CS 512: Formal Methods Spring 2016

Assignment 2: Background to SAT/SMT Solvers

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1 Problem 1

$|| \; : \; \mathcal{T} \rightarrow \mathbb{N}$

$$|t| = \begin{cases} 1 + |Lt| + |Rt| & if \quad t =< Lt, Rt > \\ 0 & otherwise \end{cases}$$

$height \; : \; \mathcal{T} \rightarrow \mathbb{N}$

$$heigh(t) = \begin{cases} max(height(Lt), height(Rt)) + 1 & if \quad t =< Lt, Rt > \\ 0 & otherwise \end{cases}$$

2 Problem 2

(a)

Poset:

Division within the natural numbers has the Reflexive, Anti-symmetric and Transitive properties.

Least Upper Bound:

Least Common Multiple of a,b $\in \mathbb{N}$ -{0} returns the smallest number, c, such that both a $\trianglelefteq$ c and b $\trianglelefteq$ c. This is the smallest multiple that both numbers divide, so $\vee$ (lcm) is the Least Upper Bound.

Greatest Lower Bound:

Greatest Common Divisor of a,b $\in \mathbb{N}$ - {0} returns the largest number, c, such that both c $\trianglelefteq$ a and c $\trianglelefteq$ b This is the largest number that divides both numbers, so $\wedge$ (gcd) is the Greatest Lower Bound.

Thus $\mathcal{A}$ satisfies all the properties of a lattice.

(b) $\mathcal{A}$ is a distributive latice. Proof:
Any number can be represented as the product of primes (prime factorization theorem)
Represent a, b and c as the product of all primes (1 ... n) raised to a power.
Example: a = $(p_1^\alpha, \ldots, p_n^\alpha)$ where each α represents the number of times that prime appears in a's prime factorization
Repeat the same for b and c.
Represent lcm ($\vee$) as the max function and gcd ($\wedge$) as the min function
Proving a $\wedge$ (b $\vee$ c) = (a $\vee$ b) $\wedge$ (a $\vee$ c) :
a $\wedge$ (b $\vee$ c) = $p_1^{min\{\alpha,max\{\beta,\gamma\}\}} \ldots p_n^{min\{\alpha,max\{\beta,\gamma\}\}}$
(a $\vee$ b) $\wedge$ (a $\vee$ c) = $p_1^{max\{min\{\alpha,\beta\},min\{\alpha,\gamma\}\}} \ldots p_n^{max\{min\{\alpha,\beta\},min\{\alpha,\gamma\}\}}$
Any ordering of α, β, γ will result in the same output for both equations (not shown to save space).
The same strategy and proof will satisfy the other constraint of: a $\vee$ (b $\wedge$ c) = (a $\vee$ b) $\wedge$ (a $\vee$ c)

3 Problem 3

Proof by induction: Base Case (1 Connective):
$\neg p = \neg p$
$\neg(p \vee q) = \neg p \wedge q$
$\neg(p \wedge q) = \neg p \vee q$

Induction Step:
A WFF ϕ with n+1 connectives can be represented in one of the following forms:

$$\begin{cases} \neg\phi \\ \phi \vee \phi \\ \phi \wedge \phi \end{cases}$$

The last 2 cases are valid in NNF. The first case is proved through De Morgan's.
$\neg(\phi \wedge \phi) \equiv \neg\phi \vee \neg\phi$ The left hand side has 4 terms. The right hand side has 5. $5 < 1.5 * 4$
The same situation happens with $\vee$ instead of $\wedge$.

4 Problem 4

Extending the HORN Clause will cause the HORN algorithm to say that certain formulas are satisfiable even if they aren't.
Here is one such example:
$(\top \rightarrow p) \wedge (\top \rightarrow \neg p)$
This is a valid HORN Formula consisting of two HORN clauses.
The algorithm will assign true to both p and ¬ p, which can't be true.
Therefore, adding ¬ to HORN Clauses will break the algorithm.