CS 512, Spring 2016: Assignment 1
Propositional Logic

Patrick Gomes

January 29, 2016

1 Problem 1

a) $P \to Q, P \to \neg Q \vdash \neg P$

1	$P \to Q$	premise
2	$P \to \neg Q$	premise
3	P	assume
4	Q	$\to$e $1, 3$
5	$\neg Q$	$\to$e $2, 3$
6	$\perp$	$\neg$e $4, 5$
7	$\neg P$	$\neg$i

b) $P \to (Q \to R), P, \neg R \vdash \neg Q$

1	$P \to (Q \to R)$	premise
2	P	premise
3	$\neg R$	premise
4	$Q \to R$	$\to$e $1, 2$
5	$\neg Q$	MT 3,4

2 Problem 2

a) $\neg P \to \neg Q \vdash Q \to P$

1	$\neg P \to \neg Q$	premise
2	Q	assume
3	$\neg\neg Q$	$\neg\neg$e 2
4	$\neg\neg P$	MT 1,3
5	P	$\neg\neg$e 4
6	$Q \to P$	$\to$i

b) $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$

1	$\neg P \vee \neg Q$	premise
2	$\neg P$	assume
3	$P \wedge Q$	assume
4	P	$\wedge$e 3
5	$\bot$	$\neg$e $2, 4$
6	$\neg(P \wedge Q)$	$\neg$i $3 - 5$
7	$\neg Q$	assume
8	$P \wedge Q$	assume
9	Q	$\wedge$e 8
10	$\bot$	$\neg$e $7, 9$
11	$\neg(P \wedge Q)$	$\neg$i $8 - 10$
12	$\neg(P \wedge Q)$	$\vee$e $1, 2 - 6, 7 - 11$

c) $\neg P, P \vee Q \vdash Q$

1	$\neg P$	premise
2	$P \vee Q$	premise
3	$\neg Q$	assume
4	P	assume
5	$\bot$	$\neg$e $1, 4$
6	Q	assume
7	$\bot$	$\neg$e $3, 6$
8	$\bot$	$\neg$e $3 - 7$
9	Q	$\neg$i

d) $P \vee Q, \neg Q \vee R \vdash P \vee R$

1	$P \vee Q$	premise
2	$\neg Q \vee R$	premise
3	P	assume
4	$P \vee R$	$\vee$i$_1$ 3
5	Q	assume
6	$\neg Q$	assume
7	$\bot$	$\neg$e $5, 6$
8	$P \vee R$	$\bot$e 7
9	R	assume
10	$P \vee R$	$\vee$i$_2$ 9
11	$P \vee R$	$\vee$e $6 - 8, 9 - 10$
12	$P \vee R$	$\vee$e $1, 2, 3 - 4, 5 - 11$

3 Problem 3

c is the number of binary connectives. n is the number of negation connectives. s is the number of propositional atoms or variables. ℓ is the length of φ, not counting paraenthesis.
s = c + 1
n = ℓ - c - s = ℓ - c - (c + 1) = ℓ - 2c - 1

4 Problem 4

Base Case (1 term):
$\phi \equiv p$
$\phi^* \equiv \neg p$
therefore: $\phi^* \equiv \neg\phi$

Defining ϕ^*:
if $\phi \equiv \psi \vee \psi \equiv \phi^* \equiv \psi^* \wedge \psi^*$
if $\phi \equiv \psi \wedge \psi \equiv \phi^* \equiv \psi^* \vee \psi^*$
if $\phi \equiv \psi \equiv \phi^* \equiv \neg\psi^*$

Induction Step:
Assume ϕ has more than 1 term and for all wff's ψ with less terms than ϕ that $\psi^* = \neg\psi$
if ϕ is of the form $\psi \vee \psi$ then $\phi^* \equiv \psi^* \wedge \psi^* \equiv \neg\psi \wedge \neg\psi \equiv \neg(\psi \vee \psi) \equiv \neg\phi$
if ϕ is of the form $\psi \wedge \psi$ then $\phi^* \equiv \psi^* \vee \psi^* \equiv \neg\psi \vee \neg\psi \equiv \neg(\psi \wedge \psi) \equiv \neg\phi$
if ϕ is of the form $\neg\psi$ then $\phi^* \equiv \neg\psi^* \equiv \neg\neg\psi \equiv \neg\phi$

Thus for any ϕ, $\phi^* \equiv \neg\phi$.

5 Problem 5

5.1 a)

$\{\neg, \wedge\}$: $\phi \vee \phi \equiv \neg(\phi \wedge \phi)$, $\phi \rightarrow \phi \equiv \neg\phi \vee \phi \equiv \neg\neg\phi \wedge \neg\phi$ Anything with $\vee$ *or* $\rightarrow$ can be represented by $\neg$ *and* $\wedge$ as shown.

$\{\neg, \rightarrow\}$: $\neg\phi \rightarrow \phi \equiv \phi \vee \phi$, which means using $\neg$ and $\rightarrow$ you can represent $\vee$, and the set $\{\neg, \vee\}$ is adequate as proved in the book.

$\{\neg, \perp\}$: $\neg\phi \equiv \phi \rightarrow \perp$ which means using $\neg$ and $\perp$ you can represent $\rightarrow$ and $\{\neg, \rightarrow\}$ is adequate as proved above.

5.2 b)

If C doesn't contain $\neg$ *or*$\perp$ then C $\subseteq \{\wedge, \vee, \rightarrow\}$. There is no way to represent $\neg$ P for a formula with only P and the connectives in C, this C isn't adequate without $\neg$ *or*$\perp$

5.3 c)

Truth Table for variables P and Q with the connectives $\{\neg, \leftrightarrow\}$

P	Q	¬P	¬Q	P↔Q	¬P↔Q	P↔ ¬Q	¬P↔ ¬Q
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	F
F	F	T	T	T	F	F	T

In no case is it possible for an odd number of T's, which means P $\wedge$ Q can not be represented by only these 2 operators, so $\{\neg, \leftrightarrow\}$ is not adequate.

6 Problem 6

Assume a formula with only 2 atoms so the possibilities are {p, ¬p, q, ¬q}.
The # operator doesn't care about the order of the elements so the possibilities with

only 2 atoms and their negations is $\#(\phi, \phi, \psi)$ or $\#(\phi, \neg\phi, \psi)$.
$\#(\phi, \phi, \psi) \equiv \phi$
$\#(\phi, \neg\phi, \psi) \equiv \psi$
ϕ and ψ are both one of the possible atoms, and their negations are also in that set. So the outcome is always equivalent to some single term. Thus, for example, it's impossible to represent p $\wedge$ q with only $\{\neg, \#\}$
.