CS 2305: Discrete Mathematics for Computing I

Lecture 06

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Quantifiers

- A quantifier specifies the extent to which a predicate is true over a range of elements from the pertinent domain
  - indicates for how many elements from the domain a given predicate is true
- There are two main types of quantifiers in Predicate Logic:

1. Universal Quantifier, “For all,” symbol: $\forall$
   - $\forall x\, P(x)$, read as “for all x P(x)”, asserts P(x) is true for every x in the domain
2. Existential Quantifier, “There exists,” symbol: $\exists$
   - $\exists x\, P(x)$, read as “for some x, P(x)”, asserts P(x) is true for some x in the domain

Universal Quantifier

$\forall x\ P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\ P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\ P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\ P(x)$ is false.

- In some instances we want to emphasize the domain
  - $\forall x \in A\ P(x)$
  - $\forall x \in A$ such that $P(x)$

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

  Examples:

  1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
  2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
  3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

- In some instances we want to emphasize the domain
  - $\exists x \in A\ P(x)$
  - $\exists x \in A$ such that $P(x)$

Uniqueness Quantifier

- $\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.

Trailing Quantifiers

- Sometimes universal and existential quantifiers are restated informally by placing the quantification at the end of the sentence. In such a case the quantifier is said to "trail" the proposition
  - $\forall x \in \mathbf{R}, x^2 \geq 0$

    For any real number x, x^2 is positive
    - x^2 is positive for any real number x
  - $\exists x \in \mathbf{R}, x^2 -3x +2 = 0$

    For some real number x, $x^2 -3x + 2 = 0$
    - $x^2 -3x +2 = 0$ for some real number x

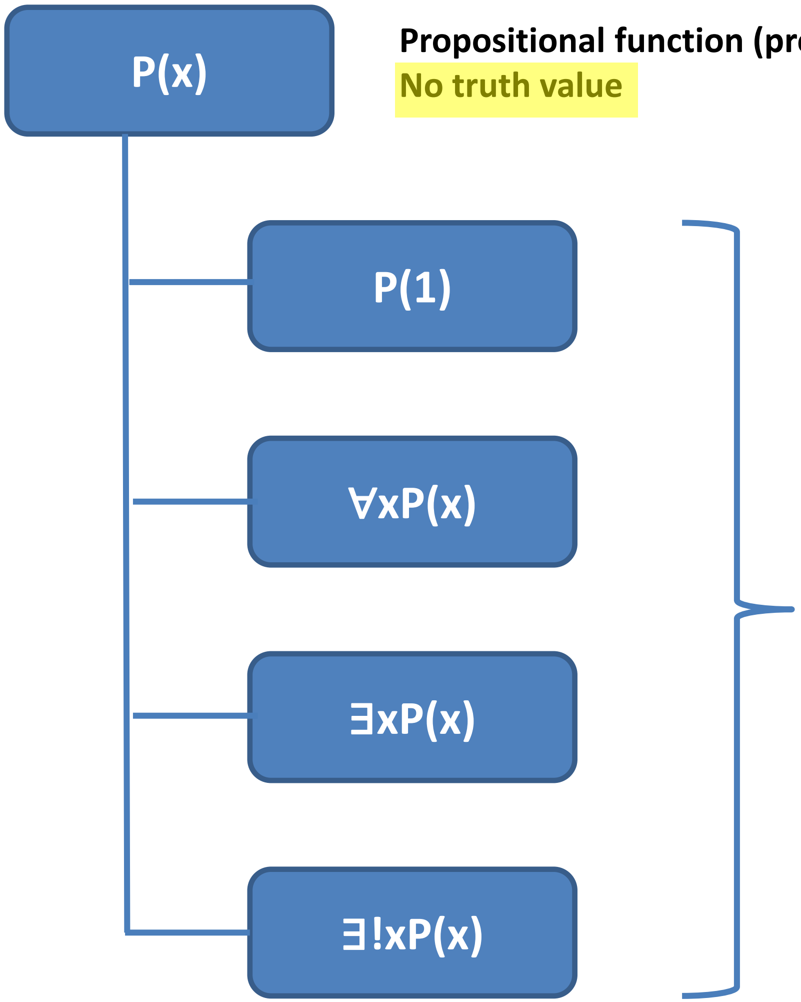
Bound and Free Variables

- When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**
- In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free

From Propositional Functions to Propositions (1)

- We already know that propositional functions become propositions when the predicate variables are assigned values
- Quantifiers provide an alternate way to convert propositional functions to propositions
  - Let P(x) be the propositional function "$x + 1 > x$". Then $\forall x\, P(x)$ is a proposition
- In general, all the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition
  - universal quantifiers
  - existential quantifiers
  - value assignments

From Propositional Functions to Propositions (2)



Propositional function (predicate) with predicate variable x

No truth value

Propositions

Easily verifiable truth value

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- $\forall x P(x)$ is an extremely strong claim
  - One way to show that $P(x)$ is false when x is in the domain is to find a single counterexample to the statement $\forall x P(x)$
  - The truth value of the quantification $\forall x Q(x)$, where Q(x) is the statement "x < 2" is false since Q(3) is false

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples**:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

☞ Quantifiers are incomplete unless the domain of discourse is known

A Word About Domains

- Generally, an implicit assumption is made that all domains of discourse for quantifiers are non-empty
- If the domain is empty, for any propositional function:
  - $\forall x P(x)$ is true
  - $\exists x P(x)$ is false

Truth Set

- If *P(x)* is a predicate and *x* has domain *D*, the **truth set** of *P(x)* is the set of all elements of *D* that make *P(x)* true when they are substituted for *x*
  - The truth set of *P(x)* is denoted as follows:
    - $\{x \in D \mid P(x)\}$.

- Suppose the domain is $\mathbf{Z}^+$, the set of positive integers. The truth set for ∃x, x is a factor of 8 is {1, 2, 4, 8}

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ……….

The set Z^+

x can be any of these values

1, 2, 4, 8

The truth set for ∃x, x is a factor of 8

These are the only values for x for which the predicate is true

Thinking about Quantifiers

- We can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.

Quantifiers Over Finite Domains (1)

- When the domain of a quantifier is finite:
  - all its elements can be listed
    - e.g. $x_1, x_2, x_3, \ldots, x_n$
  - $\forall x\, P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land \ldots \land P(x_n)$
    - Conjunction of propositions
  - $\exists x\, P(x) \equiv P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots \lor P(x_n)$
    - Disjunction of propositions

Quantifiers Over Finite Domains (2)

- P(x) is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4
  - What is the truth value of $\forall x P(x)$?
    - $\forall x P(x) \equiv P(1) \land P(2) \land P(3) \land P(4)$
      - **T** ∧ **T** ∧ **T** ∧ **F**
      - **F**
  - What is the truth value of $\exists x P(x)$?
    - $\exists x P(x) \equiv P(1) \lor P(2) \lor P(3) \lor P(4)$
      - **T** ∨ **T** ∨ **T** ∨ **F**
      - **T**

Quantifiers with Restricted Domains

- Sometimes it is not feasible to enumerate the domain of a quantifier. In such instances, an abbreviated notation is often used
  - a condition a variable must satisfy is included after the quantifier
  - Such quantifiers are called restricted quantifiers
- Examples (assumption: the domain in each case consists of the real numbers)
  - $(\forall x)_{x < 0} (x^2 > 0)$
    - For all –ve real numbers, the square of the number is positive
  - $(\forall y)_{y \neq 0} (y^3 \neq 0)$
    - For all non-zero real numbers, the cube of the number is non-zero
  - $(\exists z)_{z > 0} (z^2 = 2)$
    - There exists a non-zero number whose square is the number 2 (i.e. there is a positive square root of 2)