T-Axi (algorithmic)

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Decrement variable in context

$$
\begin{aligned}
\cdot - x &= \textbf{undefined} \\
(\Gamma, r\,x : A) - x &= \Gamma, (r-1)\,x : A \\
(\Gamma, r\,y : A) - x &= \Gamma - x, r\,y : A \\
(\Gamma, r\,x : A := e) - x &= \Gamma, (r-1)\,x : A := e \\
(\Gamma, r\,y : A := e) - x &= \Gamma - x, r\,y : A := e \\
(\Gamma, h : P) - x &= \Gamma - x, h : P
\end{aligned}
$$

Using variables

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \Rightarrow A \dashv \Gamma - x}$$

Functions

$$\frac{\Gamma, r\, x : A \vdash e \Leftarrow B \dashv \Gamma', r'\, x : A \qquad r' \leq 0}{\Gamma \vdash \lambda x.\, e \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash f \Rightarrow r\, A \to B \dashv \Gamma_1 \qquad \Gamma_1 / r = (\Gamma_2, \Gamma_3) \qquad \Gamma_2 \vdash a \Leftarrow A \dashv \Gamma_4}{\Gamma \vdash f\ a \Rightarrow B \dashv \Gamma_3 + r\, \Gamma_4}$$

Useful shorthand (when not enough space):

$$\frac{\Gamma, r\, x : A \vdash e \Leftarrow B \dashv \Gamma', 0\, x : A}{\Gamma \vdash \lambda x.\, e \Leftarrow r\, A \to B \dashv \Gamma'}$$

Box

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash e \Leftarrow A \dashv \Gamma_3}{\Gamma \vdash \mathtt{box}\ e \Leftarrow\ !_r A \dashv \Gamma_2 + r\,\Gamma_3}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow\ !_r A \dashv \Gamma_1 \quad \Gamma_1, r\,x : A \vdash e_2 \Rightarrow B \dashv \Gamma_2, r'\,x : A \quad r' \leq 0}{\Gamma \vdash \mathtt{letbox}\ x = e_1\ \mathtt{in}\ e_2 \Rightarrow B \dashv \Gamma_2}$$

Empty

$$\frac{\Gamma \vdash e \Leftarrow \mathtt{Empty} \dashv \Gamma'}{\Gamma \vdash \mathtt{Empty\text{-}elim}\ e \Leftarrow A \dashv \Gamma'}$$

Unit

$$\frac{}{\Gamma \vdash \texttt{unit} \Leftarrow \texttt{Unit} \dashv \Gamma}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \texttt{Unit} \dashv \Gamma_1 \quad \Gamma_1 \vdash e_2 \Rightarrow A \dashv \Gamma_2}{\Gamma \vdash \texttt{letunit unit} = e_1 \texttt{ in } e_2 \Rightarrow A \dashv \Gamma_2}$$

Products

$$\frac{\Gamma \vdash a \Leftarrow A \dashv \Gamma_1 \quad \Gamma_1 \vdash b \Leftarrow B \dashv \Gamma_2}{\Gamma \vdash (a, b) \Leftarrow A \otimes B \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow A \otimes B \dashv \Gamma_1 \quad \Gamma_1, 1\, x : A, 1\, y : B \vdash e_2 \Rightarrow C \dashv \Gamma_2, 0\, x : A, 0\, y : B}{\Gamma \vdash \texttt{letpair}\ (x, y) = e_1\ \texttt{in}\ e_2 \Rightarrow C \dashv \Gamma_2}$$

Sums

$$\frac{\Gamma \vdash e \Leftarrow A \dashv \Gamma'}{\Gamma \vdash \texttt{inl}\ e \Leftarrow A \oplus B \dashv \Gamma'} \qquad \frac{\Gamma \vdash e \Leftarrow B \dashv \Gamma'}{\Gamma \vdash \texttt{inr}\ e \Leftarrow A \oplus B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \Gamma_1 \vdash f \Leftarrow 1\, A \to C \dashv \Gamma_2 \quad \Gamma_1 \vdash g \Leftarrow 1\, B \to C \dashv \Gamma_3}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Leftarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

$$\frac{\Gamma \vdash e \Rightarrow A \oplus B \dashv \Gamma_1 \quad \Gamma_1 \vdash f \Rightarrow 1\, A \to C \dashv \Gamma_2 \quad \Gamma_1 \vdash g \Rightarrow 1\, B \to C \dashv \Gamma_3}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ (f, g) \Rightarrow C \dashv \Gamma_2 \sqcup \Gamma_3}$$

Let

$$\frac{\Gamma/r = (\Gamma_1, \Gamma_2) \quad \Gamma_1 \vdash e_1 \Rightarrow A \dashv \Gamma_3 \quad \Gamma_2 + r\,\Gamma_3, r\,x : A \vdash e_2 \Rightarrow B \dashv \Gamma_4, 0\,x}{\Gamma \vdash \mathtt{let}_r\ x = e_1\ \mathtt{in}\ e_2 \Rightarrow B \dashv \Gamma_4}$$

Conversion of propositions

$$\frac{}{\Gamma \vdash \top \equiv \top \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \bot \equiv \bot \dashv \Gamma}$$

$$\frac{\Gamma \vdash P_1 \hat{\equiv} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{\equiv} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \Rightarrow Q_1 \equiv P_2 \Rightarrow Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \hat{\equiv} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{\equiv} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \wedge Q_1 \equiv P_2 \wedge Q_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash P_1 \hat{\equiv} P_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash Q_1 \hat{\equiv} Q_2 \dashv \Gamma_2}{\Gamma \vdash P_1 \vee Q_1 \equiv P_2 \vee Q_2 \dashv \Gamma_2}$$

Conversion of propositions

$$\frac{\Gamma \vdash A_1 \hat{\equiv} A_2 \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \hat{\equiv} P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \forall x_1 : A_1.\, P_1 \equiv \forall x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{\equiv} A_2 \dashv \Gamma_1 \quad \Gamma_1, x_1 : A_1 \vdash P_1 \hat{\equiv} P_2\,[x_2 := x_1] \dashv \Gamma_2, x_1 : A_1}{\Gamma \vdash \exists x_1 : A_1.\, P_1 \equiv \exists x_2 : A_2.\, P_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \hat{\equiv} A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash e_1 \hat{\equiv} e_2 \Leftarrow A_1 \dashv \Gamma_2 \quad \Gamma_2 \vdash e_2 \hat{\equiv} e_2' \Leftarrow A_1 \dashv \Gamma_3}{\Gamma \vdash e_1 =_{A_1} e_1' \equiv e_2 =_{A_2} e_2' \dashv \Gamma_3}$$

Conversion of propositions

$$\frac{\Gamma \vdash N_1 \sim N_2 \dashv \Gamma'}{\Gamma \vdash N_1 \equiv N_2 \dashv \Gamma'}$$

Conversion of types

$$\frac{}{\Gamma \vdash \texttt{Unit} \equiv \texttt{Unit} \dashv \Gamma} \qquad \frac{}{\Gamma \vdash \texttt{Empty} \equiv \texttt{Empty} \dashv \Gamma}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash A_1 \otimes B_1 \equiv A_2 \otimes B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash A_1 \oplus B_1 \equiv A_2 \oplus B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma_1 \quad \Gamma_1 \vdash B_1 \triangleq B_2 \dashv \Gamma_2}{\Gamma \vdash r\, A_1 \to B_1 \equiv r\, A_2 \to B_2 \dashv \Gamma_2}$$

$$\frac{\Gamma \vdash A_1 \triangleq A_2 \dashv \Gamma'}{\Gamma \vdash\, !_r\, A_1 \equiv\, !_r\, A_2 \dashv \Gamma'}$$

Conversion of terms

$$\frac{\Gamma(x) = A}{\Gamma \vdash x \mathrel{\hat{\sim}} x \Rightarrow A \dashv \Gamma}$$

$$\frac{\Gamma, x : A \vdash f\ x \mathrel{\hat{\equiv}} g\ x \Leftarrow B \dashv \Gamma', x : A}{\Gamma \vdash f \ \equiv\ g \Leftarrow r\, A \to B \dashv \Gamma'}$$

$$\frac{\Gamma \vdash n_1 \ \sim\ n_2 \Rightarrow r\, A \to B \dashv \Gamma_1 \quad \Gamma_1 \vdash u_1 \mathrel{\hat{\equiv}} u_2 \Leftarrow A \dashv \Gamma_2}{\Gamma \vdash n_1\ u_1 \mathrel{\hat{\sim}} n_2\ u_2 \Rightarrow B \dashv \Gamma_2}$$