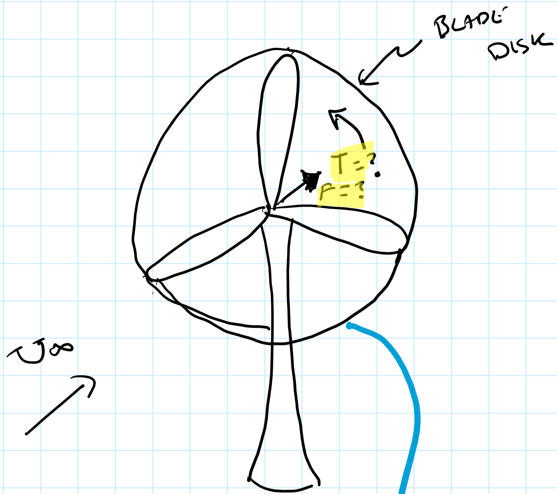
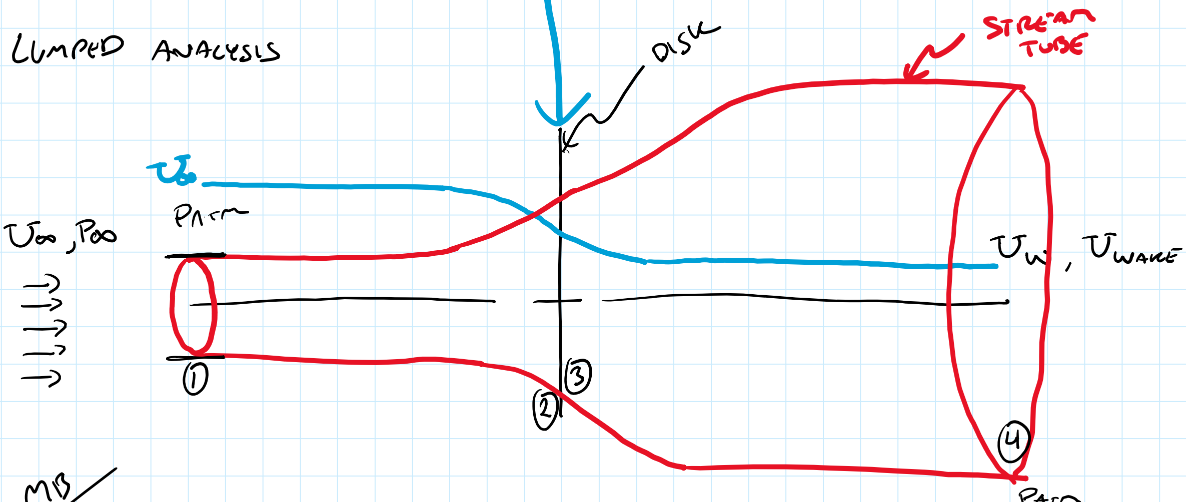
2025.10.07 - Wind Turbine Lumped analysis

Tuesday, October 7, 2025   9:58 AM



LUMPED ANALYSIS



MB

$$\rho A_\infty V_\infty = \rho A_D V_D = \rho A_W V_W$$

LM

$$\sum \vec{F} = \cancel{\frac{\partial}{\partial t}\left(\int_{CV} \rho \vec{V}\, dV\right)} + \int_{CS} \vec{V} \rho (\vec{V}\cdot\hat{n})\, dA$$

ASSUME
- STEADY

$$\sum \vec{F} = \cancel{\frac{\partial(m\vec{V})}{\partial t}}\bigg|_{CV} + \sum_{out} \dot{m}\vec{V} - \sum_{in} \dot{m}\vec{V}$$

✓ LUMPED FORM

$$\boxed{\sum F_x = \cancel{\frac{\partial}{\partial t}\left(\int_{CV} \rho u \, d\forall\right)} + \int_{CS} u\rho(\vec{V}\cdot\hat{n})\,dA}$$

$$\sum F_y = \cancel{\frac{\partial}{\partial t}\left(\int_{CV} \rho v \, d\forall\right)} + \int_{CS} v\rho(\vec{V}\cdot\hat{n})\,dA$$

$$\sum F_z = \cancel{\frac{\partial}{\partial t}\left(\int_{CV} \rho w \, d\forall\right)} + \int_{CS} w\rho(\vec{V}\cdot\hat{n})\,dA$$

$\leftarrow F_x$

$\vdots$

$$F_x = U_1(\rho A U)_1 - U_4(\rho A U)_4$$

$\dot{m} = \rho V A$

$$F_x = \dot{m}(U_1 - U_4)$$

FROM ① → ② + ③ → ④

- NO LOSSES, NO HT OR W, ρ = CONST, STEADY

∴ BERNOULLI CAN BE APPLIED

$$\boxed{P_1 + \tfrac{1}{2}\rho U_1^2 = P_2 + \tfrac{1}{2}\rho U_2^2} \quad + \quad \boxed{P_3 + \tfrac{1}{2}\rho U_3^2 = P_4 + \tfrac{1}{2}\rho U_4^2}$$

LET $P_1 = \overset{P_\infty}{P_{ATM}}$ + $P_4 = P_{ATM}$ ✓

$$\boxed{P_1 = P_4}$$

2→3

ACROSS THE BLADES

MB

$A_2 = A_3$ ∴ $\boxed{U_2 = U_3}$

USING THESE 4 EQNS

ADDED POST-LECTURE

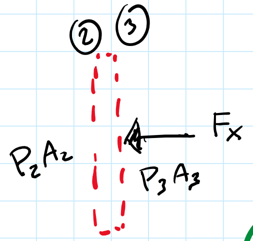
$$P_1 = P_2 + \tfrac{1}{2}\rho\left(U_2^2 - U_1^2\right) \qquad P_4 = P_3 + \tfrac{1}{2}\rho\left(U_3^2 - U_4^2\right)$$

$$P_2 + \tfrac{1}{2}\rho\left(U_2^2 - U_1^2\right) = P_3 + \tfrac{1}{2}\rho\left(U_3^2 - U_4^2\right)$$

$$P_2 - P_3 = \tfrac{1}{2}\rho\left(\cancel{U_3^2} - U_4^2 - \left(\cancel{U_2^2} - U_1^2\right)\right)$$

$$\underbrace{P_2 - P_3}_{\text{LOCAL CHANGES ACROSS THE BLADE DISK}} = \tfrac{1}{2}\rho\left(\underbrace{U_1^2 - U_4^2}_{\text{FAR AWAY INFO}}\right)$$

LM



$V_2 = V_3$
$A_2 = A_3$

$$F_x = A_2\left(P_2 - P_3\right)$$

$$F_x = \tfrac{1}{2}\rho A_2\left(U_1^2 - U_4^2\right)$$

FROM LM ABOVE

$$F_x = \dot{m}\left(U_1 - U_4\right)$$

$$F_x = \rho A_2 U_2\left(U_1 - U_4\right)$$

COMBINE THESE

$$\frac{1}{2}\cancel{\rho}\cancel{A_2}\left(U_1^2 - U_4^2\right) = \cancel{\rho}\cancel{A_2}U_2\left(U_1 - U_4\right)$$

$$\frac{\cancel{\left(U_1 - U_4\right)}\left(U_1 + U_4\right)}{2} = U_2\cancel{\left(U_1 - U_4\right)}$$

$$\boxed{U_2 = \frac{U_1 + U_4}{2}}$$

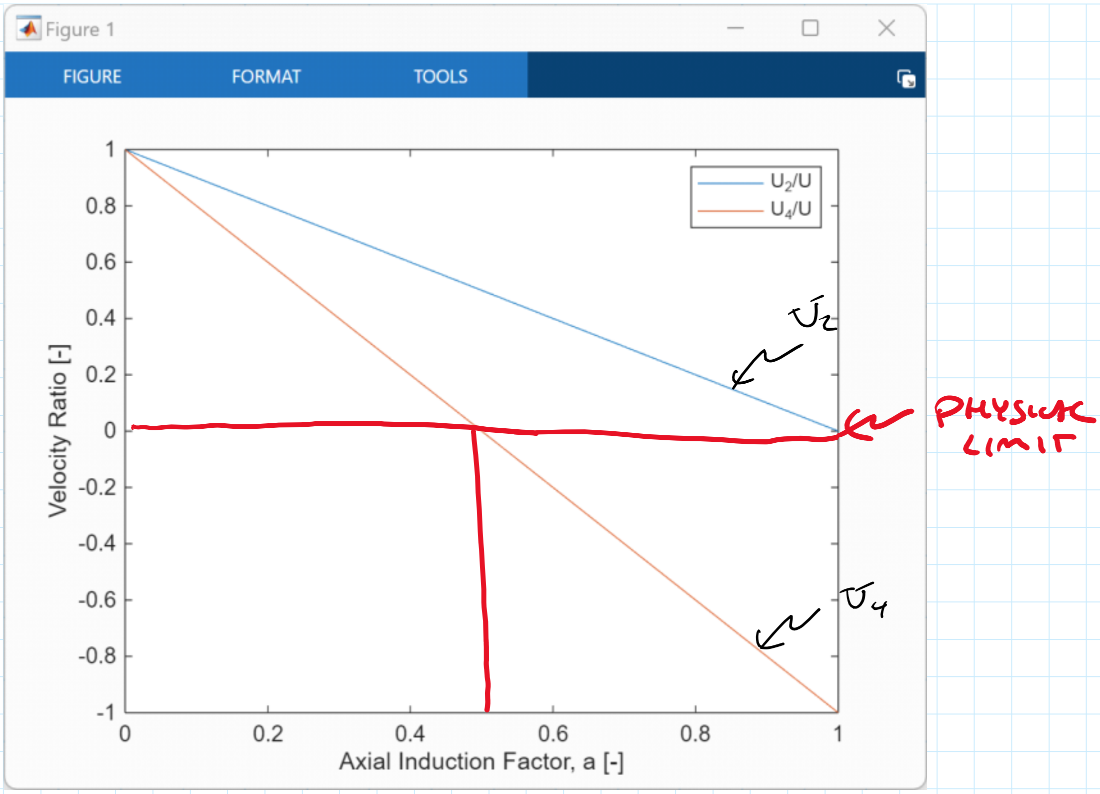
U_2 IS THE AVERAGE OF U_∞ + U_w

DEFINE THE AXIAL INDUCTION FACTOR

$$a \equiv \frac{U_1 - U_2}{U_1}$$

THE AMOUNT THAT THE WIND "SLOWS DOWN" PRIOR TO THE BLADES

$U_2 = U_1(1-a)$ + $U_4 = U_1(1-2a)$



$a < 0.5$

LET'S LOOK @ POWER

EB

$$\cancel{\frac{dE}{dt}\Big|_{CV}} = \cancel{\dot{Q}} - \dot{W} + \sum_{in} \dot{m}_i \left(\cancel{h_i} + \frac{V_i^{\,2}}{2} + \cancel{gz_i}\right) - \sum_{out} \dot{m}_o \left(\cancel{h_o} + \frac{V_o^{\,2}}{2} + \cancel{gz_o}\right)$$

$$\dot{W} = \text{POWER} = \tfrac{1}{2} \underbrace{\rho A_2 U_2}_{\dot{m}} \left(U_1^2 - U_4^2\right)$$

USING DEFN a

$$\dot{W} = \tfrac{1}{2} \rho A \underbrace{U_2}_{U_1(1-a)} \underbrace{(U_1 + U_4)}_{U_4 = U_1(1-2a)} \underbrace{(U_1 - U_4)}_{U_4 = U_1(1-2a)}$$

ADDED IT - NE

$$= \tfrac{1}{2} \rho A U_1 (1-a) \left(U_1 + U_1(1-2a)\right) \left(U_1 - U_1(1-2a)\right)$$

ADDED POST-LECTURE

$$\left.\begin{aligned}
&= \tfrac{1}{2}\rho A U_1 (1-a)\big(U_1 + U_1(1-2a)\big)\big(U_1 - U_1(1-2a)\big) \\
&= \tfrac{1}{2}\rho A U_1 (1-a)\left(U_1 + U_1 - 2U_1 a\right)\left(U_1 - U_1 + 2U_1 a\right) \\
&= \tfrac{1}{2}\rho A U_1 (1-a)\, U_1 (2-2a)\, U_1 (2a) \\
&= \tfrac{1}{2}\rho A U_1^3 (1-a)(2-2a)(2a) \\
&= \tfrac{1}{2}\rho A U_1^3 (1-a)\, 2(1-a)(2a)
\end{aligned}\right\}$$

$$\boxed{\dot{W} = \tfrac{1}{2}\rho A U_1^3\, 4a(1-a)^2}$$

LOOKING @ THE EFFICIENCY OF A TURBINE

$$\eta \equiv \frac{\text{WANT}}{\$} = \frac{\text{OUTPUT}}{\text{INPUT}}$$

COEFFICIENT OF POWER, C_P [-]

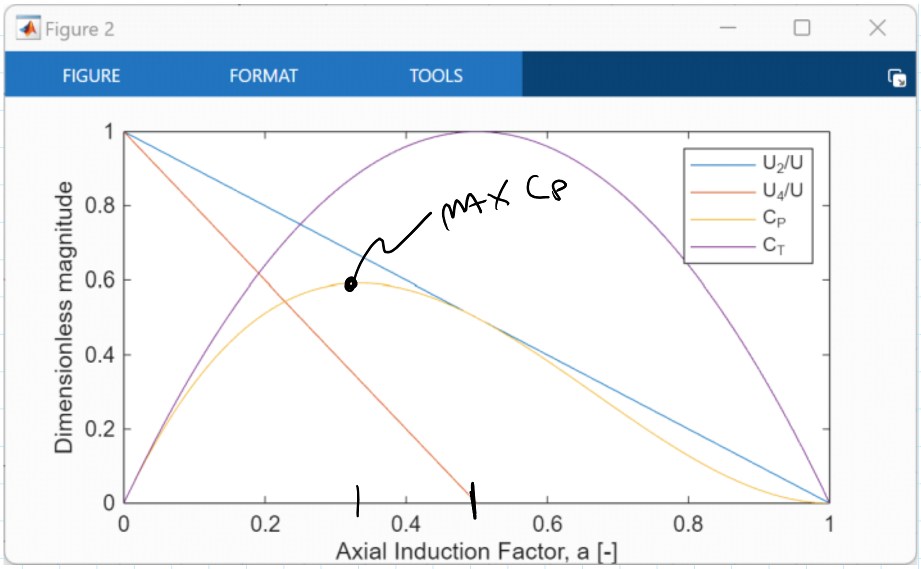
$$C_P \equiv \frac{\dot{W}}{\tfrac{1}{2}\rho U_\infty^3 A}$$

SWEPT AREA OF THE BLADE DISK

WIND

$$C_P = \frac{\cancel{\tfrac{1}{2}}\,\cancel{\rho}\,\cancel{A}\,\cancel{U_1^3}\, 4a(1-a)^2}{\cancel{\tfrac{1}{2}}\,\cancel{\rho}\,\cancel{U_1^3}\,\cancel{A}}$$

$$C_P = 4a(1-a)^2$$

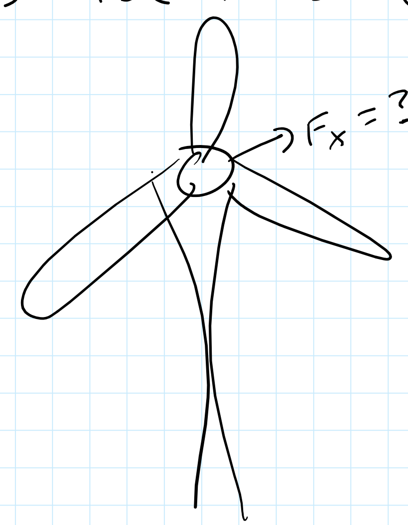


$@\ a = 1/3 \qquad C_{p,max} = \frac{16}{27} = 0.5926$

LIMIT OF PERFORMANCE OF A WIND TURBINE

KNOWN AS THE BETZ LIMIT

CAN DO A SIMILAR ANALYSIS FOR THRUST (F_x)



USING PREVIOUS ANALYSIS

$$F_x = \tfrac{1}{2} \rho A \left(U_1^2 - U_4^2\right)$$

$U_4 = U_1(1-2a)$

$U_4 = U_1(1-2a)$

ADDED POST-LECTURE

$$= \tfrac{1}{2}\rho A\left(U_1^2 - U_1^2(1-2a)^2\right)$$

$$= \tfrac{1}{2}\rho A\left(1-(1-2a)^2\right)U_1^2$$

$$= \tfrac{1}{2}\rho A\left(1-(1-4a+4a^2)\right)U_1^2$$

$$= \tfrac{1}{2}\rho A U^2(4a-4a^2$$

$$= \tfrac{1}{2}\rho A U^2 4a(1-a)$$

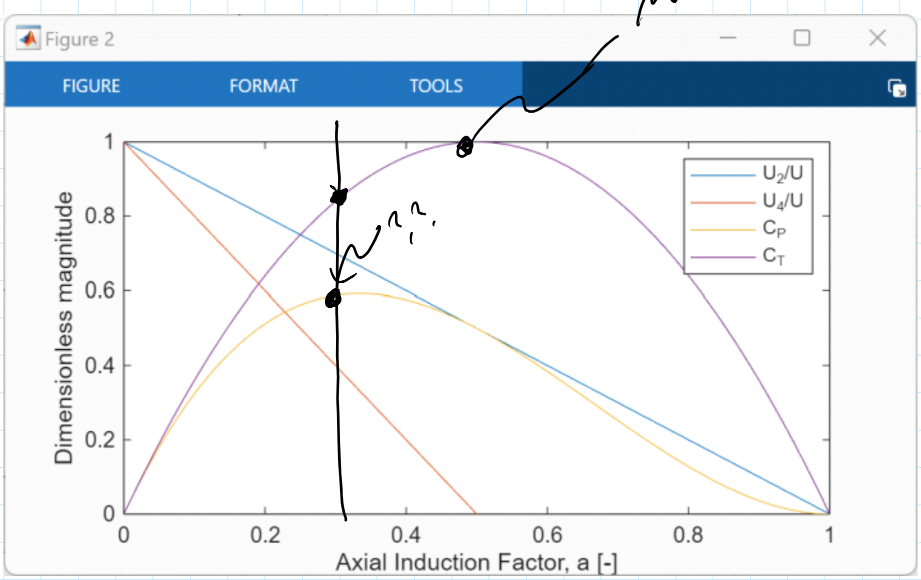
$$F_x = \tfrac{1}{2}\rho A U^2 4a(1-a)$$

COEFFICIENT OF THRUST, C_T [-]

$$C_T = \frac{F_x}{\tfrac{1}{2}\rho U^2 A} = \frac{\text{ACT THRUST}}{\text{DYNAMIC THRUST}}$$

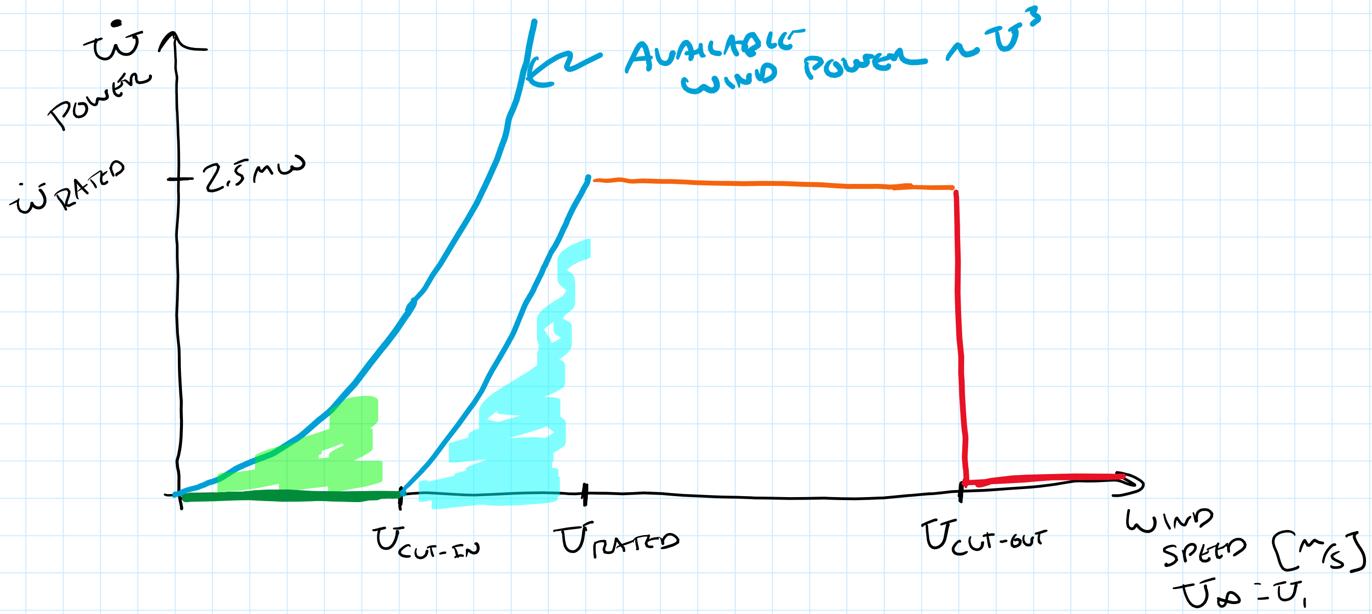
$$= \frac{\cancel{\tfrac{1}{2}}\cancel{\rho}\cancel{A}\cancel{U^2}4a(1-a)}{\cancel{\tfrac{1}{2}}\cancel{\rho}\cancel{U^2}\cancel{A}}$$

MAX C_T



WANT TO MAXIMIZE C_P + LIVE WITH C_T

WIND TURBINE PERFORMANCE / CONTROL / OPERATION



REGION #1 "PARKED"

$U_{WIND} < U_{CUT\text{-}IN}$

NOT ROTATING BECAUSE THE AMOUNT OF OUTPUT POWER IS NOT SUFFICIENT TO OVERCOME LOSSES

REGION #2 "MAXIMIZE POWER"

$U_{CUT\text{-}IN} < U_{WIND} < U_{RATED}$

GOAL IS TO ADJUST THE PITCH TO MAXIMIZE POWER OUTPUT

REGION #3 "MAINTAIN"

$U_{RATED} < U_{WIND} < U_{CUT\text{-}OUT}$

GOAL IS TO MAINTAIN RATED POWER BY PITCHING THE BLADES

REGION #4 "SHUT DOWN"

$$U_{WIND} > U_{CUT\text{-}OUT}$$

BLADES ARE PITCHED IN ORDER TO PREVENT STRUCTURAL DAMAGE. NO ROTATIONAL VELOCITY $\therefore \dot{\omega} = 0$