Data Structures
Chapter 2

1. Recurrence Relations
2. **Discrete Math**
   - Logarithms
   - Summations
   - Inductive Proofs
3. Structure

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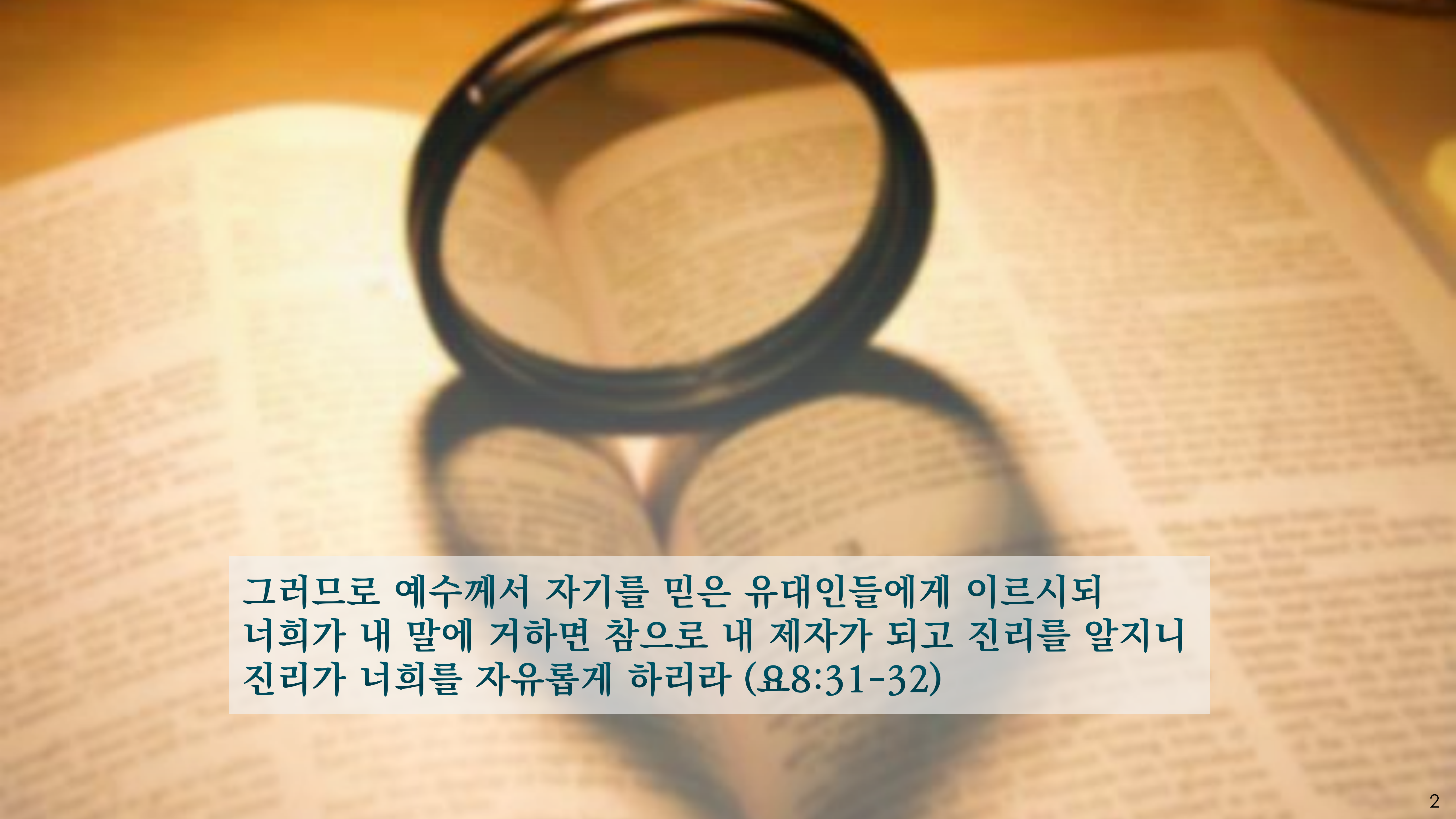


그러므로 예수께서 자기를 믿은 유대인들에게 이르시되
너희가 내 말에 거하면 참으로 내 제자가 되고 진리를 알지니
진리가 너희를 자유롭게 하리라 (요8:31-32)

1. Logarithms

- **Exponents:**
  - X^Y, or "**X to the Y^th power**", "X to the power of Y"
    X multiplied by itself Y times

- **Some useful identities:**
  - $X^0 = 1$, provided $x \neq 0$.
  - $X^A X^B = X^{A+B}$
  - $X^A / X^B = X^{A-B}$
  - $X^{-B} = \frac{1}{X^B}$
  - $X^{1/n} = \sqrt[n]{X}$
  - $X^N + X^N = 2X^N$
  - $2^N + 2^N = 2 \cdot 2^N = 2^{N+1}$

1. Logarithms

- **Logarithms**
  - ***definition***: $\boldsymbol{X^A = B}$ if and only if $\boldsymbol{log_X B = A}$
  - ***intuition***: $\boldsymbol{log_X B}$ means: the power $\boldsymbol{X}$ must be raised to, to get $\boldsymbol{B}$
  - This is the same as asserting $\boldsymbol{X^{log_X B} = B}$.
- Examples:
  - $log_X 1 = 0$
  - $log_X X = 1$
  - $\log_2 16 = 4$
  - $\log_{10} 1000 = 3$

- Most people use base 10, written as log_{10} or log, base e or ln .

- In computer science, we typically use base 2, would write it as $\boldsymbol{log_2}$ or lg.
  Between us, however, we simply use $\boldsymbol{log}$ instead of log_2 or $\boldsymbol{lg}$.

Example 1

- How many bits does it take to encode $1{,}000{,}000$ different values?
  - Each bit can take on one of two values (0 or 1).
  - Therefore, n bits can represent 2^n values. (ex. 8bits: 0~255)
  - So, encoding 1,000,000 values will require $log_2 n = 20$ bits.
    To be exact, $\lceil log_2(1{,}000{,}000) \rceil$ = 20.   // “a little under 20”

- $\lceil x \rceil$ → **Ceiling** function: the smallest integer $\geq$ x.
  - Ex.

$$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$$

- $\lfloor x \rfloor$ → **Floor** function: the largest integer $\leq$ x.
  - Ex.

$$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$$

Example 1

Powers of 2:

- A bit is 0 or 1 (just two different "letters" or "symbols")
- A sequence of *n* bits can represent 2^n distinct things
    - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java: an **`int`** is 32 bits and signed, so "max int" is "about 2 billion"

a **`long`** is 64 bits and signed, so "max long" is $2^{63}-1$

Example 2

- If we have an alphabetically **sorted list of 100 names**, how many records do we need to look at to find a given individual?
  - Since the list is sorted, we can use **binary search**.
  - Look at the middle element: if it's after than the name we're looking for, search the first half of the list. If it's before the name we're looking for, look at the second half of the list.
  - Each check cuts the size of the list in half;
    how many times can we do this?
  - If we think backwards, in terms of doubling the list, we'll need n doublings to generate a list of length 2^n.
  - If $2^n = 128$, then $n = log_2 128 = 7$.

Example 3:

- Let's suppose that we begin with a value N, divide it by 2, then the result that we divide it by 2, and so on, until reaching 1 or less.
  - N, N/2, N/4, ……., 4, 2, 1

- **Question:** How many times did we divide before reaching 1 or less?
  - Think of it from the other direction: How many times do I have to multiply by 2 to reach N?
  - 1, 2, 4, ….., N/4, N/2, N
    Call this **k** number of times, then $N = 2^{\boldsymbol{k}}$, or $\boldsymbol{k} = \log(N)$.

- **Exercise:** How is this related to the idea of binary search?
  - (I leave this one for you to think about it, as an exercise.)

Logarithmic Operators:

- $\log ab = \log a + \log b$
- $\log \frac{a}{b} = \log a \; - \; \log b$
- $\log a^{b} \; = b \log a$
- $log_{a}n = \; \frac{log_{b}n}{log_{b}a} = \frac{\log n}{\log a}$

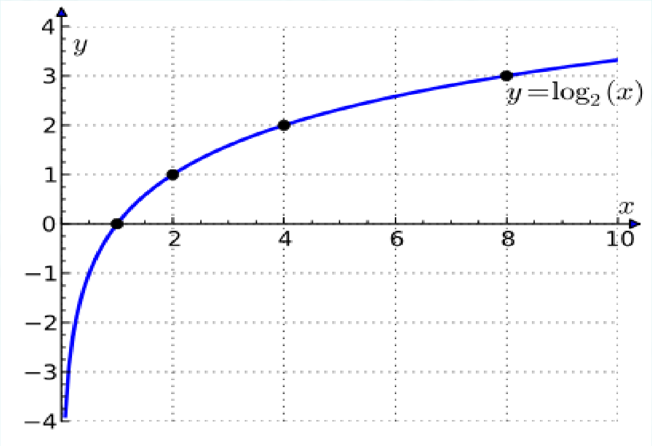
  (this is used to change bases.)
- $log_{a}a = 1, for \; all \; a > 0$
- $log_{a}1 = 0, for \; all \; a > 0$

1. Logarithms – Logarithmic Operators

Logarithmic Operators:

- $\log ab = \log a + \log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^b = b \log a$
- $log_a n = \frac{log_b n}{log_b a} = \frac{\log n}{\log a}$
  (this is used to change bases.)
- $log_a a = 1, for\ all\ a > 0$
- $log_a 1 = 0, for\ all\ a > 0$

- Evaluate $log_4 4 + log_2 2 + log_{10} 1$
- Evaluate $log_2 \frac{1}{2}$
- Plot $y = log_2 x$



Example 4

- Solve for x. $x^{x^{x^{\cdots}}} = 2$

$$x^{x^{x^{x^{\cdots}}}} = 2$$
$$\log x^{x^{x^{x^{\cdots}}}} = \log 2$$
$$x^{x^{x^{x^{\cdots}}}} \log x = \log 2$$
$$2 \log x = \log 2$$
$$\log x = \frac{1}{2} \log 2 = \log 2^{\frac{1}{2}}$$
$$x = 2^{\frac{1}{2}} = \sqrt{2}.$$

Example 5

- Evaluate $log_4 2 + log_4 32$
  - $log_4 2 + log_4 32 = log_4 2 \ * 32 = log_4 64 = (log_4 4^3) = 3$

- Evaluate $log_2 400$ (all most all calculators don't do base 2…. wow!)
  - $log_2 400 = \frac{log_e 400}{log_e 2} = \frac{5.991}{0.69} = 8.68$
- What is x?
  - $2^x = 2^{10} + 2^{10}$
  - $x \log 2 = \log (2^{10} + 2^{10})$
    $x = \log (2*2^{10})$,
    $x = \log 2 + \log 2^{10}$,
    $x = 1 + 10 = 11$

Example: Logarithm is fun

- Solve $y = \frac{log_{\mathrm{e}}(\frac{\mathrm{x}}{\mathrm{m}} - \mathrm{sa})}{r^2}$ for X-mas

Example 6

- Logarithms can also be very useful for comparing very large or small numbers.
- **Exercise**: $10^{100} > 2^{256}$ is true or not?
  Neither number can be calculated directly without risking overflow.

- Hint:
  Since the logarithm function is monotonically increasing, if $a < b$, then $\log a < \log b$.
-
-

2. Summations

- In analyzing a program's performance, we'll need to add up the number of times an operation is taken.
- It is typically written as:

$$\sum_{i=1}^{n} f(i)$$

which is equivalent to $f(1) + f(2) + \ldots + f(n)$

- **Example:**
  - summation of 1 … 10?
  - summation of 1 … n?

2. Summations

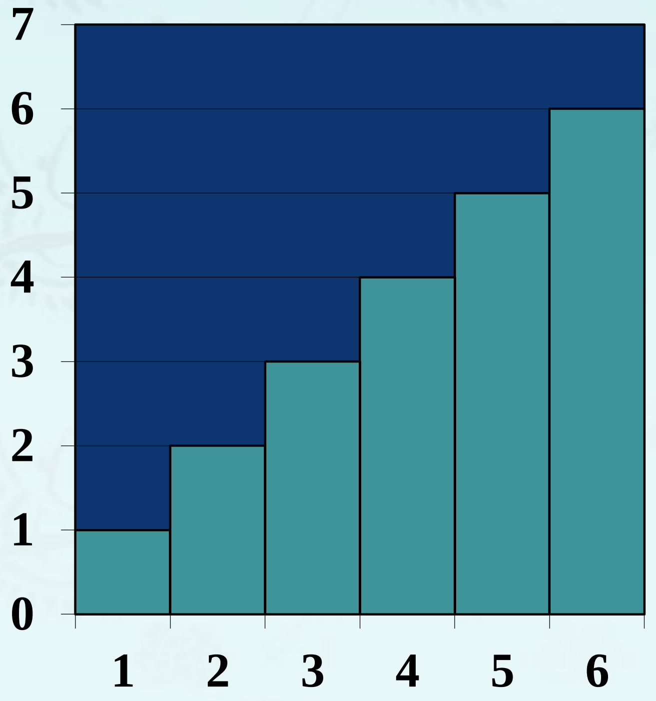
- We will be particularly interested in the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

2. Summations – Arithmetic Sum

- There is a simple visual proof of this fact for the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



2. Summations – Arithmetic Sum

- Some common summations and their closed-form solutions:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}$$

2. Summations – Geometric Sum

We are also be interested in the sum, it is so called **geometric** sum;

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \dots\dots + a^{n-1} + a^n$$

- **Proof:** Let's use S to denote the sum:
- $\boldsymbol{S} = \ 1 + a^1 + a^2 + \ \dots\dots + a^{n-1} + a^n$
- $\boldsymbol{aS} = \qquad a^1 + a^2 + \ \dots\dots + a^{n-1} + a^n + a^{n+1}$

$$= S + \ a^{n+1} \ - 1$$

From $aS = S + \ a^{n+1} \ - 1$, **we solve for $\boldsymbol{S}$, obtaining:**

$$\boldsymbol{S} \ = \sum_{i=0}^{n} a^i = \frac{a^{n+1} \ - \ 1}{a \ - 1}$$

2. Summations – Geometric Sum

We are also be interested in the sum, it is so called **geometric** sum;

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n = \frac{a^{n+1} - 1}{a - 1}$$

Exercise: $\sum_{i=0}^{n-1} a^i =$

Exercise: $\sum_{i=0}^{n} 2^i =$

2. Summations – Geometric Sum

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \dots \dots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $a < 1$ **and** n **goes to infinity**, the sum becomes

$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

Exercise: Compute the infinite "geometric" series intuitively and using arithmetic sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \dots \dots$$

2. Summations – Geometric Sum

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \dots\dots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $\boldsymbol{a} < \mathbf{1}$ **and** $\boldsymbol{n}$ **goes to infinity**, the sum becomes

$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

Exercise: Compute the infinite "geometric" series intuitively and using arithmetic sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots$$

2. Summations – Geometric Sum

Infinite geometric series....

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $\boldsymbol{a} < $ **1 and $\boldsymbol{n}$ goes to infinity**, the sum becomes

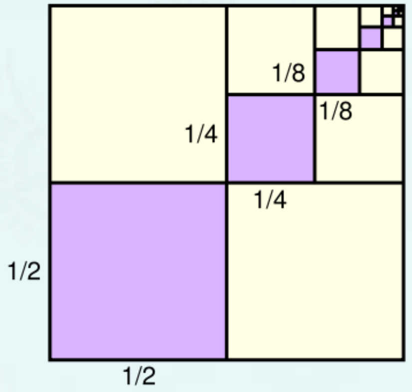
$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

Exercise: Compute the sum of the areas of the purple squares.
Intuitively? Using arithmetic sum?

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \ldots\ldots$$

$=$

$=$



3. Inductive Proof

- The three most common forms of proof are:
  1. Direct proof (sometimes called a constructive proof)
  2. Indirect proof or proof by contradiction
  3. **Inductive proof**
  - It is very similar to recursion.
  - You establish a **base case** that is proved directly.
  - This is followed by an **inductive step** which shows how the **hypothesis** holds for **larger cases**.

3. Inductive Proof

- **Inductive proof**
  - In data structures and algorithms, we often want to prove that something holds over a range of values
  - The **base case** will prove the theorem for the initial c values.
  - The **inductive step** will show that, if the theorem holds for $\boldsymbol{n-1}$, then it holds for $\boldsymbol{n}$. **Alternatively**, you may assume that it is true for n which is the inductive hypothesis first and then prove it for $\boldsymbol{n+1}$. or for **2n**.

This is so called the inductive hypothesis

3. Inductive Proof

Inductive proof Example: prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case**: Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1$.
- **Inductive step**: We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)(n)}{2}$$

- Assuming this is true, adding the $\boldsymbol{n}$**th term** yields:

$$\sum_{i=1}^{n} i = \frac{(n-1)(n)}{2} + n = \frac{(n-1)(n)}{2} + \frac{2n}{2} = \frac{n(n+1)}{2}$$

Therefore, we have proved it.

3. Inductive Proof

Inductive proof Example: prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case**: Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1.$
- **Inductive step**: Assume that it holds for n, then for $2n$; that is :

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = \boldsymbol{n(2n+1)}$$

Using the **induction hypothesis** that the left side above can be rewritten and rearranged algebraically:

$$\sum_{i=1}^{2n} i = \frac{n(n+1)}{2} + [(n+1) + (n+2) + \ldots + (n+n)]$$

$$= \frac{n(n+1)}{2} + n * n + \frac{n(n+1)}{2}$$

$$= n(n+1) + n * n$$

$$= \boldsymbol{n(2n+1)}$$

Therefore, we have proved it.

3. Inductive Proof

Inductive proof Exercise: prove $\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$

- **Base case**: Let $n = 1$. $\frac{2+3+1}{6} = 1$.
- **Inductive step**: We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6}$$

- Assuming this is true, adding the $\boldsymbol{n}$**th term** yields:

$$\sum_{i=1}^{n-1} i^2 + n^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6} + \frac{6n^2}{6}$$

$$= \frac{2n^3+3n^2+n}{6}$$

Therefore, we have proved it.

Exercise: Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

Summary

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