

Notes on Differential Calculus

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1. Differentiability

Definition 1.1. Let $f : (a, b) \rightarrow \mathbb{R}^n$, and let $f_i = \pi_i \circ f$ be its components. Then, f is differentiable at $t_0 \in (a, b)$ if the following limit exists.

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

Remark. The vector $f'(t_0)$ represents the tangent to the curve f at the point $f(t_0)$. The full tangent line is the parametric curve $f(t) + f'(t_0)(t - t_0)$.

Definition 1.2. Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \rightarrow \mathbb{R}^m$. Then, f is differentiable at $x \in U$ if there exists a linear transformation $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x) - \lambda h}{\|h\|} = 0.$$

The derivative of f at x is denoted by $\lambda = Df(x)$.

Remark. In a neighbourhood of x , we may approximate

$$f(x + h) \approx f(x) + Df(x)(h).$$

Remark. The statement that this quantity goes to zero means that each of the m components must also go to zero. For each of these limits, there are n axes along which we can let $h \rightarrow 0$. As a result, we obtain $m \times n$ limits, which allow us to identify the $m \times n$ components of the matrix representing the linear transformation λ (in the standard basis). These are the partial derivatives of f , and the matrix of λ is the Jacobian matrix of f evaluated at x .

Example. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. By choosing $\lambda = T$, we see that T is differentiable everywhere, with $DT(x) = T$ for every choice of $x \in \mathbb{R}^n$. This is made obvious by the fact that the best linear approximation of a linear map at some point is the map itself; indeed, the ‘approximation’ is exact.

Lemma 1.1. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, with derivative $Df(x)$, then

1. f is continuous at x .
2. The linear transformation $Df(x)$ is unique.

Proof. We prove the second part. Suppose that λ, μ satisfy the requirements for $Df(x)$; it can be shown that $\lim_{h \rightarrow 0} (\lambda - \mu)h / \|h\| = 0$. Now, if $\lambda v \neq \mu v$ for some non-zero vector $v \in \mathbb{R}^n$, then

$$\lambda v - \mu v = \frac{\lambda(tv) - \mu(tv)}{\|tv\|} \cdot \|v\| \rightarrow 0,$$

a contradiction. ■

2. Chain rule

Exercise I. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then, there exists $M > 0$ such that for all $x \in \mathbb{R}^n$, we have

$$\|Tx\| \leq M\|x\|.$$

Solution. Set $v_i = T(e_i)$ where e_i are the standard unit basis vectors of \mathbb{R}^n . Then,

$$\|Tx\| = \left\| \sum_i x_i v_i \right\| \leq \sum_i \|x_i v_i\| \leq \max \|v_i\| \sum_i |x_i|.$$

Since each $|x_i| \leq \|x\|$, set $M = n \max \|v_i\|$ and write

$$\|Tx\| \leq \max \|v_i\| \sum_i |x_i| \leq \max \|v_i\| \cdot n \|x\| = M\|x\|.$$

Theorem 2.1 (Chain Rule). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$ where f is differentiable at $a \in \mathbb{R}^n$ and g is differentiable at $f(a) \in \mathbb{R}^m$. Then, $g \circ f$ is differentiable, with $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$. Note that this means that the Jacobian matrices simply multiply.

Proof. Set $b = f(a) \in \mathbb{R}^m$, $\lambda = Df(a)$, $\mu = Dg(f(a))$. Define

$$\begin{aligned}\varphi : \mathbb{R}^n &\rightarrow \mathbb{R}^m, & \varphi(x) &= f(x) - f(a) - \lambda(x - a), \\ \psi : \mathbb{R}^m &\rightarrow \mathbb{R}^k, & \psi(y) &= g(y) - g(b) - \mu(y - b).\end{aligned}$$

We claim that

$$\lim_{x \rightarrow a} \frac{g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x - a)}{\|x - a\|} = 0.$$

Write the numerator as

$$g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x - a) = \psi(f(x)) + \mu(\varphi(x)).$$

Note that

$$\lim_{x \rightarrow a} \frac{\varphi(x)}{\|x - a\|} = 0, \quad \lim_{y \rightarrow b} \frac{\psi(y)}{\|y - b\|} = 0.$$

Thus, find $M > 0$ such that

$$\|\mu(\varphi(x))\| \leq \|\varphi(x)\|$$

for all $x \in \mathbb{R}^n$, hence

$$\lim_{x \rightarrow a} \frac{\|\mu(\varphi(x))\|}{\|x - a\|} \leq \lim_{\{x \rightarrow a\}} \frac{M\|\varphi(x)\|}{\|x - a\|} = 0.$$

Now write

$$\lim_{f(x) \rightarrow b} \frac{\psi(f(x))}{\|f(x) - b\|} = 0,$$

hence for any $\varepsilon > 0$, there is a neighbourhood of b on which

$$\|\psi(f(x))\| \leq \varepsilon \|f(x) - b\| = \varepsilon \|\varphi(x) + \lambda(x - a)\|.$$

Apply the triangle inequality and find $M' > 0$ such that

$$\|\psi(f(x))\| \leq \varepsilon \|\varphi(x)\| + \varepsilon M' \|x - a\|.$$

Thus,

$$\lim_{x \rightarrow a} \frac{\|\psi(f(x))\|}{\|x - a\|} \leq \lim_{x \rightarrow a} \frac{\varepsilon \|\varphi(x)\|}{\|x - a\|} + \varepsilon M' = \varepsilon M'.$$

Since $\varepsilon > 0$ was arbitrary, this limit is zero, completing the proof. ■

3. Partial derivatives

Definition 3.1. Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \rightarrow \mathbb{R}$. The partial derivative of f with respect to the coordinate x_j at some $a \in U$ is defined by the following limit, if it exists.

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \rightarrow 0} \frac{f(a + he_j) - f(a)}{h}.$$

Lemma 3.1. If $f : U \rightarrow \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}^n$, then

$$Df(a)(x_1, \dots, x_n) = x_1 \frac{\partial f}{\partial x_1}(a) + \dots + x_n \frac{\partial f}{\partial x_n}(a).$$

Example. Consider

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} (xy)/(x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$; it is not even continuous there. However, both partial derivatives of f exist at $(0, 0)$.

Lemma 3.2. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then the matrix representation of $Df(a)$ in the standard basis is given by

$$[Df(a)] = \left[\frac{\partial f_i}{\partial x_j}(a) \right]_{ij}.$$

Lemma 3.3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$, and let $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$ be differentiable at $f(a) \in \mathbb{R}^m$. Then, the matrix representation of $D(g \circ f)(a)$ in the standard basis is the product

$$[D(g \circ f)(a)] = [Dg(f(a))][Df(a)] = \left[\sum_{\ell=1}^m \frac{\partial g_i}{\partial y_\ell}(f(a)) \frac{\partial f_\ell}{\partial x_j}(a) \right]_{ij}.$$

In other words,

$$\frac{\partial}{\partial x_j}(g \circ f)_{i(a)} = \sum_{\ell=1}^m \frac{\partial g_i}{\partial y_\ell}(f(a)) \frac{\partial f_\ell}{\partial x_j}(a).$$

Example. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be differentiable, and let $\Gamma(f) = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\}$ be the graph of f . Now, let $\gamma : [-1, 1] \rightarrow \Gamma(f)$ be a differentiable curve, represented by

$$\gamma(t) = (g(t), h(t), f(g(t), h(t))).$$

Then, we can compute the derivative

$$\gamma'(a) = \left(g'(a), h'(a), g'(a) \frac{\partial f}{\partial x} + h'(a) \frac{\partial f}{\partial y} \Big|_{(g(a), h(a))} \right)$$

Exercise II. Consider the inner product map, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. What is its derivative?

Solution. We treat the inner product as a map $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, which acts as

$$\langle \mathbf{x}, \mathbf{y} \rangle := g(x_1, \dots, x_n, y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n.$$

Now, note that

$$\frac{\partial g}{\partial x_i} = y_i, \quad \frac{\partial g}{\partial y_i} = x_i.$$

Thus,

$$\begin{aligned} Dg(\mathbf{a}, \mathbf{b})(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i}(\mathbf{a}, \mathbf{b}) + \sum_{i=1}^n y_i \frac{\partial g}{\partial y_i}(\mathbf{a}, \mathbf{b}) \\ &= \sum_{i=1}^n x_i b_i + \sum_{i=1}^n y_i a_i \\ &= \langle \mathbf{x}, \mathbf{b} \rangle + \langle \mathbf{y}, \mathbf{a} \rangle. \end{aligned}$$

In other words, the matrix representation of the derivative of the inner product map at the point (\mathbf{a}, \mathbf{b}) is given by $[\mathbf{b}^\top \mathbf{a}^\top]$.

Exercise III. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ be a differentiable curve. What is the derivative of the real map $t \mapsto \|\gamma(t)\|^2$?

Solution. We write this map as $t \mapsto \langle \gamma(t), \gamma(t) \rangle$. Consider the scheme

$$\mathbb{R} \rightarrow \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad t \mapsto \begin{pmatrix} \gamma(t) \\ \gamma(t) \end{pmatrix} \mapsto \langle \gamma(t), \gamma(t) \rangle.$$

Pick a point $t \in \mathbb{R}$, whence the derivative of the map at t is

$$(\gamma(t)^\top \quad \gamma(t)^\top) \begin{pmatrix} \gamma'(t) \\ \gamma'(t) \end{pmatrix} = 2\langle \gamma(t), \gamma'(t) \rangle.$$

Remark. Consider the surface $S^{n-1} \subset \mathbb{R}^n$, and pick an arbitrary differentiable curve $\gamma : \mathbb{R} \rightarrow S^{n-1}$. Now, the tangent vector $\gamma'(t)$ is tangent to the sphere S^{n-1} at any point $\gamma(t)$. We claim that this tangent drawn at $\gamma(t)$ is always perpendicular to the position vector $\gamma(t)$. This is made trivial by our exercise: the map $t \mapsto \|\gamma(t)\|^2 = 1$ is a constant map since γ is a curve on the unit sphere. This means that it has zero derivative, forcing $\langle \gamma(t), \gamma'(t) \rangle = 0$.

3.1. Directional derivatives

Definition 3.1.1. Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \rightarrow \mathbb{R}$. The directional derivative of f along a direction $\mathbf{v} \in \mathbb{R}^n$ at a point $\mathbf{a} \in U$ is defined by the following limit, if it exists.

$$\nabla_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}.$$

Example. Consider

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} x^3/(x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$. However, all directional derivatives of f exist at $(0, 0)$. Indeed, consider a direction $(\cos \theta, \sin \theta)$, and examine the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} [f(t \cos \theta, t \sin \theta) - f(0, 0)] = \cos^3 \theta.$$

Definition 3.1.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable. The gradient of f is defined as the map

$$\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto \left[\frac{\partial f}{\partial x_i}(x) \right]_i.$$

Remark. The gradient at a point $x \in \mathbb{R}^n$ is thought of as a vector. In contrast, the derivative is thought of as a linear transformation. Otherwise, we see that $\nabla f(x) = [Df(x)]$.

Definition 3.1.3. Let $C^1(\mathbb{R}^n)$ be the set of real-valued differentiable functions on \mathbb{R}^n . Fix a point $a \in \mathbb{R}^n$, then fix a tangent vector $v \in \mathbb{R}^n$. Then, the map

$$\nabla_v : C^1(\mathbb{R}^n) \rightarrow \mathbb{R}, \quad f \mapsto Df(a)(v)$$

is a linear functional. The quantity $\nabla_v f$ is called the directional derivative of f in the direction v at the point a .

Remark. We can represent ∇_v as the operator

$$\nabla_v(\cdot) = D(\cdot)(a)(v) = \sum_i v_i \frac{\partial}{\partial x_i} \Big|_a = v \cdot \nabla(\cdot).$$

Lemma 3.1.1. The directional derivatives ∇_v form a vector space called the tangent space, attached to the point $a \in \mathbb{R}^n$. This can be identified with the vector space \mathbb{R}^n by the natural map $\nabla_v \mapsto v$. The standard basis can be informally denoted by the vectors

$$\nabla_{e_1} := \frac{\partial}{\partial x_1}, \quad \dots, \quad \nabla_{e_n} := \frac{\partial}{\partial x_n}.$$

3.2. Differentiation on manifolds *

Definition 3.2.1. A homeomorphism is a continuous, bijective map whose inverse is also continuous.

Lemma 3.2.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. Denote the graph of f as

$$\Gamma(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}.$$

Then, $\Gamma(f)$ is a smooth manifold.

Proof. Consider the homeomorphism

$$\varphi : \Gamma(f) \rightarrow \mathbb{R}^n, \quad (x, f(x)) \mapsto x.$$

This is clearly bijective, continuous (restriction of a projection map), with a continuous inverse (from the continuity of f). Call this homeomorphism φ a coordinate map on $\Gamma(f)$. ■

Definition 3.2.2. Let $f : M \rightarrow \mathbb{R}$ where M is a smooth manifold, with a coordinate map $\varphi : M \rightarrow \mathbb{R}^n$. We say that f is differentiable at a point $a \in M$ if $f \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $\varphi(a)$.

Definition 3.2.3. Let $f : M \rightarrow \mathbb{R}$ where M is a smooth manifold, let $\varphi : M \rightarrow \mathbb{R}^n$ be a coordinate map, and let $a \in M$. Let $\gamma : \mathbb{R} \rightarrow M$ be a curve such that $\gamma(0) = a$, and further let γ be differentiable in the sense that $\varphi \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is differentiable. The directional derivative of f at a along γ is defined as

$$\left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0} = \lim_{h \rightarrow 0} \left. \frac{f(\gamma(t+h)) - f(\gamma(t))}{h} \right|_{t=0}.$$

Note that we are taking the derivative of $f \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ in the conventional sense.

Lemma 3.2.2. Let γ_1 and γ_2 be two curves in M such that $\gamma_1(0) = \gamma_2(0) = a$, and

$$\left. \frac{d}{dt} \varphi \circ \gamma_1(t) \right|_{t=0} = \left. \frac{d}{dt} \varphi \circ \gamma_2(t) \right|_{t=0}.$$

In other words, γ_1 and γ_2 pass through the same point a at $t = 0$, and have the same velocities there. Then, the directional derivatives of f at a along γ_1 and γ_2 are the same.

Definition 3.2.4. Let M be a smooth manifold, and let $a \in M$. Consider the following equivalence relation on the set of all curves γ in M such that $\gamma(0) = a$.

$$\gamma_1 \sim \gamma_2 \implies \frac{d}{dt}\varphi \circ \gamma_1(t)|_{t=0} = \frac{d}{dt}\varphi \circ \gamma_2(t)|_{t=0}.$$

Each resultant equivalence class of curves is called a tangent vector at $a \in M$. Note that all these curves in a particular equivalence class pass through a with the same velocity vector.

The collection of all such tangent vectors, i.e. the space of all curves through a modulo the equivalence relation which identifies curves with the same velocity vector through a , is called the tangent space to M at a , denoted $T_a M$.

Remark. Each tangent vector $v \in T_a M$ acts on a differentiable function $f : M \rightarrow \mathbb{R}$ yielding a (well-defined) directional derivative at a .

$$v : C^1(M) \rightarrow \mathbb{R}, \quad f \mapsto \frac{d}{dt}f(\gamma_{v(t)})|_{t=0}.$$

Thus, the tangent space represents all the directions in which taking a derivative of f makes sense.

Remark. The tangent space $T_a M$ is a vector space. Upon fixing f , the map $Df(a) : T_a M \rightarrow \mathbb{R}, v \mapsto vf(a)$ is a linear functional on the tangent space.

Remark. Given a tangent vector $v \in T_a M$, it can be identified with its corresponding velocity vector in \mathbb{R}^n . Thus, the tangent space $T_a M$ can be identified with the geometric tangent plane drawn to the manifold M at the point a .

4. Mean value theorem

Consider a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and fix $a \in \mathbb{R}^n$. Define the functions

$$g_i : \mathbb{R} \rightarrow \mathbb{R}, \quad g_i(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n).$$

Then, each g_i is differentiable, with

$$g_i'(x) = \frac{\partial f}{\partial x_i}(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n).$$

By applying the Mean Value Theorem on some interval $[c, d]$, we can find $\alpha \in (c, d)$ such that $g_i(d) - g_i(c) = g_i'(\alpha)(d - c)$. In other words,

$$f(\dots, d, \dots) - f(\dots, c, \dots) = \frac{\partial f}{\partial x_i}(\dots, \alpha, \dots)(d - c).$$

Theorem 4.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $a \in \mathbb{R}^n$. Then, f is differentiable at a if all the partial derivatives $\partial f / \partial x_j$ exist in a neighbourhood of a and are continuous at a .

Proof. Without loss of generality, let $m = 1$. We claim that

$$\lim_{h \rightarrow 0} \frac{1}{\|h\|} \left\| f(a+h) - f(a) - \sum_{i=0}^n \frac{\partial f}{\partial x_i}(a) h_i \right\| = 0.$$

Examine

$$\begin{aligned} f(a+h) - f(a) &= f(a_1 + h_1, \dots, a_n + h_n) - f(a_1, \dots, a_n) \\ &= f(a_1 + h_1, \dots, a_n + h_n) - f(a_1 + h_1, \dots, a_{n-1} + h_{n-1}, a_n) + \\ &\quad f(a_1 + h_1, \dots, a_{n-1} + h_{n-1}, a_n) - f(a_1 + h_1, \dots, a_{n-1}, a_n) + \\ &\quad \dots \\ &\quad f(a_1 + h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) \\ &= \frac{\partial f}{\partial x_n}(c_n) h_n + \dots + \frac{\partial f}{\partial x_1}(c_1) h_1. \end{aligned}$$

The last step follows from the Mean Value Theorem. As $h \rightarrow 0$, each $c_i \rightarrow a$. Thus,

$$\begin{aligned} \frac{1}{\|h\|} \left\| f(a+h) - f(a) - \sum_{i=0}^n \frac{\partial f}{\partial x_i}(a) h_i \right\| &= \frac{1}{\|h\|} \left\| \sum_{i=0}^n \left(\frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right) h_i \right\| \\ &\leq \sum_{i=0}^n \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right| \frac{|h_i|}{\|h\|} \\ &\leq \sum_{i=0}^n \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right|. \end{aligned}$$

Taking the limit $h \rightarrow 0$, observe that $(\partial f / \partial x_i)(c_i) \rightarrow (\partial f / \partial x_i)(a)$ by the continuity of the partial derivatives, completing the proof. ■

Corollary 4.1.1. All polynomial functions on \mathbb{R}^n are differentiable.

Theorem 4.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable with continuous partial derivatives, and let $a \in \mathbb{R}^n$ be a point of local maximum. Then, $Df(a) = 0$.

Proof. We need only show that each

$$\frac{\partial f}{\partial x_i}(a) = 0.$$

This must be true, since a is also a local maximum of each of the restrictions g_i as defined earlier. ■

5. Inverse and implicit function theorems

Theorem 5.1 (Inverse function theorem). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable on a neighbourhood of $a \in \mathbb{R}^n$, and let $\det(Df(a)) \neq 0$. Then, there exist neighbourhoods U of a and W of $f(a)$ such that the restriction $f : U \rightarrow W$ is invertible. Furthermore, f^{-1} is continuous on U and differentiable on U .

Lemma 5.2. Consider a continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and let M denote the surface defined by the zero set of f . Then, M can be represented as the graph of a differentiable function $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ at those points where $Df \neq 0$.

Proof. Without loss of generality, suppose that $\partial f / \partial x_n \neq 0$ at some point $a \in M$. It can be shown that the map

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto (x_1, x_2, \dots, x_{n-1}, f(x))$$

is invertible in a neighbourhood W of a , with a continuous and differentiable inverse of the form

$$G : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad u \mapsto (u_1, u_2, \dots, u_{n-1}, g(u)).$$

Since $F \circ G$ must be the identity map on W , we demand

$$(x_1, x_2, \dots, x_{n-1}, f(x_1, x_2, \dots, x_{n-1}, g(x))) = (x_1, x_2, \dots, x_{n-1}, x_n).$$

Thus, the zero set of f in this neighbourhood of a satisfies $x_n = 0$, hence

$$f(x_1, x_2, \dots, x_{n-1}, g(x_1, x_2, \dots, x_{n-1}, 0)) = 0.$$

In other words, the part of the surface M in the neighbourhood of a is precisely the set of points

$$(x_1, x_2, \dots, x_{n-1}, g(x_1, x_2, \dots, x_{n-1}, 0)).$$

Simply set

$$h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \quad x \mapsto g(x_1, x_2, \dots, x_{n-1}, 0),$$

whence the surface M is locally represented by the graph of h . ■

Remark. Note that by using

$$f(x_1, \dots, x_{n-1}, h(x_1, \dots, x_{n-1})) = 0$$

on the surface, we can use the chain rule to conclude that for all $1 \leq i < n$, we have

$$\frac{\partial f}{\partial x_i}(a) + \frac{\partial f}{\partial x_n}(a) \frac{\partial h}{\partial x_i}(a_1, \dots, a_{n-1}) = 0.$$

Theorem 5.3 (Implicit function theorem). Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be continuously differentiable in an open set containing (a, b) , with $f(a, b) = 0$. Let $\det (\partial f^j / \partial x_{n+k}(a, b)) \neq 0$. Then, there exists an open set $U \subset \mathbb{R}^n$ containing a , an open set $V \subset \mathbb{R}^m$ containing b , and a differentiable function $g : U \rightarrow V$ such that $f(x, g(x)) = 0$.

Remark. The condition on the determinant can be rephrased as $\text{rank } Df(a, b) = m$.

Theorem 5.4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable, and let M be the surface defined by its zero set. Furthermore, let $\nabla f(a) \neq 0$ for some $a \in M$; thus, M can be locally represented by a graph on \mathbb{R}^{n-1} . Then, $\nabla f(a)$ is normal to the tangent vectors drawn at a to M ; in fact, the perpendicular space of $\nabla f(a)$ is precisely the tangent space $T_a M$.

Proof. Consider a tangent vector drawn at a to M , represented by the differentiable curve $\gamma : \mathbb{R} \rightarrow M$, $\gamma(0) = a$; note that we use the identification $\gamma'(0) = v \in \mathbb{R}^n$. Then, calculate

$$\frac{d}{dt}f(\gamma(t))|_{t=0} = Df(\gamma(0))(\gamma'(0)) = Df(a)(v).$$

On the other hand, we have $f(\gamma(t)) = 0$ identically. Thus,

$$v \cdot \nabla f(a) = Df(a)(v) = 0$$

as claimed. ■

6. Taylor's theorem

Theorem 6.1 (Clairaut). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ have continuous second order partial derivatives. Then,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Theorem 6.2 (Taylor). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ have continuous second order partial derivatives, and let $(x_0, y_0) \in \mathbb{R}^2$. Then, there exists $\varepsilon > 0$ such that for all $\|(x - x_0, y - y_0)\| < \varepsilon$,*

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(y - y_0)^2 \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + R(x, y), \end{aligned}$$

where as $(x, y) \rightarrow (x_0, y_0)$, the remainder term vanishes as

$$\frac{|R(x, y)|}{\|(x - x_0, y - y_0)\|^2} \rightarrow 0.$$

All partial derivatives here are evaluated at (x_0, y_0) .

Proof. This follows from applying the Taylor's Theorem in one variable to the real function $g : \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto f((1 - t)(x_0, y_0) + t(x, y))$. ■

7. Critical points and extrema

Definition 7.1. We say that $a \in \mathbb{R}^n$ is a critical point of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ if all $\partial f / \partial x^j = 0$ there.

Lemma 7.1. *All points of extrema of a differentiable function are critical points.*

Proof. We already know that $Df(a) = 0$ where a is either a point of maximum or minimum. ■

Example. In order to find a point of extrema of a C^2 -smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we first identify a critical point (x_0, y_0) . Next, we must find a neighbourhood of (x_0, y_0) which contains no other critical points – to do this, apply Taylor’s Theorem. Indeed, we see that

$$f(x, y) = f(x_0, y_0) + A(x - x_0)^2 + 2B(x - x_0)(y - y_0) + C(y - y_0)^2 + R_2.$$

For non-degeneracy of solutions, we demand $AC - B^2 \neq 0$, i.e. at (x_0, y_0) , we want

$$\left[\frac{\partial^2 f}{\partial x \partial y} \right]^2 \neq \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2}.$$

If $AC - B^2 > 0$ and $\partial^2 f / \partial x^2 > 0$, then we have found a point of minima; if $\partial^2 f / \partial x^2 < 0$, then we have found a point of maximum. If $AC - B^2 < 0$, then we have found a saddle point.

Example. Suppose that we wish to maximize the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, given an equation of constraint $g = 0$, where $g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Using the method of Lagrange multipliers, we look for solutions of the system

$$\begin{cases} \nabla f(x, y) + \lambda \nabla g(x, y) = 0 \\ g(x, y) = 0. \end{cases}$$