Formulas in Applied Linear Optimization in KTH

## An iteration in the simplex method

1. Compute simplex multipliers $y$ and reduced costs $s$ from
$$\begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix} \begin{pmatrix} y \\ s_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}.$$
2. If $(s_N)_t < 0$, compute search direction $p$ from
$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} 0 \\ e_t \end{pmatrix}.$$
3. Compute maximal steplength $\alpha_{\max}$ and limiting constraint $r$
$$\alpha_{\max} = \min_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}, \quad r = \operatorname{argmin}_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}.$$
4. Let $x = x + \alpha_{\max} p$.
5. Replace $(x_N)_t = 0$ by $(x_B)_r = 0$ among active constraints.

## An iteration in the simplex method, alternatively

1. Compute simplex multipliers $y$ and reduced costs $s$ from
$$B^T y = c_B, \quad s_N = c_N - N^T y.$$
2. If $(s_N)_t < 0$, compute search direction $p$ from
$$p_N = e_t, \quad Bp_B = -N_t.$$
3. Compute maximal steplength $\alpha_{\max}$ and limiting constraint $r$
$$\alpha_{\max} = \min_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}, \quad r = \operatorname{argmin}_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}.$$
4. Let $x = x + \alpha_{\max} p$.
5. Replace $(x_N)_t = 0$ by $(x_B)_r = 0$ among active constraints.

## The simplex method

- Primal simplex method described here. The basic solution $x$ is feasible to ($PLP$). The corresponding $y$ and $s$ fulfill $A^T y + s = c$ and complementarity slack $x^T s = 0$. Optimal when $s \geq 0$.
- The simplex method needs an initial basic feasible solution. Can be obtained by a Phase I problem.
- The simplex method terminates if $\alpha_{\max} > 0$ in all iterations. The case $\alpha_{\max} = 0$ may occur if more than $n$ constraints are active. Referred to as degeneracy. An anticycling strategy is required to guarantee convergence if $\alpha_{\max} = 0$.
- The simplex method takes a polynomial number of iterations "in general".
- One may construct "evil" (pathological) problems where the simplex method requires an exponential number of iterations or cycles.

Formulas in Applied Linear Optimization in KTH

## An iteration in the simplex method

- Compute simplex multipliers $y$ and reduced costs $s$ from
$$B^T y = c_B, \quad s_N = c_N - N^T y.$$
- If $(s_N)_t < 0$, compute search direction $p$ from
$$p_N = e_t, \quad Bp_B = -N_t.$$
- Compute maximal steplength $\alpha_{\max}$ and limiting constraint $r$ from
$$\alpha_{\max} = \min_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}, \quad r = \operatorname*{argmin}_{i:(p_B)_i<0} \frac{(x_B)_i}{-(p_B)_i}.$$
- Let $x = x + \alpha_{\max} p$.
- Replace $(x_N)_t = 0$ by $(x_B)_r = 0$ among active constraints.

## Linear programming, optimality conditions

LP problems:

$$(PLP) \quad \begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \qquad (DLP) \quad \begin{array}{ll} \max & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

Partition $A = (B\ N)$. Let $x_N = 0$ and $s_B = 0$. The optimality conditions become

$$\begin{aligned} &Bx_B = b, \\ &B^T y = c_B, \quad N^T y + s_N = c_N, \\ &x_B \geq 0, \quad s_N \geq 0. \end{aligned}$$

In the simplex method, all conditions except $s_N \geq 0$ are fulfilled throughout.

## Sensitivity analysis

What happens if problem data is changed? For example if $b$ and/or $c$ are/is changed to $\tilde{b}$ and $\tilde{c}$ respectively.

For a given basis feasible solution the answer can be given "immediately" as long as the basis gives primal and dual feasibility, respectively.

$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{x}_B \\ \tilde{x}_N \end{pmatrix} = \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix}, \quad \tilde{x}_B \geq 0,$$
$$\begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \tilde{s}_N \end{pmatrix} = \begin{pmatrix} \tilde{c}_B \\ \tilde{c}_N \end{pmatrix}, \quad \tilde{s}_N \geq 0.$$

Formulas in Applied Linear Optimization in KTH

## Primal simplex method and dual simplex method

$$(PLP) \quad \begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0. \end{array} \qquad (DLP) \quad \begin{array}{ll} \max & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

Partition $A = (B \; N)$. Let $x_N = 0$ and $s_B = 0$. The optimality conditions become

$$\begin{aligned} B x_B &= b, \\ B^T y &= c_B, \quad N^T y + s_N = c_N, \\ x_B &\geq 0, \quad s_N \geq 0. \end{aligned}$$

### Primal simplex method

All conditions except $s_N \geq 0$ are fulfilled throughout.

### Dual simplex method

All conditions except $x_B \geq 0$ are fulfilled throughout.

## An iteration in the dual simplex method

1. Compute primal basic solution $x$ from
$$\begin{pmatrix} B & N \\ 0 & I \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$
2. If $(x_B)_t < 0$, compute search direction $q$, $\eta_N$ from
$$\begin{pmatrix} B^T & 0 \\ N^T & I \end{pmatrix} \begin{pmatrix} q \\ \eta_N \end{pmatrix} = - \begin{pmatrix} e_t \\ 0 \end{pmatrix}.$$
3. Compute maximal steplength $\alpha_{\max}$ and limiting constraint $r$
$$\alpha_{\max} = \min_{i:(\eta_N)_i < 0} \frac{(s_N)_i}{-(\eta_N)_i}, \qquad r = \operatorname*{argmin}_{i:(\eta_N)_i < 0} \frac{(s_N)_i}{-(\eta_N)_i}.$$
4. Let $y = y + \alpha_{\max} q$, $s_B = s_B + \alpha_{\max} e_t$, $s_N = s_N + \alpha_{\max} \eta_N$.
5. Replace $(s_B)_t = 0$ by $(s_N)_r = 0$ among active constraints.