}))}$$

# Context clean-up

$\Gamma, rx : A \vdash; e \Leftarrow A \dashv \Gamma', 0x : A$  is a shorthand for  
 $\Gamma, rx : A \vdash; e \Leftarrow A \dashv \Gamma', r'x : A$  with the additional condition  
 $r' \sqsubseteq 0$  when  $i = c$

$\Gamma, rx : A \vdash; e \Rightarrow A \dashv \Gamma', 0x : A$  is a shorthand for  
 $\Gamma, rx : A \vdash; e \Rightarrow A \dashv \Gamma', r'x : A$  with the additional condition  
 $r' \sqsubseteq 0$  when  $i = c$

# Subsumption and annotations

$$\frac{\Gamma \vdash_i e \Rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \stackrel{\cong}{=} B \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash_i (e : A) \Rightarrow A \dashv \Gamma_2} \text{ANNOT}$$

# Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash_c x \Rightarrow A \dashv \Gamma - x}$$

$$\frac{\Gamma(x) = A}{\Gamma \vdash_{nc} x \Rightarrow A \dashv \Gamma}$$

# Functions

$$\frac{\Gamma, r_A^i x : A \vdash_i e \Leftarrow B \dashv \Gamma', 0x : A}{\Gamma \vdash_i \lambda x. e \Leftarrow rA \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i f \Rightarrow rA \rightarrow B \dashv \Gamma_1 \quad \Gamma_1/r = (\Gamma_2, \Gamma_3) \quad \Gamma_2 \vdash_i a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash_i f a \Rightarrow B \dashv \Gamma_3 + r\Gamma_4}$$

## Box

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash_i e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash_i \text{box } e \Leftarrow !_r A \dashv \Gamma_2 + r \Gamma_3}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow !_r A \dashv \Gamma_1 \quad \Gamma_1, r_A^i x : A \vdash_i e_2 \Rightarrow B \dashv \Gamma_2, 0 x : A}{\Gamma \vdash_i \text{let box } x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_2}$$

# Empty

$$\frac{\Gamma \vdash; e \Leftarrow \text{Empty} \dashv \Gamma'}{\Gamma \vdash; \text{Empty-elim } e \Leftarrow A \dashv \Gamma'}$$

# Unit

$$\frac{}{\Gamma \vdash; \text{unit} \Leftarrow \text{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow \text{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash; \text{let unit} = e_1 \text{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

# Products

$$\frac{\Gamma \vdash_i e_1 \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash_i e_2 \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash_i (e_1, e_2) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_i e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1_A^i x : A, 1_B^i y : B \vdash_i e_2 \Rightarrow C \dashv \Gamma_2, 0x : A, 0y}{\Gamma \vdash_i \text{let } (x, y) = e_1 \text{ in } e_2 \Rightarrow C \dashv \Gamma_2}$$

# Sums

$$\frac{\Gamma \vdash_i e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash_i \text{inl } e \Leftarrow A \oplus B \dashv \Gamma'} \quad \frac{\Gamma \vdash_i e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash_i \text{inr } e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{c} \Gamma_1 \vdash_i f \Leftarrow 1A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Leftarrow 1B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash_i e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \begin{array}{c} \Gamma_1 \vdash_i f \Rightarrow 1A \rightarrow C \dashv \Gamma_2 \\ \Gamma_1 \vdash_i g \Rightarrow 1B \rightarrow C \dashv \Gamma_3 \end{array}}{\Gamma \vdash_i \text{case } e \text{ of } (f, g) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

## Let

$$\frac{\Gamma / r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash; e_1 \Rightarrow A \dashv \Gamma_3 \quad (\Gamma_2 + r \Gamma_3), r_A^i x : A \vdash; e_2 \Rightarrow B \dashv \Gamma_4}{\Gamma \vdash; \text{let}_r x = e_1 \text{ in } e_2 \Rightarrow B \dashv \Gamma_4}$$

# Polymorphism

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda a. e \Leftarrow \forall a : \text{Type}_r. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall a : \text{Type}_r. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e A \Rightarrow B[a := A] \dashv \Gamma_2}$$

# Polymorphism (implicit arguments)

$$\frac{\Gamma, a : \text{Type}_r \vdash_i e \Leftarrow B \dashv \Gamma', a : \text{Type}_r}{\Gamma \vdash_i \Lambda \{a\}. e \Leftarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash_i e \Rightarrow \forall \{a : \text{Type}_r\}. B \dashv \Gamma_1 \quad \Gamma_1 \vdash A \Rightarrow \text{Type}_r \dashv \Gamma_2}{\Gamma \vdash_i e @ A \Rightarrow B [a := A] \dashv \Gamma_2}$$

# Subsumption and annotations

$$\frac{\Gamma \vdash p \Rightarrow P' \dashv \Gamma_1 \quad \Gamma_1 \vdash P \triangleq P' \dashv \Gamma_2}{\Gamma \vdash p \Leftarrow P \dashv \Gamma_2} \text{SUBSUMPTION}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{proving } P \mathbf{ by } p \Rightarrow P \dashv \Gamma_2} \text{ANNOT}$$

# Assumptions and implication

$$\frac{\Gamma(h) = P}{\Gamma \vdash h \Rightarrow P \dashv \Gamma} \quad \frac{(h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} \Leftarrow P \dashv \Gamma}$$

$$\frac{\Gamma, h : P \vdash q \Leftarrow Q \dashv \Gamma', h : P}{\Gamma \vdash \mathbf{assume} \ h \ \mathbf{in} \ q \Leftarrow P \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow P \Rightarrow Q \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2}{\Gamma \vdash \mathbf{apply} \ q \ p \Rightarrow Q \dashv \Gamma_2}$$

# Propositional logic

$$\frac{}{\Gamma \vdash \text{trivial} \Leftarrow \top \dashv \Gamma}$$

$$\frac{\Gamma \vdash p \Leftarrow \perp \dashv \Gamma'}{\Gamma \vdash \text{absurd } p \Leftarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma_1 \quad \Gamma_1 \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \text{both } p \ q \Leftarrow P \wedge Q \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \text{and-left } pq \Rightarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma'}{\Gamma \vdash \text{and-right } pq \Rightarrow Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Leftarrow P \dashv \Gamma'}{\Gamma \vdash \text{or-left } p \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Leftarrow Q \dashv \Gamma'}{\Gamma \vdash \text{or-right } q \Leftarrow P \vee Q \dashv \Gamma'}$$

$$\frac{\Gamma \vdash pq \Rightarrow P \vee Q \dashv \Gamma_1 \quad \Gamma_1 \vdash r_1 \Leftarrow P \Rightarrow R \dashv \Gamma_2 \quad \Gamma_1 \vdash r_2 \Leftarrow Q \Rightarrow R \dashv \Gamma_3}{\Gamma \vdash \text{cases } pq \ r_1 \ r_2 \Leftarrow R \dashv \Gamma_2 \sqcup \Gamma_3}$$

# Positive conjunction

$$\frac{\Gamma \vdash pq \Rightarrow P \wedge Q \dashv \Gamma_1 \quad \Gamma_1, h_1 : P, h_2 : Q \vdash r \Leftarrow R \dashv \Gamma_2, h_1 : P, h_2 : Q}{\Gamma \vdash \mathbf{destruct} \ pq \ \mathbf{as} \ h_1 \ h_2 \ \mathbf{in} \ r \Leftarrow R \dashv \Gamma_2}$$

To make the system more checking, it makes sense to turn conjunction positive and get rid of the projections.

# Utilities

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P \dashv \Gamma_2 \quad \Gamma_2, h : P \vdash q \Leftarrow Q \dashv \Gamma_3, h : P}{\Gamma \vdash \mathbf{lemma} \ h : P \ \mathbf{by} \ p \ \mathbf{in} \ q \Leftarrow Q \dashv \Gamma_3}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma \vdash q \Leftarrow P \Rightarrow Q \dashv \Gamma_2 \quad \Gamma_2 \vdash p \Leftarrow P \dashv \Gamma_3}{\Gamma \vdash \mathbf{suffices} \ P \ \mathbf{by} \ q \ \mathbf{in} \ p \Leftarrow Q \dashv \Gamma_2}$$

# Quantifiers

$$\frac{\Gamma \vdash p \Leftarrow P[x := y] \dashv \Gamma'}{\Gamma \vdash \text{pick-any } y \text{ in } p \Leftarrow \forall x : A. P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \forall x : A. P \dashv \Gamma_1 \quad \Gamma_1 \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash \text{stantiate } p \text{ with } e \Rightarrow P[x := e] \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash_{\text{nc}} e \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash p \Leftarrow P[x := e] \dashv \Gamma_2}{\Gamma \vdash \text{witness } e \text{ such that } p \Leftarrow \exists x : A. P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1 \quad \Gamma_1, y : A, h : P[x := y] \vdash q \Leftarrow Q \dashv \Gamma_2}{\Gamma \vdash \text{pick-witness } y \ h \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2}$$

# Equality

$$\frac{\Gamma \vdash e_1 \hat{=} e_2 \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \mathbf{refl} \Leftarrow e_1 =_A e_2 \dashv \Gamma'}$$

$$\frac{\Gamma \vdash q \Rightarrow e_1 =_A e_2 \dashv \Gamma_1 \quad \begin{array}{c} \Gamma_1, x : A \vdash P \Leftarrow \text{prop} \dashv \Gamma_2, x : A \\ \Gamma_2 \vdash p \Leftarrow P[x := e_2] \dashv \Gamma_3 \end{array}}{\Gamma \vdash \mathbf{rewrite} \ q \ \mathbf{at} \ x.P \ \mathbf{in} \ p \Rightarrow P[x := e_1] \dashv \Gamma_3}$$

$$\frac{\Gamma, x : A \vdash p \Leftarrow f =_B g \dashv \Gamma', x : A}{\Gamma \vdash \mathbf{funext} \ x \ \mathbf{in} \ p \Leftarrow f =_{rA \rightarrow B} g \dashv \Gamma'}$$

# Classical logic

$$\frac{\Gamma, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma', h : \neg P}{\Gamma \vdash \text{by-contradiction } h \text{ in } q \Leftarrow P \dashv \Gamma'}$$

$$\frac{\Gamma \vdash P \Leftarrow \text{prop} \dashv \Gamma_1 \quad \Gamma_1, h : \neg P \vdash q \Leftarrow \perp \dashv \Gamma_2, h : \neg P}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q \Rightarrow P \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash_{\text{nc}} \text{choose } p \Rightarrow A \dashv \Gamma'}$$

$$\frac{\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma'}{\Gamma \vdash \text{choose-spec } p \Rightarrow P[x := \text{choose } p] \dashv \Gamma'}$$

# Classical logic

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P [x := y] \vdash q \Leftarrow Q \dashv \Gamma_2, y : A := \text{choose } p, h$$

---

$$\Gamma \vdash \text{choose-witness } y h \text{ for } p \text{ in } q \Leftarrow Q \dashv \Gamma_2$$

$$\Gamma \vdash p \Rightarrow \exists x : A. P \dashv \Gamma_1$$

$$\Gamma, y : A := \text{choose } p, h : P [x := y] \vdash_{\text{nc}} e \Rightarrow B \dashv \Gamma_2, y : A := \text{choose } p, h$$

---

$$\Gamma \vdash_{\text{nc}} \text{choose-witness } y h \text{ for } p \text{ in } e \Rightarrow B \dashv \Gamma_2$$

# Conversion of types

$$\frac{\Gamma \vdash \text{Unit} \equiv \text{Unit} \Rightarrow \text{Type} \dashv \Gamma}{\Gamma \vdash \text{Empty} \equiv \text{Empty} \Rightarrow \text{Type} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \hat{=} B_2 \Rightarrow \text{Type} \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 \Rightarrow \text{Type} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \hat{=} B_2 \Rightarrow \text{Type} \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 \Rightarrow \text{Type} \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \hat{=} B_2 \Rightarrow \text{Type} \dashv \Gamma_2}{\Gamma \vdash r A_1 \rightarrow B_1 \equiv r A_2 \rightarrow B_2 \Rightarrow \text{Type} \dashv \Gamma_2}$$

# Conversion of propositions

$$\frac{}{\Gamma \vdash T \equiv T \dashv \Gamma} \quad \frac{}{\Gamma \vdash \perp \equiv \perp \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \hat{=} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{=} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \hat{=} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{=} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \hat{=} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{=} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2 \dashv \Gamma_2}$$

# Conversion of propositions

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \hat{=} P_2 [x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1. P_1 \equiv \forall x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \hat{=} P_2 [x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1. P_1 \equiv \exists x_2 : A_2. P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{=} A_2 \Rightarrow \text{Type} \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \hat{=} e_2 \Leftarrow A_1 \dashv \Gamma_2 \quad \Gamma_2 \vdash e'_1 \hat{=} e'_2 \Leftarrow A_2}{\Gamma \vdash e_1 =_{A_1} e'_1 \equiv e_2 =_{A_2} e'_2 \dashv \Gamma_3}$$

# Conversion of propositions

$$\frac{\Gamma \vdash N_1 \sim N_2 \dashv \Gamma'}{\Gamma \vdash N_1 \equiv N_2 \dashv \Gamma'}$$

# Conversion of terms

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \hat{\sim} x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma, x : A \vdash f \ x \stackrel{\text{def}}{=} g \ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \equiv g \Leftarrow rA \rightarrow B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash n_1 \sim n_2 \Rightarrow rA \rightarrow B \dashv \Gamma_1 \quad \Gamma_1 \vdash u_1 \stackrel{\text{def}}{=} u_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1 \ u_1 \hat{\sim} \ n_2 \ u_2 \Rightarrow B \dashv \Gamma_2}$$