

T-Axi (declarative)

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Quantities

Quantities:

$r ::= 0 \mid 1 \mid \omega$

ω is the default quantity, so when there's nothing to indicate quantity, it means it's ω .

Quantities are ordered:

$$\omega \leq r$$

$$0 \leq 0$$

$$1 \leq 1$$

Operations on quantities

Addition of quantities:

$$0 + r = r$$

$$\omega + r = \omega$$

$$1 + 0 = 1$$

$$1 + 1 = \omega$$

$$1 + \omega = \omega$$

Multiplication:

$$0 \cdot r = 0$$

$$1 \cdot r = r$$

$$\omega \cdot 0 = 0$$

$$\omega \cdot 1 = \omega$$

$$\omega \cdot \omega = \omega$$

Types

Types:

$$A, B ::= r A \rightarrow B \mid !_r A \mid A \otimes B \mid A \oplus B \mid \text{Unit} \mid \text{Empty}$$

Terms

Terms:

$e ::=$

$x \mid \lambda_r x : A. e \mid e_1 e_2 \mid$
 $\text{box}_r e \mid \text{letbox } x : A = e_1 \text{ in } e_2$
 $(e_1, e_2) \mid \text{letpair } (x, y) = e_1 \text{ in } e_2 \mid$
 $\text{inl}_A e \mid \text{inr}_A e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{letunit}_A \text{ unit} = e_1 \text{ in } e_2 \mid$
 $\text{Empty-elim}_A e \mid$
 $\text{let}_r x : A = e_1 \text{ in } e_2 \mid$
 $\text{choose}_A p \mid \text{choose-witness } x \ h \text{ for } p \text{ in } e$

$\text{choose}_A p$ and $\text{choose-witness } x \ h \text{ for } p \text{ in } e$ are noncomputable terms, whereas all others are computable.

Propositions

Propositions:

$P, Q ::=$

$\top \mid \perp \mid P \Rightarrow Q \mid P \wedge Q \mid P \vee Q \mid$

$\forall x : A. P \mid \exists x : A. P \mid$

$e_1 =_A e_2$

Notations:

$\neg P$ stands for $P \Rightarrow \perp$

$P \Leftrightarrow Q$ stands for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proofterms

Proofterms (P, Q are propositions, e are terms, h are variables):

$p, q ::=$

$h \mid$ **assumption** \mid **trivial** \mid **absurd** p
assume $h : P$ **in** $q \mid$ **apply** $p_1 p_2 \mid$
both $p_1 p_2 \mid$ **and-left** $p \mid$ **and-right** $p \mid$
or-left $p \mid$ **or-right** $p \mid$ **cases** $p_1 p_2 p_3 \mid$
lemma $h : P$ **by** p **in** $q \mid$ **proving** P **by** $p \mid$
suffices P **by** q **in** $p \mid$
pick-any $x : A$ **in** $e \mid$ **instantiate** p **with** $e \mid$
witness e **such that** $p \mid$ **pick-witness** $x h$ **for** p_1 **in** $p_2 \mid$
refl $e \mid$ **rewrite** p_1 **in** $p_2 \mid$ **funext** $x : A$ **in** p
by-contradiction $h : \neg P$ **in** $q \mid$
choose-spec $p \mid$ **choose-witness** $x h$ **for** p **in** q

Contexts

Contexts:

$$\Gamma ::= \cdot \mid \Gamma, r x : A \mid \Gamma, r x : A := e \mid \Gamma, h : P$$

Operations on contexts:

$\Gamma_1 + \Gamma_2$ – context addition

$r \Gamma$ – context scaling

$\Gamma_1 \leq \Gamma_2$ – context subusage/ordering

$|\Gamma|$ – cartesianization

Context addition

$$\cdot + \cdot = \cdot$$

$$(\Gamma_1, r_1 x : A) + (\Gamma_2, r_2 x : A) = (\Gamma_1 + \Gamma_2), (r_1 + r_2) x : A$$

$$(\Gamma_1, r_1 x : A := e) + (\Gamma_2, r_2 x : A := e) = (\Gamma_1 + \Gamma_2), (r_1 + r_2) x : A := e$$

$$(\Gamma_1, p : P) + (\Gamma_2, h : P) = (\Gamma_1 + \Gamma_2), h : P$$

Context scaling

$$s \cdot = \cdot$$

$$s(\Gamma, rx : A) = s\Gamma, (s \cdot r)x : A$$

$$s(\Gamma, rx : A := e) = s\Gamma, (s \cdot r)x : A := e$$

$$s(\Gamma, h : P) = s\Gamma, h : P$$

Context ordering

$$\overline{\cdot \leq \cdot}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1 x : A \leq \Gamma_2, r_2 x : A}$$

$$\frac{\Gamma_1 \leq \Gamma_2 \quad r_1 \leq r_2}{\Gamma_1, r_1 x : A := e \leq \Gamma_2, r_2 x : A := e}$$

$$\frac{\Gamma_1 \leq \Gamma_2}{\Gamma_1, h : P \leq \Gamma_2, h : P}$$

Cartesianization

Cartesianization turns a context into a traditional context that doesn't mention any quantities.

$$\begin{aligned}
 |\cdot| &= \cdot \\
 |\Gamma, r x : A| &= |\Gamma|, x : A \\
 |\Gamma, r x : A := e| &= |\Gamma|, x : A := e \\
 |\Gamma, h : P| &= |\Gamma|, h : P
 \end{aligned}$$

Judgements

Well-formed context judgement: $\Gamma \text{ ctx}$

Cartesian context judgement: $\Gamma \text{ cartesian}$

Well-formed type judgement: $\Gamma \vdash A \text{ type}$

Typing judgement: $\Gamma \vdash e : A$

Well-formed proposition judgement: $\Gamma \vdash P \text{ prop}$

Proof judgement: $\Gamma \vdash p : P$

Sanity checks

We'll set up the system so that:

- If $\Gamma \vdash e : A$, then $|\Gamma| \vdash A$ type.
- If $\Gamma \vdash p : P$, then $\Gamma \vdash P$ prop.
- If $\Gamma \vdash P$ prop, then Γ cartesian.
- If $\Gamma \vdash A$ type, then Γ cartesian.
- If Γ cartesian, then Γ ctx.

Well-formed contexts

$$\frac{}{\cdot \text{ ctx}}$$

$$\frac{\Gamma \text{ ctx} \quad |\Gamma| \vdash A \text{ type} \quad x \notin \Gamma}{\Gamma, r x : A \text{ ctx}}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash e : A \quad x \notin \Gamma}{\Gamma, r x : A := e \text{ ctx}} \text{ DUBIOUS}$$

$$\frac{\Gamma \text{ ctx} \quad |\Gamma| \vdash P \text{ prop} \quad h \notin \Gamma}{\Gamma, h : P \text{ ctx}}$$

Cartesian contexts

$$\frac{\Gamma \text{ ctx} \quad \Gamma = |\Gamma|}{\Gamma \text{ cartesian}}$$

Well-formed types

$$\frac{\Gamma \text{ cartesian}}{\Gamma \vdash \text{Unit type}}$$

$$\frac{\Gamma \text{ cartesian}}{\Gamma \vdash \text{Empty type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash r A \rightarrow B \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash !_r A \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \otimes B \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \oplus B \text{ type}}$$

Using variables

$$\frac{\Gamma \text{ ctx} \quad \Gamma = \Gamma_1, r x : A, \Gamma_2 \quad r \leq 1 \quad \Gamma_1 \leq 0 \Gamma_1 \quad \Gamma_2 \leq 0 \Gamma_2}{\Gamma \vdash x : A}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma = \Gamma_1, r x : A := e, \Gamma_2 \quad r \leq 1 \quad \Gamma_1 \leq 0 \Gamma_1 \quad \Gamma_2 \leq 0 \Gamma_2}{\Gamma \vdash x : A}$$

Functions

$$\frac{\Gamma, rx : A \vdash e : B}{\Gamma \vdash \lambda_r x : A. e : r A \rightarrow B}$$

$$\frac{\Gamma \leq \Gamma_1 + r \Gamma_2 \quad \Gamma_1 \vdash e_1 : r A \rightarrow B \quad \Gamma_2 \vdash e_2 : A}{\Gamma \vdash e_1 e_2 : B}$$

Box

$$\frac{\Gamma \leq_r \Gamma' \quad \Gamma' \vdash e : A}{\Gamma \vdash \text{box}_r e : !_r A}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : !_r A \quad \Gamma_2, rx : A \vdash e_2 : B}{\Gamma \vdash \text{letbox } x : A = e_1 \text{ in } e_2 : B}$$

Empty

$$\frac{|\Gamma| \vdash A \text{ type} \quad \Gamma \vdash e : \text{Empty}}{\Gamma \vdash \text{Empty-elim}_A e : A}$$

Unit

$$\frac{\Gamma \text{ ctx} \quad \Gamma \leq 0 \Gamma}{\Gamma \vdash \text{unit} : \text{Unit}}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : \text{Unit} \quad \Gamma_2 \vdash e_2 : A}{\Gamma \vdash \text{letunit}_A \text{ unit} = e_1 \text{ in } e_2 : A}$$

Products

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash a : A \quad \Gamma_2 \vdash b : B}{\Gamma \vdash (a, b) : A \otimes B}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : A \otimes B \quad \Gamma_2, 1x : A, 1y : B \vdash e_2 : C}{\Gamma \vdash \text{letpair } (x, y) = e_1 \text{ in } e_2 : C}$$

Sums

$$\frac{\Gamma \vdash e : A \quad |\Gamma| \vdash B \text{ type}}{\Gamma \vdash \text{inl}_B e : A \oplus B} \quad \frac{|\Gamma| \vdash A \text{ type} \quad \Gamma \vdash e : B}{\Gamma \vdash \text{inl}_A e : A \oplus B}$$

$$\frac{\Gamma \leq \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e : A \oplus B \quad \Gamma_2 \vdash f : 1 A \rightarrow C \quad \Gamma_2 \vdash g : 1 B \rightarrow C}{\Gamma \vdash \text{case } e \text{ of } (f, g) : C}$$

Let

$$\frac{\Gamma \leq r \Gamma_1 + \Gamma_2 \quad \Gamma_1 \vdash e_1 : A \quad \Gamma_2, r x : A := e_1 \vdash e_2 : B}{\Gamma \vdash \text{let}_r x : A = e_1 \text{ in } e_2 : B}$$

Well-formed propositions

$$\frac{\Gamma \text{ cartesian}}{\Gamma \vdash \top \text{ prop}} \quad \frac{\Gamma \text{ cartesian}}{\Gamma \vdash \perp \text{ prop}}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \Rightarrow Q \text{ prop}}$$

$$\frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \wedge Q \text{ prop}} \quad \frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash P \vee Q \text{ prop}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash P \text{ prop}}{\Gamma \vdash \forall x : A. P \text{ prop}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A \vdash P \text{ prop}}{\Gamma \vdash \exists x : A. P \text{ prop}}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 =_A e_2 \text{ prop}}$$

Substitution

The notation is $P[x := e]$ for substitution in propositions.

Assumptions and implication

$$\frac{\Gamma \vdash P \text{ prop} \quad (h : P) \in \Gamma}{\Gamma \vdash h : P} \quad \frac{\Gamma \vdash P \text{ prop} \quad (h : P) \in \Gamma}{\Gamma \vdash \mathbf{assumption} : P}$$

$$\frac{\Gamma, h : P \vdash q : Q}{\Gamma \vdash \mathbf{assume} \ h : P \ \mathbf{in} \ q : P \Rightarrow Q}$$

$$\frac{\Gamma \vdash q : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \mathbf{apply} \ q \ p : Q}$$

Propositional logic

$$\frac{\Gamma \text{ cartesian}}{\Gamma \vdash \mathbf{trivial} : \top} \quad \frac{\Gamma \vdash Q \text{ prop} \quad \Gamma \vdash p : \perp}{\Gamma \vdash \mathbf{absurd} \ p : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash \mathbf{both} \ p \ q : P \wedge Q}$$

$$\frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \mathbf{and-left} \ pq : P} \quad \frac{\Gamma \vdash pq : P \wedge Q}{\Gamma \vdash \mathbf{and-right} \ pq : Q}$$

$$\frac{\Gamma \vdash p : P \quad \Gamma \vdash Q \text{ prop}}{\Gamma \vdash \mathbf{or-left} \ p : P \vee Q} \quad \frac{\Gamma \vdash P \text{ prop} \quad \Gamma \vdash q : Q}{\Gamma \vdash \mathbf{or-right} \ q : P \vee Q}$$

$$\frac{\Gamma \vdash pq : P \vee Q \quad \Gamma \vdash r_1 : P \Rightarrow R \quad \Gamma \vdash r_2 : Q \Rightarrow R}{\Gamma \vdash \mathbf{cases} \ pq \ r_1 \ r_2 : R}$$

Utilities

$$\frac{\Gamma \vdash p : P \quad \Gamma, h : P \vdash q : Q}{\Gamma \vdash \mathbf{lemma} \ h : P \ \mathbf{by} \ p \ \mathbf{in} \ q : Q}$$

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \mathbf{proving} \ P \ \mathbf{by} \ p : P}$$

$$\frac{\Gamma \vdash pq : P \Rightarrow Q \quad \Gamma \vdash p : P}{\Gamma \vdash \mathbf{suffices} \ P \ \mathbf{by} \ pq \ \mathbf{in} \ p : Q}$$

Quantifiers

$$\frac{\Gamma, x : A \vdash p : P}{\Gamma \vdash \text{pick-any } x : A \text{ in } p : \forall x : A. P}$$

$$\frac{\Gamma \vdash p : \forall x : A. P \quad \Gamma \vdash e : A}{\Gamma \vdash \text{instantiate } p \text{ with } e : P[x := e]}$$

$$\frac{\Gamma \vdash e : A \quad \Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash p : P[x := e]}{\Gamma \vdash \text{witness } e \text{ such that } p : \exists x : A. P}$$

$$\frac{\Gamma \vdash p : \exists x : A. P \quad \Gamma \vdash Q \text{ prop} \quad \Gamma, x : A, h : P \vdash q : Q}{\Gamma \vdash \text{pick-witness } x \text{ } h \text{ for } p \text{ in } q : Q}$$

Equality

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathbf{refl} \ e : e =_A e}$$

$$\frac{\Gamma \vdash q : e_1 =_A e_2 \quad \Gamma, x : A \vdash P \text{ prop} \quad \Gamma \vdash p : P[x := e_2]}{\Gamma \vdash \mathbf{rewrite} \ q \text{ in } p : P[x := e_1]}$$

$$\frac{\Gamma, x : A \vdash p : f \ x =_B g \ x}{\Gamma \vdash \mathbf{funext} \ x : A \text{ in } p : f =_{rA \rightarrow B} g}$$

Classical Logic

$$\frac{\Gamma, h : \neg P \vdash q : \perp}{\Gamma \vdash \text{by-contradiction } h : \neg P \text{ in } q : P}$$

$$\frac{|\Gamma| \vdash p : \exists x : A. P}{\Gamma \vdash \text{choose } p : A}$$

$$\frac{\Gamma \vdash p : \exists x : A. P}{\Gamma \vdash \text{choose-spec } p : P [x := \text{choose } p]}$$

$$\frac{\Gamma \vdash p : \exists x : A. P \quad \Gamma \vdash Q \text{ prop} \quad \Gamma, x : A := \text{choose } p, h : P \vdash q : Q}{\Gamma \vdash \text{choose-witness } x \text{ for } p \text{ in } q : Q}$$

$$\frac{|\Gamma| \vdash p : \exists x : A. P \quad |\Gamma| \vdash B \text{ type} \quad \Gamma, x : A := \text{choose } p, h : P \vdash e : B}{\Gamma \vdash \text{choose-witness } x \text{ for } p \text{ in } e : B}$$