1. Prime numbers

Definition 1.1 (Prime numbers)**.** A natural number is called a *prime number* if it is greater than 1 and cannot be written as the product of two smaller natural numbers.

Remark. The numbers 2, 3, and 17 are prime. Corollary 1.1.1 shows that this list is not exhaustive!

Theorem 1.1 (Euclid)**.** *There are infinitely many primes.*

Proof. Suppose to the contrary that $p_1, p_2, \ldots, p_n$ is a finite enumeration of all primes. Set $P = p_1 p_2 \ldots p_n$. Since $P + 1$ is not in our list, it cannot be prime. Thus, some prime factor p_j divides $P + 1$. Since p_j also divides P, it must divide the difference $(P + 1) - P = 1$, a contradiction. □

Corollary 1.1.1. *There is no largest prime number.*

Corollary 1.1.2. *There are infinitely many composite numbers.*

Theorem 1.2. *There are arbitrarily long stretches of composite numbers.*

Proof. For any $n > 2$, consider

$$n! + 2, \quad n! + 3, \quad \ldots, \quad n! + n.$$

□