Beam Mechanics

Wind & Loading

Wind profile (power law):

$$U(z) = U_{\text{ref}} \left(\frac{z}{z_{\text{ref}}} \right)^{\epsilon}$$

$\epsilon \approx 0.1 - 0.3$ (terrain dep.)
Distributed drag:

$$q(z) = \frac{1}{2} \rho C_D(z) D(z) U(z)^2$$

Nacelle load:

$$q_n(z) = \begin{cases} \frac{F_t}{h_n}, & z \in [z_t, z_t + h_n] \\ 0, & \text{else} \end{cases}$$

Shear & Moment

Differential relations:

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -q(z), \quad \frac{\mathrm{d}M}{\mathrm{d}z} = -V(z)$$

Integration:

$$V(z) = \int_z^H q(\xi)\,\mathrm{d}\xi, \quad M(z) = \int_z^H V(\xi)\,\mathrm{d}\xi$$

Deflection

Curvature:

$$\kappa(z) = \frac{M(z)}{EI(z)}$$

Slope:

$$\theta(z) = \int_0^z \kappa(\xi)\,\mathrm{d}\xi$$

Deflection (BC: $w(0) = 0$, $\theta(0) = 0$):

$$w(z) = \int_0^z \theta(\xi)\,\mathrm{d}\xi$$

Integration method: trapezoidal (cumtrapz)

Stress

Bending stress (at base):

$$\sigma_b = \frac{Mc}{I}$$

$c = R_{\text{outer}}$ for tubes
Principal stresses (uniaxial):

$$\sigma_1 = \sigma_b, \quad \sigma_2 = 0, \quad \tau_{\max} = \frac{\sigma_b}{2}$$

Directional factor:

$$\sigma_{\text{case2}} = \sigma_{\text{case1}} \cos(\Delta\psi)$$

Max occurs at: $z = 0$ (tower base)

Aerodynamics

Basic Parameters

Tip-speed ratio (fundamental): $\lambda = \frac{\omega R}{V_\infty}$ (Typical $\lambda = 8$–13)
Angular velocity: $\omega = \frac{2\pi n}{60}$ [rad/s] (n in RPM)
Local TSR: $\lambda_r = \lambda r / R$

Induction Factors

Axial induction: $a = \frac{1}{3}$
Tangential induction:

$$a' = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\lambda_r^2} a(1-a)}$$

Flow Angles

Inflow angle:

$$\phi = \tan^{-1}\left(\frac{1-a}{(1+a')\lambda_r} \right)$$

Angle of attack: $\alpha = \phi - (\theta + \beta)$ (θ = twist, β = pitch)
Relative velocity: $V_{\mathbf{rel}} = \sqrt{(V_\infty(1-a))^2 + (\omega r(1+a'))^2}$

Force Coefficients

Normal:

$$C_n = C_L \cos\phi + C_D \sin\phi$$

Tangential:

$$C_t = C_L \sin\phi - C_D \cos\phi$$

Elemental Loads

Thrust:

$$\mathrm{d}T = \frac{1}{2} \rho V_{\text{rel}}^2 c C_n$$

Torque:

$$\mathrm{d}Q = \frac{1}{2} \rho V_{\text{rel}}^2 c C_t r$$

Power:

$$\mathrm{d}P = \mathrm{d}Q \cdot \omega$$

Total Loads

Integration (trapezoidal):

$$T = B \int_{r_h}^{R} \mathrm{d}T\,\mathrm{d}r = B \cdot \mathrm{trapz}(r, \mathrm{d}T)$$

$$P = B \int_{r_h}^{R} \mathrm{d}P\,\mathrm{d}r = B \cdot \mathrm{trapz}(r, \mathrm{d}P)$$

B = number of blades

Performance Coefficients

Power coefficient (key metric):

$$C_P = \frac{P}{\frac{1}{2} \rho A V_\infty^3}$$

Thrust coefficient:

$$C_T = \frac{T}{\frac{1}{2} \rho A V_\infty^2}$$

Swept area: $A = \pi R^2$ *Betz limit:* $C_{P,\max} = 0.5926$ (theoretical max)
Power from C_P: $P = C_P \left(\frac{1}{2} \rho A V_\infty^3 \right)$

Material Strength

Process: (1) Calculate max stress $\sigma_{\max}$, (2) Obtain material properties S_y, S_{ut}, (3) Compute n = strength/stress, (4) Check $n \geq 2.0$

Static Failure

Max Normal Stress:

$$n_{\mathbf{MNST}} = \frac{S_y}{\sigma_{\max}}$$

Max Shear Stress:

$$n_{\text{MSST}} = \frac{S_y/2}{\sigma_{\max}/2} = \frac{S_y}{\sigma_{\max}}$$

Distortion Energy:

$$n_{\mathbf{DET}} = \frac{S_y}{\sqrt{\sigma_{\max}^2 + 3\tau_{\max}^2}}$$

Equivalent stress:

$$\sigma_{\text{eq}} = \sqrt{\sigma^2 + 3\tau^2}$$

For uniaxial: all yield $n = S_y / \sigma_{\max}$

Fatigue Analysis

Mean stress:

$$\sigma_m = \frac{1}{2}(\sigma_{\max} + \sigma_{\min})$$

Alternating stress:

$$\sigma_a = \frac{1}{2} |\sigma_{\max} - \sigma_{\min}|$$

Base endurance:

$$S_e' = C_{se} S_{ut}$$

Modified endurance:

$$S_e = C_{\text{size}} C_{\text{surf}} C_{\text{rel}} S_e'$$

Goodman safety factor:

$$n_G = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

Requires: $n_G \geq 2.0$

Other Relations

Bernoulli (Incompressible)

Along a streamline, steady, inviscid:

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \mathbf{const}$$

Equivalent energy form:

$$p + \frac{1}{2} \rho V^2 + \rho g z = \mathbf{const}$$

Reynolds Number

Definition:

$$\mathrm{Re} = \frac{\rho V_{\text{rel}} D}{\mu}$$

Key Constants

Typical values:
- $C_{se} = 0.5$ (endurance factor)
- $C_{\text{size}} \approx 0.9$ (size factor)
- $C_{\text{surf}} \approx 0.7$ (surface factor)
- $C_{\text{rel}} = 1.0$ (reliability)

Air Properties

Typical values:
- $\rho = 1.225$ kg/m^3 (sea level)
- $\mu = 1.81 \times 10^{-5}$ Pa·s

Power Relations

Power from C_P:

$$P = C_P \cdot \frac{1}{2} \rho A V_\infty^3$$

Mohr's Circle

Radius:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

Center:

$$C = \frac{\sigma_x + \sigma_y}{2}$$

Second Moment of Area

Solid cylinder:

$$I = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$$

Ring (hollow):

$$I = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{4}(R_o^4 - R_i^4)$$

Rectangle:

$$I = \frac{bh^3}{12}$$

Material Properties

- S_y = yield strength [Pa]
- S_{ut} = ultimate strength [Pa]
- E = elastic modulus [Pa]
- I = second moment of area [m^4]

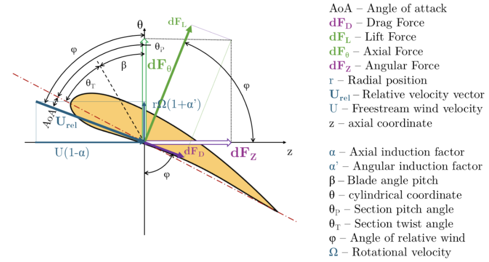
Typical steel: $S_y \approx 345$ MPa, $S_{ut} \approx 450$ MPa, $E \approx 200$ GPa

Assumptions

- Steady-state, linear-elastic
- Pure bending (uniaxial)
- Rigid nacelle
- No blade interference
- **Lumped/Actuator-disk (aero): uniform axisymmetric inflow/outflow**
- **Inviscid, incompressible, steady far-field; pressure jump at disk**
- **No swirl in far-wake (for basic BEM); momentum applies**
- **Betz-optimal:** $a = 1/3$ (upper bound for ideal power extraction)

Visual Notes

Blade element schematic (r, $\mathrm{d}r$, forces)



Tower shear, moment, slope, deflection distributions