

Hint-based unification for STLC

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Terms, contexts and judgements

Terms:

$e ::=$

$x \mid (e : H) \mid$
 $\lambda x. e \mid e_1 \ e_2 \mid$
 $(e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid$
 $\text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid$
 $\text{unit} \mid \text{exfalse } e$

Contexts:

$\Gamma ::= \cdot \mid \Gamma, x : H$

Judgements:

$\Gamma \vdash e \leftarrow H \Rightarrow H' \dashv \Gamma'$ – in context Γ , term e checks with hint H
 and infers hint H' in output context Γ'

Output contexts – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2} \text{VAR}$$

$$\frac{\Gamma \vdash e \leftarrow A \sqcup B \Rightarrow C \dashv \Gamma'}{\Gamma \vdash (e : A) \leftarrow B \Rightarrow C \dashv \Gamma'} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \leftarrow \text{hint}(e) \Rightarrow A \dashv \Gamma' \quad e \text{ constructor}}{\Gamma \vdash e \leftarrow ? \Rightarrow A \dashv \Gamma'} \text{HOLE}$$

Output contexts – type-directed rules

$$\frac{\Gamma, x : A \vdash e \leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x. e \leftarrow A \rightarrow B \Rightarrow A' \rightarrow B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \leftarrow ? \rightarrow B \Rightarrow A \rightarrow B' \dashv \Gamma' \quad \Gamma' \vdash a \leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f \ a \leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \leftarrow A \times ? \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \text{outl } e \leftarrow A \Rightarrow A' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow ? \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \text{outr } e \leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Output contexts – type-directed rules

$$\frac{\Gamma \vdash e \leftarrow A \Rightarrow A' \vdash \Gamma'}{\Gamma \vdash \text{inl } e \leftarrow A + B \Rightarrow A' + B \vdash \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow B \Rightarrow B' \vdash \Gamma'}{\Gamma \vdash \text{inr } e \leftarrow A + B \Rightarrow A + B' \vdash \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow ? + ? \Rightarrow A + B \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash f \leftarrow A \rightarrow C_1 \Rightarrow A' \rightarrow C_2 \vdash \Gamma_2 \\ \Gamma_2 \vdash g \leftarrow B \rightarrow C_2 \Rightarrow B' \rightarrow C_3 \vdash \Gamma_3 \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \leftarrow C_1 \Rightarrow C_3 \vdash \Gamma_3}$$

$$\frac{}{\Gamma \vdash \text{unit} \leftarrow \mathbf{1} \Rightarrow \mathbf{1} \vdash \Gamma} \quad \frac{\Gamma \vdash e \leftarrow \mathbf{0} \Rightarrow \mathbf{0} \vdash \Gamma'}{\Gamma \vdash \text{exfalse } e \leftarrow A \Rightarrow A \vdash \Gamma'}$$

Order on contexts

$$\overline{\cdot \sqsubseteq \cdot}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \quad A \sqsubseteq B}{\Gamma_1, x : A \sqsubseteq \Gamma_2, x : B}$$

Order on contexts – properties

\sqsubseteq is a partial order, whose structure is inherited from the order on hints.

- Reflexivity: $\Gamma \sqsubseteq \Gamma$
- Transitivity: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_3 \implies \Gamma_1 \sqsubseteq \Gamma_3$
- Weak antisymmetry: $\Gamma_1 \sqsubseteq \Gamma_2 \implies \Gamma_2 \sqsubseteq \Gamma_1 \implies \Gamma_1 = \Gamma_2$

Metatheory

If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, then:

- (Information increase) $\Gamma \sqsubseteq \Gamma'$ (proof: induction)
- (Information increase) $A \sqsubseteq B$ (proof: induction)
- (Context squeeze) If $\Gamma \sqsubseteq \Delta \sqsubseteq \Gamma'$, then $\Delta \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: induction)
- (Hint squeeze) If $A \sqsubseteq A'$ and $A' \sqsubseteq B$, then $\Gamma \vdash e \Leftarrow A' \Rightarrow B \dashv \Gamma'$ (proof: induction)

Metatheory – decidability and determinism

- (Decidability) For Γ, e, A it is decidable whether there exist B and Γ' such that $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$ (proof: the rules are literally the algorithm)
- (Determinism) If $\Gamma \vdash e \Leftarrow A \Rightarrow B_1 \dashv \Gamma_1$ and $\Gamma \vdash e \Leftarrow A \Rightarrow B_2 \dashv \Gamma_2$, then $B_1 = B_2$ and $\Gamma_1 = \Gamma_2$ (proof: induction)

Metatheory – soundness

(Soundness) If $\Gamma \vdash e \Leftarrow A \Rightarrow B \dashv \Gamma'$, B is a type and Γ' only contains types, then $\Gamma' \vdash e : B$.

Output contexts – abridged basic rules

$$\overline{\Gamma_1, x : A, \Gamma_2 \vdash x \Leftarrow B \Rightarrow A \sqcup B \dashv \vdash \Gamma_1, x : A \sqcup B, \Gamma_2}^{\text{VAR}}$$

$$\frac{e \Leftarrow A \sqcup B \Rightarrow C}{(e : A) \Leftarrow B \Rightarrow C}^{\text{ANNOT}}$$

$$\frac{e \Leftarrow \text{hint}(e) \Rightarrow A \quad e \text{ constructor}}{e \Leftarrow ? \Rightarrow A}^{\text{HOLE}}$$

Output contexts – abridged type-directed rules

$$\frac{\Gamma, x : A \vdash e \leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x. e \leftarrow A \rightarrow B \Rightarrow A' \rightarrow B' \dashv \Gamma'}$$

$$\frac{f \leftarrow ? \rightarrow B \Rightarrow A \rightarrow B' \quad a \leftarrow A \Rightarrow A'}{f \ a \leftarrow B \Rightarrow B'}$$

$$\frac{a \leftarrow A \Rightarrow A' \quad b \leftarrow B \Rightarrow B'}{(a, b) \leftarrow A \times B \Rightarrow A' \times B'}$$

$$\frac{e \leftarrow A \times ? \Rightarrow A' \times B}{\text{outl } e \leftarrow A \Rightarrow A'} \quad \frac{e \leftarrow ? \times B \Rightarrow A \times B'}{\text{outr } e \leftarrow B \Rightarrow B'}$$

Output contexts – abridged type-directed rules

$$\frac{e \Leftarrow A \Rightarrow A'}{\text{inl } e \Leftarrow A + B \Rightarrow A' + B}$$

$$\frac{e \Leftarrow B \Rightarrow B'}{\text{inr } e \Leftarrow A + B \Rightarrow A + B'}$$

$$\frac{e \Leftarrow ? + ? \Rightarrow A + B \quad \begin{array}{l} f \Leftarrow A \rightarrow C_1 \Rightarrow A' \rightarrow C_2 \\ g \Leftarrow B \rightarrow C_2 \Rightarrow B' \rightarrow C_3 \end{array}}{\text{case } e \text{ of } (f, g) \Leftarrow C_1 \Rightarrow C_3}$$

$$\frac{}{\text{unit } \Leftarrow \mathbf{1} \Rightarrow \mathbf{1}}$$

$$\frac{e \Leftarrow \mathbf{0} \Rightarrow \mathbf{0}}{\text{exfalse } e \Leftarrow A \Rightarrow A}$$

Types, holes, terms and contexts

Holes:

$$H ::= \alpha \mid H_1 \rightarrow H_2 \mid H_1 \times H_2 \mid H_1 + H_2 \mid \mathbf{1} \mid \mathbf{0}$$

Terms:

$$e ::=$$
$$\begin{aligned} & x \mid (e : H) \mid \\ & \lambda x. e \mid e_1 \ e_2 \mid \\ & (e_1, e_2) \mid \text{outl } e \mid \text{outr } e \mid \\ & \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } (e_1, e_2) \mid \\ & \text{unit} \mid \text{exfalse } e \end{aligned}$$

Note: the terms are the same as in STLC with Hints.

Contexts and judgements

Contexts:

$$\Gamma ::= \cdot \mid \Gamma, x : A \mid \Gamma, \alpha \mid \Gamma, \alpha := H$$

We introduce unification variables, denoted with Greek letter $(\alpha, \beta, \gamma, \dots)$. When extending the context with a unification variable, it is set to some hint, which at the beginning will be just $?$.

Judgements:

$\Gamma \vdash e \xleftarrow{\text{blue}} H \xrightarrow{\text{red}} H' \dashv \Gamma'$ – in context Γ , term e checks with hint H and infers hint H' in output context Γ'

Unification – basic rules

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x \leftarrow B \Rightarrow A \sqcup B \dashv \Gamma_1, x : A \sqcup B, \Gamma_2} \text{VAR}$$

$$\frac{\Gamma \vdash e \leftarrow A \sqcup B \Rightarrow C \dashv \Gamma'}{\Gamma \vdash (e : A) \leftarrow B \Rightarrow C \dashv \Gamma'} \text{ANNOT}$$

$$\frac{\Gamma \vdash e \leftarrow \text{hint}(e) \Rightarrow A \dashv \Gamma' \quad e \text{ constructor}}{\Gamma \vdash e \leftarrow ? \Rightarrow A \dashv \Gamma'} \text{HOLE}$$

Unification – type-directed rules

$$\frac{\Gamma, x : A \vdash e \leftarrow B \Rightarrow B' \dashv \Gamma', x : A'}{\Gamma \vdash \lambda x. e \leftarrow A \rightarrow B \Rightarrow A' \rightarrow B' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \leftarrow ? \rightarrow B \Rightarrow A \rightarrow B' \dashv \Gamma' \quad \Gamma' \vdash a \leftarrow A \Rightarrow A' \dashv \Gamma''}{\Gamma \vdash f \ a \leftarrow B \Rightarrow B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash a \leftarrow A \Rightarrow A' \dashv \Gamma' \quad \Gamma' \vdash b \leftarrow B \Rightarrow B' \dashv \Gamma''}{\Gamma \vdash (a, b) \leftarrow A \times B \Rightarrow A' \times B' \dashv \Gamma''}$$

$$\frac{\Gamma \vdash e \leftarrow A \times ? \Rightarrow A' \times B \dashv \Gamma'}{\Gamma \vdash \text{outl } e \leftarrow A \Rightarrow A' \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow ? \times B \Rightarrow A \times B' \dashv \Gamma'}{\Gamma \vdash \text{outr } e \leftarrow B \Rightarrow B' \dashv \Gamma'}$$

Unification – type-directed rules

$$\frac{\Gamma \vdash e \leftarrow A \Rightarrow A' \vdash \Gamma'}{\Gamma \vdash \text{inl } e \leftarrow A + B \Rightarrow A' + B \vdash \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow B \Rightarrow B' \vdash \Gamma'}{\Gamma \vdash \text{inr } e \leftarrow A + B \Rightarrow A + B' \vdash \Gamma'}$$

$$\frac{\Gamma \vdash e \leftarrow ? + ? \Rightarrow A + B \vdash \Gamma_1 \quad \begin{array}{l} \Gamma_1 \vdash f \leftarrow A \rightarrow C_1 \Rightarrow A' \rightarrow C_2 \vdash \Gamma_2 \\ \Gamma_2 \vdash g \leftarrow B \rightarrow C_2 \Rightarrow B' \rightarrow C_3 \vdash \Gamma_3 \end{array}}{\Gamma \vdash \text{case } e \text{ of } (f, g) \leftarrow C_1 \Rightarrow C_3 \vdash \Gamma_3}$$

$$\frac{}{\Gamma \vdash \text{unit} \leftarrow \mathbf{1} \Rightarrow \mathbf{1} \vdash \Gamma} \quad \frac{\Gamma \vdash e \leftarrow \mathbf{0} \Rightarrow \mathbf{0} \vdash \Gamma'}{\Gamma \vdash \text{exfalse } e \leftarrow A \Rightarrow A \vdash \Gamma'}$$