

# Reinforcement learning

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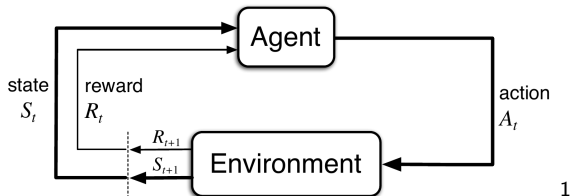
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- 1 Introduction
- 2 Tabular model-based algorithms
- 3 Tabular model-free algorithms
- 4 Non-tabular model-free algorithms
- 5 References

# Reinforcement learning



<sup>1</sup>Image from [Sutton and Barto, 2018]

# Reinforcement learning

- An agent interacts with an environment during a sequence of discrete time steps  $t = 0, 1, 2, \dots$
- At each time step  $t$ , the agent receives some representation of the state  $s_t \in \mathcal{S}$
- The agent then selects an action  $a_t \in \mathcal{A}(s_t)$
- One time step later, the agent receives a reward  $r_{t+1} \in \mathbb{R}$  and a new state  $s_{t+1} \in \mathcal{S}$
- A policy  $\pi : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$  is a function such that  $\pi(s, a)$  represents the probability that  $a_t = a$  given that  $s_t = s$

## Discounted return

- The discounted return  $u_t$  after time step  $t$  is given by

$$u_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k r_{t+1+k},$$

where  $0 \leq \gamma < 1$  is the discount factor

- A reward received  $k$  time steps into the future is only worth  $\gamma^{k-1}$  times what it would be worth if it were received on the next step
- If necessary, a state can transition only to itself and yield no rewards
- The objective of the agent is to maximize the discounted return

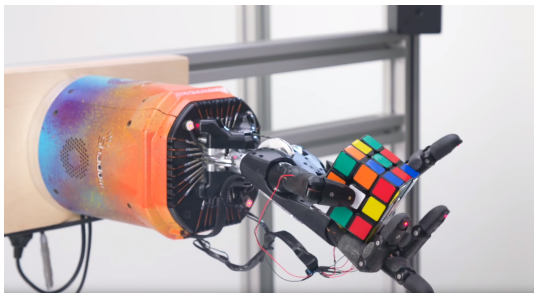
# Applications: games

- Atari [Mnih et al., 2015], Dota 2 [Brockman et al., 2019a], chess and Go [Silver et al., 2018]



# Applications: robotics

- Rubik's cube manipulation [Brockman et al., 2019b]



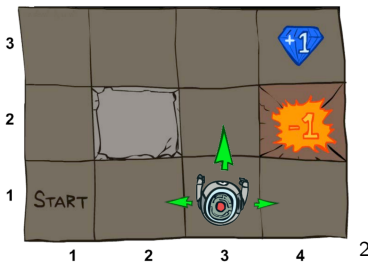
# Applications: others

- Practical:
  - Logistics
  - Finance
  - Marketing
- Theoretical:
  - Every task with a computable description can be formulated as a reinforcement learning problem [Hutter, 2004].



## Example: grid world

- States  $\mathcal{S} = \{1, 2, \dots, 12\}$ , actions  $\mathcal{A} = \{1, 2, 3, 4\}$
- Reward 1 on action at goal, reward  $-1$  on action at trap, and reward 0 on action at other states
- Actions at goal and trap transition to an absorbing state
- Discount factor  $\gamma = 0.9$



<sup>2</sup>Image from [Klein et al., 2019]

# Markov decision process

- A state that summarizes the entire past with all that is relevant for decision making has the Markov property
- In a Markov decision process, for any sequence of states, action and rewards  $s_t, a_t, r_t, \dots, r_1, s_0, a_0$  (history) and all  $s' \in \mathcal{S}, r' \in \mathbb{R}$ ,

$$P(S_{t+1} = s', R_{t+1} = r' \mid s_t, a_t, r_t, \dots, r_1, s_0, a_0) = \\ P(S_{t+1} = s', R_{t+1} = r' \mid s_t, a_t)$$

- The conditional joint probability distribution over states and rewards on the right side defines the one-step dynamics of the problem

$$\mathcal{P}_{ss'}^a$$

- The probability  $\mathcal{P}_{ss'}^a$  of transitioning from state  $s$  to state  $s'$  given action  $a$  is given by

$$\begin{aligned}\mathcal{P}_{ss'}^a &= P(S_{t+1} = s' \mid S_t = s, A_t = a) \\ &= \sum_{r'} P(S_{t+1} = s', R_{t+1} = r' \mid S_t = s, A_t = a).\end{aligned}$$

- Note that  $\mathcal{P}_{ss'}^a$  is independent of the current time step

$$\mathcal{R}_{ss'}^a$$

- The expected reward on transitioning from state  $s$  to state  $s'$  given the action  $a$  is given by

$$\begin{aligned}\mathcal{R}_{ss'}^a &= \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a, S_{t+1} = s'] \\ &= \frac{1}{\mathcal{P}_{ss'}^a} \sum_{r'} r' P(S_{t+1} = s', R_{t+1} = r' \mid A_t = a, S_t = s)\end{aligned}$$

- Note that  $\mathcal{R}_{ss'}^a$  is independent of the current time step

## Value function $V^\pi$

- The value  $V^\pi(s)$  of a state  $s \in \mathcal{S}$  is the expected (discounted) return of starting in state  $s$  and following the policy  $\pi$

$$V^\pi(s) = \mathbb{E}_\pi[U_t \mid S_t = s] = \mathbb{E}_\pi \left[ \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s \right] \quad (1)$$

## Action value function $Q^\pi$

- The value  $Q^\pi(s, a)$  of taking an action  $a \in \mathcal{A}$  when in state  $s \in \mathcal{S}$  and afterwards following the policy  $\pi$  is given by

$$\begin{aligned} Q^\pi(s, a) &= \mathbb{E}_\pi[U_t \mid S_t = s, A_t = a] \\ &= \mathbb{E}_\pi \left[ \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a \right] \end{aligned}$$

# Recursivity of the value function

## Theorem (Recursivity of the value function)

*For any policy  $\pi$  and state  $s \in \mathcal{S}$ ,*

$$V^\pi(s) = \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^\pi(s')]. \quad (2)$$

# Notation

- We denote random variables by upper case letters and assignments to these variables by corresponding lower case letters
- We omit the subscript that typically relates a probability function to random variables when there is no risk of ambiguity
- For example, let  $X$  and  $Y$  be discrete random variables. In the same context, we will let  $p(x|y)$  denote  $P(X = x|Y = y)$  and  $p(y|x)$  denote  $P(Y = y|X = x)$
- This notation where the arguments select between different probability functions is standard in machine learning



# Recursivity of the value function

## Proof.

Note that Equation 1 can be rewritten as

$$V^\pi(s) = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s) \sum_{k=0}^T \gamma^k r_{t+k+1},$$

where the dependency on the policy  $\pi$  becomes implicit. By marginalization,

$$V^\pi(s) = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} \left[ \sum_{a_t} \sum_{s_{t+1}} p(r_{t+1:T+t+1}, a_t, s_{t+1} \mid S_t = s) \right] \sum_{k=0}^T \gamma^k r_{t+k+1}.$$

# Recursivity of the value function

## Proof. (cont.)

By the chain rule of probability,

$$V^\pi(s) = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} \sum_{a_t} \sum_{s_{t+1}} p(a_t | S_t = s) p(s_{t+1} | S_t = s, a_t) p(r_{t+1:T+t+1} | S_t = s, a_t, s_{t+1}) \sum_{k=0}^T \gamma^k r_{t+k+1}.$$

By the distributive property and reordering the three outermost summations,

$$V^\pi(s) = \sum_{a_t} p(a_t | S_t = s) \sum_{s_{t+1}} p(s_{t+1} | S_t = s, a_t) \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} | S_t = s, a_t, s_{t+1}) \sum_{k=0}^T \gamma^k r_{t+k+1}. \quad (3)$$

## Recursivity of the value function

### Proof. (cont.)

Let  $E$  denote the limit in the previous equation, such that

$$E = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) \sum_{k=0}^T \gamma^k r_{t+k+1}.$$

By isolating the first term in the innermost summation,

$$E = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) \left[ r_{t+1} + \sum_{k=1}^T \gamma^k r_{t+k+1} \right].$$

# Recursivity of the value function

## Proof. (cont.)

By the linearity of expectation,  $E = E_1 + E_2$ , where

$$\begin{aligned} E_1 &= \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) r_{t+1} \\ &= \mathbb{E}_{\pi} [R_{t+1} \mid S_t = s, a_t, s_{t+1}] = \mathcal{R}_{ss_{t+1}}^{a_t}, \end{aligned}$$

and

$$E_2 = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) \sum_{k=1}^T \gamma^k r_{t+k+1}.$$

# Recursivity of the value function

## Proof. (cont.)

By changing the indices in the innermost summation,

$$E_2 = \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) \sum_{k=0}^{T-1} \gamma^{k+1} r_{t+k+2}.$$

By moving a constant factor of  $\gamma$  outside of the innermost summation,

$$E_2 = \gamma \lim_{T \rightarrow \infty} \sum_{r_{t+1:T+t+1}} p(r_{t+1:T+t+1} \mid S_t = s, a_t, s_{t+1}) \sum_{k=0}^{T-1} \gamma^k r_{t+k+2}.$$

## Recursivity of the value function

### Proof. (cont.)

Because  $R_{t+2:T+t+1} \perp\!\!\!\perp S_t, A_t \mid S_{t+1}$  due to the Markov property,

$$E_2 = \gamma \lim_{T \rightarrow \infty} \mathbb{E}_\pi \left[ \sum_{k=0}^{T-1} \gamma^k R_{t+k+2} \mid s_{t+1} \right] = \gamma V^\pi(s_{t+1}).$$

Returning to Equation 3,

$$V^\pi(s) = \sum_{a_t} p(a_t \mid S_t = s) \sum_{s_{t+1}} p(s_{t+1} \mid S_t = s, a_t) [\mathcal{R}_{ss_{t+1}}^{a_t} + \gamma V^\pi(s_{t+1})].$$

By making the dependency on  $\pi$  explicit and renaming variables,

$$V^\pi(s) = \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^\pi(s')],$$



# Recursivity of the action value function

## Theorem (Recursivity of the action value function)

*For any policy  $\pi$ , state  $s \in \mathcal{S}$ , and action  $a \in \mathcal{S}$ ,*

$$Q^\pi(s, a) = \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma \sum_{a'} \pi(s', a') Q^\pi(s', a')].$$

## Relationship between $V^\pi$ and $Q^\pi$

### Theorem (Relationship between $V^\pi$ and $Q^\pi$ )

*For any policy  $\pi$ , state  $s \in \mathcal{S}$ , and action  $a \in \mathcal{S}$ ,*

$$V^\pi(s) = \sum_a \pi(s, a) Q^\pi(s, a),$$

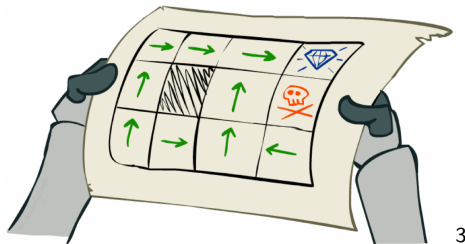
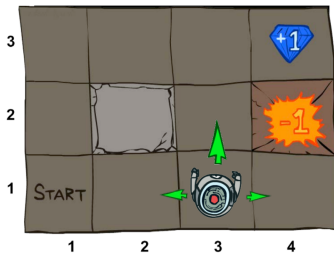
*and*

$$Q^\pi(s, a) = \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^\pi(s')].$$



# Optimal policies

- Let  $\pi \geq \pi'$  if and only if  $V^\pi(s) \geq V^{\pi'}(s)$  for all  $s \in \mathcal{S}$ .
- A policy  $\pi^*$  is optimal if  $\pi^* \geq \pi$  for any policy  $\pi$
- An optimal policy always exists, but is not necessarily unique



<sup>3</sup>Image from [Klein et al., 2019]

# Optimal value functions

## Theorem (Bellman optimality equations)

*For any action  $a \in \mathcal{A}$  and state  $s \in \mathcal{S}$ , the optimal state value function  $V^*$  and the optimal action value function  $Q^*$  are given by*

$$V^*(s) \triangleq \max_{\pi} V^{\pi}(s) = \max_a \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^*(s')],$$

*and*

$$Q^*(s, a) \triangleq \max_{\pi} Q^{\pi}(s, a) = \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma \max_{a'} Q^*(s', a')].$$

# Reinforcement learning algorithms

- Reinforcement learning algorithms aim to find an optimal policy  $\pi^*$  for a given environment
- For any state  $s \in \mathcal{S}$ , an optimal policy  $\pi^*$  can be found given either  $V^*$  or  $Q^*$
- In the case of  $Q^*$ , for any  $s \in \mathcal{S}$ , it suffices to choose an  $a$  such that  $Q^*(s, a)$  is maximal
- In the case of  $V^*$ , for any  $s \in \mathcal{S}$ , it suffices to choose one of the actions  $a$  that maximizes the right hand side of the Bellman optimality equation

- 1 Introduction
- 2 Tabular model-based algorithms**
- 3 Tabular model-free algorithms
- 4 Non-tabular model-free algorithms
- 5 References

# Dynamic programming

- Dynamic programming algorithms can be used to compute optimal policies given a perfect model of the environment (one-step dynamics) when the sets of states and actions are finite
- The problem of finding the optimal value functions has optimal substructure: it can be solved by breaking it into sub-problems and then recursively finding the solutions to the sub-problems

# Policy evaluation

- Policy evaluation is an iterative algorithm to compute the state value function  $V^\pi$  for an arbitrary policy  $\pi$
- It relies on creating a sequence  $V_0, V_1, \dots$  of estimates of  $V^\pi$  given by

$$V_{k+1}(s) = \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V_k(s')],$$

for all  $s \in \mathcal{S}$

- The initial value estimate  $V_0$  can be arbitrary
- The sequence  $V_0(s), V_1(s), \dots$  converges to  $V^\pi(s)$  for all  $s \in \mathcal{S}$

## In-place policy evaluation

- Instead of computing the new estimate  $V_{k+1}$  using the old estimate  $V_k$ , it is also possible to change a single estimate  $V$  in-place using

$$V(s) \leftarrow \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V(s')],$$

for all  $s \in \mathcal{S}$

- The estimate  $V(s)$  also converges to  $V^\pi(s)$  for all  $s \in \mathcal{S}$  after repeated passes over all states

# In-place policy evaluation

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**Algorithm 1** Iterative policy evaluation (in-place)

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**Input:** policy  $\pi$ , one-step dynamics functions  $\mathcal{P}$  and  $\mathcal{R}$ , discount factor  $\gamma$ , tolerance  $\theta$ .

**Output:** Value function  $V = V^\pi$  when  $\theta \rightarrow 0$ .

```
1: for each  $s \in \mathcal{S}$  do
2:    $V(s) \leftarrow 0$ 
3: end for
4: repeat
5:    $\Delta \leftarrow 0$ 
6:   for each  $s \in \mathcal{S}$  do
7:      $v \leftarrow V(s)$ 
8:      $V(s) \leftarrow \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V(s')]$ 
9:      $\Delta \leftarrow \max(\Delta, |v - V(s)|)$ 
10:  end for
11: until  $\Delta < \theta$ 
```

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# Convergence of iterative policy evaluation

## Definition (Norm)

Consider a vector space  $Z$  over a field  $F$ . A function  $\|\cdot\| : Z \rightarrow [0, \infty)$  is a norm if

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|,$$

$$\|a\mathbf{v}\| = |a|\|\mathbf{v}\|,$$

$$\|\mathbf{v}\| = 0 \implies \mathbf{v} = \mathbf{0},$$

for all  $\mathbf{u}, \mathbf{v} \in Z$  and  $a \in F$ .

# Convergence of iterative policy evaluation

## Definition (Euclidean norm)

The Euclidean norm  $\|\cdot\|_2 : \mathbb{R}^d \rightarrow [0, \infty)$  is given by

$$\|\mathbf{v}\|_2 = \sqrt{\sum_i v_i^2}.$$

## Definition (Maximum norm)

The maximum norm  $\|\cdot\|_\infty : \mathbb{R}^d \rightarrow [0, \infty)$  is given by

$$\|\mathbf{v}\|_\infty = \max_i |v_i|.$$

# Convergence of iterative policy evaluation

## Definition (Convergence of a sequence)

A sequence  $(\mathbf{v}_n)_{n \geq 0} = \mathbf{v}_0, \mathbf{v}_1, \dots$  is said to converge to a vector  $\mathbf{v}$  in the norm  $\|\cdot\|$ , denoted  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$ , if

$$\lim_{n \rightarrow \infty} \|\mathbf{v}_n - \mathbf{v}\| = 0.$$

# Convergence of iterative policy evaluation

## Definition (Bellman operator)

Consider a reinforcement learning task with states  $\mathcal{S} = \{1, 2, \dots, |\mathcal{S}|\}$ , actions  $\mathcal{A} = \{1, 2, \dots, |\mathcal{A}|\}$ , and discount factor  $\gamma < 1$ . For any vector  $\mathbf{v} \in \mathbb{R}^{|\mathcal{S}|}$  and state  $s \in \mathcal{S}$ , the Bellman operator  $T^\pi : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  for the policy  $\pi$  is given by

$$T^\pi(\mathbf{v})_s = \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma v_{s'}].$$

# Convergence of iterative policy evaluation

## Theorem (Convergence of iterative policy evaluation)

Consider the Bellman operator  $T^\pi : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  for the policy  $\pi$ . Given an arbitrary  $\mathbf{v}_0 \in \mathbb{R}^{|\mathcal{S}|}$ , consider also the sequence  $(\mathbf{v}_n)_{n \geq 0}$  where  $\mathbf{v}_{k+1} = T^\pi(\mathbf{v}_k)$ . Finally, consider the vector  $\mathbf{v}^\pi \in \mathbb{R}^{|\mathcal{S}|}$  such that  $v_s^\pi = V^\pi(s)$ , for any  $s \in \mathcal{S}$ . For any  $n \geq 0$ ,

$$\mathbf{v}_n \xrightarrow{\|\cdot\|_\infty} \mathbf{v}^\pi,$$

$$\mathbf{v}^\pi = T^\pi(\mathbf{v}^\pi),$$

$$\|\mathbf{v}_n - \mathbf{v}^\pi\|_\infty \leq \gamma^n \|\mathbf{v}_0 - \mathbf{v}^\pi\|_\infty.$$

# Convergence of iterative policy evaluation

## Definition (Cauchy sequence)

Consider a normed vector space  $(Z, \|\cdot\|)$ . A sequence  $(\mathbf{v}_n)_{n \geq 0} = \mathbf{v}_0, \mathbf{v}_1, \dots$  of vectors in this space is Cauchy if

$$\lim_{n \rightarrow \infty} \sup_{m \geq n} \|\mathbf{v}_n - \mathbf{v}_m\| = 0.$$

In other words, if a sequence  $(\mathbf{v}_n)_{n \geq 0}$  is Cauchy, then for every  $\epsilon > 0$  there is an  $N$  such that for every  $n \geq N$ , we have  $\sup_{m \geq n} \|\mathbf{v}_n - \mathbf{v}_m\| < \epsilon$ .

# Convergence of iterative policy evaluation

## Definition (Banach space)

A Banach space is a normed vector space  $(Z, \|\cdot\|)$  where if  $(\mathbf{v}_n)_{n \geq 0}$  is a Cauchy sequence then  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$  for some vector  $\mathbf{v}$ .

For any  $d$ , both  $(\mathbb{R}^d, \|\cdot\|_2)$  and  $(\mathbb{R}^d, \|\cdot\|_\infty)$  are Banach spaces, although we omit the corresponding proofs.

# Convergence of iterative policy evaluation

## Definition (L-contraction)

Consider a normed vector space  $(Z, \|\cdot\|)$  and a function  $T : Z \rightarrow Z$ . The function  $T$  is  $L$ -Lipschitz if

$$\|T(\mathbf{u}) - T(\mathbf{v})\| \leq L\|\mathbf{u} - \mathbf{v}\|,$$

for all  $\mathbf{u}, \mathbf{v} \in Z$ . If  $L < 1$ , then  $T$  is also an  $L$ -contraction.



# Convergence of iterative policy evaluation

## Lemma

*Consider a normed vector space  $(Z, \|\cdot\|)$ , an  $L$ -Lipschitz function  $T : Z \rightarrow Z$ , and a sequence  $(\mathbf{v}_n)_{n \geq 0} = \mathbf{v}_0, \mathbf{v}_1, \dots$  of vectors in this space. If  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$ , then  $T(\mathbf{v}_n) \xrightarrow{\|\cdot\|} T(\mathbf{v})$ .*

# Convergence of iterative policy evaluation

## Proof.

For any  $n \geq 0$ , by the definition of an  $L$ -Lipschitz function,

$$0 \leq \|T(\mathbf{v}_n) - T(\mathbf{v})\| \leq L\|\mathbf{v}_n - \mathbf{v}\|.$$

Since  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$ ,

$$\lim_{n \rightarrow \infty} L\|\mathbf{v}_n - \mathbf{v}\| = L \lim_{n \rightarrow \infty} \|\mathbf{v}_n - \mathbf{v}\| = 0.$$

By the squeeze theorem,

$$\lim_{n \rightarrow \infty} \|T(\mathbf{v}_n) - T(\mathbf{v})\| = 0.$$



# Convergence of iterative policy evaluation

## Theorem (Banach's fixed point theorem)

*If  $(Z, \|\cdot\|)$  is a Banach space and  $T : Z \rightarrow Z$  is an  $L$ -contraction, then  $T$  has a unique fixed point  $\mathbf{v}$ . Furthermore, for any  $\mathbf{v}_0 \in Z$ , let  $\mathbf{v}_{n+1} = T(\mathbf{v}_n)$ . For any  $n \geq 0$ ,*

$$\begin{aligned}\mathbf{v}_n &\xrightarrow{\|\cdot\|} \mathbf{v}, \\ \|\mathbf{v}_n - \mathbf{v}\| &\leq L^n \|\mathbf{v}_0 - \mathbf{v}\|.\end{aligned}$$

# Convergence of iterative policy evaluation

## Proof.

We first show that the sequence  $(\mathbf{v}_n)_{n \geq 0}$  is Cauchy, which guarantees that  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$  for some vector  $\mathbf{v}$ .

As a first step, we show that  $\|\mathbf{v}_{n+k} - \mathbf{v}_n\| \leq L^n \|\mathbf{v}_k - \mathbf{v}_0\|$  for any  $n, k \geq 0$ . The case  $n = 0$  is trivial. Suppose that the inductive hypothesis is true for some  $n$ , and consider the case  $n + 1$ :

$$\begin{aligned} \|\mathbf{v}_{n+k+1} - \mathbf{v}_{n+1}\| &= \|T(\mathbf{v}_{n+k}) - T(\mathbf{v}_n)\| && \text{(definition of the sequence)} \\ &\leq L \|\mathbf{v}_{n+k} - \mathbf{v}_n\| && \text{(definition of } L\text{-contraction)} \\ &\leq L^{n+1} \|\mathbf{v}_k - \mathbf{v}_0\|, && \text{(inductive hypothesis)} \end{aligned}$$

as we wanted to show.

# Convergence of iterative policy evaluation

## Proof. (cont.)

For  $k \geq 1$ ,  $\|\mathbf{v}_k - \mathbf{v}_0\| = \|\mathbf{v}_k + (-\mathbf{v}_{k-1} + \mathbf{v}_{k-1}) + \dots + (-\mathbf{v}_1 + \mathbf{v}_1) - \mathbf{v}_0\|$ .  
Therefore,

$$\|\mathbf{v}_k - \mathbf{v}_0\| = \left\| \sum_{i=1}^k \mathbf{v}_i - \mathbf{v}_{i-1} \right\| \quad (\text{reorganizing terms})$$

$$\leq \sum_{i=1}^k \|\mathbf{v}_i - \mathbf{v}_{i-1}\| \quad (\text{triangle inequality})$$

$$\leq \sum_{i=1}^k L^{i-1} \|\mathbf{v}_1 - \mathbf{v}_0\| \quad (\text{earlier result})$$

$$\leq \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L}. \quad \left( \lim_{n \rightarrow \infty} \sum_{i=0}^n L^i = \frac{1}{1 - L} \right)$$

# Convergence of iterative policy evaluation

## Proof. (cont.)

We are now close to showing that  $(\mathbf{v}_n)_{n \geq 0}$  is Cauchy. For any  $n, k \geq 0$ , combining the previous two results,

$$0 \leq \|\mathbf{v}_{n+k} - \mathbf{v}_n\| \leq L^n \|\mathbf{v}_k - \mathbf{v}_0\| \leq L^n \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L}.$$

Therefore, for any fixed  $n \geq 0$ ,

$$0 \leq \sup_{k \geq 0} \|\mathbf{v}_{n+k} - \mathbf{v}_n\| \leq \sup_{k \geq 0} L^n \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L} = L^n \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L}.$$

# Convergence of iterative policy evaluation

Proof. (cont.)

Because  $0 \leq L < 1$ ,

$$\lim_{n \rightarrow \infty} L^n \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L} = \frac{\|\mathbf{v}_1 - \mathbf{v}_0\|}{1 - L} \lim_{n \rightarrow \infty} L^n = 0.$$

Therefore, by the squeeze theorem,

$$\lim_{n \rightarrow \infty} \sup_{k \geq 0} \|\mathbf{v}_{n+k} - \mathbf{v}_n\| = 0,$$

which completes the proof that  $(\mathbf{v}_n)_{n \geq 0}$  is Cauchy. Let  $\mathbf{v}$  denote the vector such that  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$ .

# Convergence of iterative policy evaluation

## Proof. (cont.)

Our next step is to show that  $\mathbf{v}$  is a fixed point of  $T$ . For any  $n$ ,

$$\begin{aligned} 0 \leq \|T(\mathbf{v}) - \mathbf{v}\| &= \|T(\mathbf{v}) + (-T(\mathbf{v}_n) + T(\mathbf{v}_n)) - \mathbf{v}\| && \text{(introducing zeros)} \\ &\leq \|T(\mathbf{v}) - T(\mathbf{v}_n)\| + \|T(\mathbf{v}_n) - \mathbf{v}\| && \text{(triangle inequality)} \\ &\leq L\|\mathbf{v} - \mathbf{v}_n\| + \|\mathbf{v}_{n+1} - \mathbf{v}\| && (L\text{-contraction}) \end{aligned}$$

Because  $\mathbf{v}_n \xrightarrow{\|\cdot\|} \mathbf{v}$ ,

$$\lim_{n \rightarrow \infty} L\|\mathbf{v} - \mathbf{v}_n\| + \|\mathbf{v}_{n+1} - \mathbf{v}\| = 0.$$



# Convergence of iterative policy evaluation

## Proof. (cont.)

Therefore, by the squeeze theorem

$$\|T(\mathbf{v}) - \mathbf{v}\| = \lim_{n \rightarrow \infty} \|T(\mathbf{v}) - \mathbf{v}\| = 0.$$

By the definition of a norm,  $T(\mathbf{v}) - \mathbf{v} = \mathbf{0}$ , which implies  $T(\mathbf{v}) = \mathbf{v}$ , completing the proof.

# Convergence of iterative policy evaluation

## Proof. (cont.)

Our next step is to show that the fixed point of  $T$  is unique. Suppose that  $T(\mathbf{u}) = \mathbf{u}$  and  $T(\mathbf{v}) = \mathbf{v}$  for some vectors  $\mathbf{u}$  and  $\mathbf{v}$ . In that case,

$$\|\mathbf{u} - \mathbf{v}\| = \|T(\mathbf{u}) - T(\mathbf{v})\| \leq L\|\mathbf{u} - \mathbf{v}\|.$$

If we suppose that  $\|\mathbf{u} - \mathbf{v}\| > 0$ , dividing the inequation by  $\|\mathbf{u} - \mathbf{v}\|$  leads to the conclusion that  $L \geq 1$ . However,  $T$  is an  $L$ -contraction, contradicting our supposition. Therefore,  $\|\mathbf{u} - \mathbf{v}\| \leq 0$ , which implies that  $\mathbf{u} = \mathbf{v}$ .

# Convergence of iterative policy evaluation

## Proof. (cont.)

Our last step is to show that  $\|\mathbf{v}_n - \mathbf{v}\| \leq L^n \|\mathbf{v}_0 - \mathbf{v}\|$ , for any  $n$ . The case  $n = 0$  is trivial. Suppose that the inductive hypothesis is true for some  $n$ , and consider the case  $n + 1$ :

$$\begin{aligned}\|\mathbf{v}_{n+1} - \mathbf{v}\| &= \|T(\mathbf{v}_n) - T(\mathbf{v})\| && \text{(definition of fixed point)} \\ &\leq L \|\mathbf{v}_n - \mathbf{v}\| \leq L^{n+1} \|\mathbf{v}_0 - \mathbf{v}\|, && \text{(inductive hypothesis)}\end{aligned}$$

as we wanted to show. □

# Convergence of iterative policy evaluation

## Theorem (Convergence of iterative policy evaluation)

Consider the Bellman operator  $T^\pi : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  for the policy  $\pi$ . Given an arbitrary  $\mathbf{v}_0 \in \mathbb{R}^{|\mathcal{S}|}$ , consider also the sequence  $(\mathbf{v}_n)_{n \geq 0}$  where  $\mathbf{v}_{k+1} = T^\pi(\mathbf{v}_k)$ . Finally, consider the vector  $\mathbf{v}^\pi \in \mathbb{R}^{|\mathcal{S}|}$  such that  $v_s^\pi = V^\pi(s)$ , for any  $s \in \mathcal{S}$ . For any  $n \geq 0$ ,

$$\mathbf{v}_n \xrightarrow{\|\cdot\|_\infty} \mathbf{v}^\pi,$$

$$\mathbf{v}^\pi = T^\pi(\mathbf{v}^\pi),$$

$$\|\mathbf{v}_n - \mathbf{v}^\pi\|_\infty \leq \gamma^n \|\mathbf{v}_0 - \mathbf{v}^\pi\|_\infty.$$

# Convergence of iterative policy evaluation

## Proof.

As a first step, we show that  $\mathbf{v}^\pi$  is a fixed point of  $T^\pi$ . For any  $s \in \mathcal{S}$ , by the definition of  $T^\pi$  and  $V^\pi$ ,

$$T^\pi(\mathbf{v}^\pi)_s = \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma v_{s'}^\pi] = v_s^\pi.$$

Our next step is to show that the Bellman operator  $T^\pi : \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}^{|\mathcal{S}|}$  is a  $\gamma$ -contraction. Because  $(\mathbb{R}^{|\mathcal{S}|}, \|\cdot\|_\infty)$  is a Banach space and  $\mathbf{v}^\pi$  is a fixed point of  $T^\pi$ , the desired results follow from Banach's fixed point theorem.

# Convergence of iterative policy evaluation

## Proof. (cont.)

Note that, for any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{|\mathcal{S}|}$  and every state  $s \in \mathcal{S}$ ,

$$\begin{aligned} T^\pi(\mathbf{u})_s - T^\pi(\mathbf{v})_s &= \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma u_{s'}] \\ &\quad - \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma v_{s'}] \\ &= \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a \gamma u_{s'} - \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a \gamma v_{s'} \\ &= \gamma \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a [u_{s'} - v_{s'}]. \end{aligned}$$

# Convergence of iterative policy evaluation

## Proof.

By the definition of maximum norm, for any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{|\mathcal{S}|}$ ,

$$\begin{aligned} & \|T^\pi(\mathbf{u}) - T^\pi(\mathbf{v})\|_\infty \\ &= \gamma \max_s \left| \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a [u_{s'} - v_{s'}] \right| && \text{(definition of maximum norm)} \\ &\leq \gamma \max_s \sum_a \sum_{s'} \left| \pi(s, a) \mathcal{P}_{ss'}^a [u_{s'} - v_{s'}] \right| && \text{(triangle inequality)} \\ &= \gamma \max_s \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a |u_{s'} - v_{s'}| && \text{(multiplicativity)} \\ &\leq \gamma \max_s \sum_a \sum_{s'} \pi(s, a) \mathcal{P}_{ss'}^a \|\mathbf{u} - \mathbf{v}\|_\infty && \text{(definition of maximum norm)} \\ &= \gamma \|\mathbf{u} - \mathbf{v}\|_\infty \max_s \sum_a \pi(s, a) \sum_{s'} \mathcal{P}_{ss'}^a && \text{(distributivity)} \\ &= \gamma \|\mathbf{u} - \mathbf{v}\|_\infty && \text{(unit measure).} \end{aligned}$$

# Deterministic policies

- A deterministic policy  $\pi$  is one such that, for all  $s \in \mathcal{S}$ ,  $\pi(s, a) = 1$  for some  $a \in \mathcal{A}$  and  $\pi(s, b) = 0$  for all  $b \neq a$
- In this case, we abuse notation and represent a policy by a function  $\pi : \mathcal{S} \rightarrow \mathcal{A}$  from states to actions



# Policy improvement

- Let  $\pi$  and  $\pi'$  be any pair of deterministic policies such that, for all  $s \in \mathcal{S}$ ,  $Q^\pi(s, \pi'(s)) \geq V^\pi(s)$
- The policy improvement theorem guarantees that  $V^{\pi'}(s) \geq V^\pi(s)$  for all  $s \in \mathcal{S}$
- For all  $s \in \mathcal{S}$ , a policy  $\pi$  may be improved to a policy  $\pi'$  by letting

$$\pi'(s) = \arg \max_a Q^\pi(s, a) = \arg \max_a \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V^\pi(s')]$$

# Policy iteration

- Policy evaluation and policy improvement can be interleaved
- This process produces the sequence

$$\pi_0, V^{\pi_0}, \pi_1, V^{\pi_1}, \pi_2, V^{\pi_2}, \dots$$

- If  $\pi_t = \pi_{t+1}$ , then  $\pi_t$  is optimal by the uniqueness of  $V^*$
- The initial policy  $\pi_0$  can be arbitrary

# Value iteration

- A more efficient alternative iteratively improves the estimates for the value of each state under an optimal policy
- It relies on creating a sequence  $V_0, V_1, \dots$  of estimates given by:

$$V_{k+1}(s) = \max_a \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V_k(s')]$$

- The initial estimate  $V_0$  can be arbitrary
- The sequence  $V_0(s), V_1(s), \dots$  converges to  $V^*(s)$  for all  $s \in \mathcal{S}$
- In-place value iteration has the same guarantees

# Value iteration

---

## Algorithm 2 Value iteration (in-place)

---

**Input:** one-step dynamics ( $\mathcal{P}$  and  $\mathcal{R}$ ), discount factor  $\gamma$ , and tolerance  $\theta$ .

**Output:** optimal deterministic policy  $\pi$  when  $\theta \rightarrow 0$ .

```
1: for each  $s \in \mathcal{S}$  do
2:    $V(s) \leftarrow 0$ 
3: end for
4: repeat
5:    $\Delta \leftarrow 0$ 
6:   for each  $s \in \mathcal{S}$  do
7:      $v \leftarrow V(s)$ 
8:      $V(s) \leftarrow \max_a \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V(s')]$ 
9:      $\Delta \leftarrow \max(\Delta, |v - V(s)|)$ 
10:  end for
11: until  $\Delta < \theta$ 
12: for each  $s \in \mathcal{S}$  do
13:    $\pi(s) = \arg \max_a \sum_{s'} \mathcal{P}_{ss'}^a [\mathcal{R}_{ss'}^a + \gamma V(s')]$ 
14: end for
```

---

## Unknown one-step dynamics

- It is always possible to estimate the one-step dynamics by interacting with the environment
- For any  $s, s' \in \mathcal{S}$  and  $a \in \mathcal{A}$ , the simplest (maximum likelihood) estimate for  $\mathcal{P}_{ss'}^a$  is given by

$$\hat{\mathcal{P}}_{ss'}^a = \frac{N(s', s, a)}{N(s, a)},$$

where  $N(s, a) > 0$  is the number of times that action  $a$  was taken at state  $s$ , and  $N(s', s, a)$  is the number of times that state  $s'$  was observed after action  $a$  was taken at state  $s$

- The estimates  $\hat{\mathcal{P}}_{ss'}^a$  and  $\hat{\mathcal{R}}_{ss'}^a$  can be combined with value iteration
- Robust alternative: posterior sampling for reinforcement learning [Osband et al., 2013]

# Exercises

- 1 Implement the environment described in p. 9. Note: a reward is obtained on going from (and not to) the trap/goal state to the absorbing state
- 2 Implement policy evaluation, policy improvement, policy iteration, and value iteration<sup>4</sup>
- 3 Consider an optimal policy for the environment described in p. 9. Using policy evaluation, manually compute the sequence  $V_0, V_1, \dots, V_k$  for a small  $k$ . Let  $V_0(s) = 0$  for all  $s \in \mathcal{S}$ . Compare this sequence to the sequence obtained by your implementation
- 4 Show that adding a constant to all the rewards in any given reinforcement learning problem simply adds a constant to the value of each state
- 5 (\*) Show the recursivity of the action value function (Th. 2, p. 23)
- 6 (\*) Show the relationship between  $V^\pi$  and  $Q^\pi$  (Th. 3, p. 24)
- 7 (\*) Show the recursivity of the Bellman optimality equations (Th. 4, p. 26)

---

<sup>4</sup>You may want to study <https://github.com/paulorauber/rl>

- 1 Introduction
- 2 Tabular model-based algorithms
- 3 Tabular model-free algorithms**
- 4 Non-tabular model-free algorithms
- 5 References

# Monte Carlo control

- Monte Carlo control methods find an optimal policy without estimating the one-step dynamics by interleaving policy evaluation and policy improvement
- These methods require an episodic problem, where there is a transition to an absorbing state after a finite number of time steps
- Policy evaluation for  $\pi$  consists of experiencing several episodes and averaging the returns that follow every possible state action pair  $(s, a)$  to obtain an estimate of  $Q^\pi(s, a)$ .
- In practice, “policy improvement” based on  $\pi$  is performed before a reliable estimate of  $Q^\pi$  is available



# Exploration

- For a given state  $s$ , an  $\epsilon$ -greedy policy with respect to an estimate  $Q$  of the action value function chooses a random action with probability  $\epsilon$ , and an action  $\arg \max_a Q(s, a)$  with probability  $1 - \epsilon$
- Monte Carlo control typically relies on  $\epsilon$ -greedy policies to ensure that the environment is explored sufficiently
- Exploration/exploitation trade-off: should the agent explore in order to learn about potentially new sources of reward or exploit the well-known sources of reward?

# Monte Carlo control

---

## Algorithm 3 Monte Carlo control algorithm

---

**Input:** set of states  $\mathcal{S}$ , number of episodes  $N$ , probability of choosing random action  $\epsilon$ .

**Output:** deterministic policy  $\pi$ , optimal when  $N \rightarrow \infty$ .

```
1: for each  $s \in \mathcal{S}$  do
2:   for each action  $a \in \mathcal{A}(s)$  do
3:      $Q(s, a) \leftarrow 0$ 
4:      $n(s, a) \leftarrow 0$ 
5:   end for
6: end for
7: for each  $i$  in  $\{1, \dots, N\}$  do
8:   Experience a new episode  $e$  following an  $\epsilon$ -greedy policy based on  $Q$ .
9:   for each state-action pair  $(s, a)$  in the episode  $e$  do
10:     $u \leftarrow$  return following  $(s, a)$  in the episode  $e$ .
11:     $n(s, a) \leftarrow n(s, a) + 1$ 
12:     $Q(s, a) \leftarrow Q(s, a) + \frac{1}{n(s, a)}[u - Q(s, a)]$ 
13:   end for
14: end for
15: for each state  $s \in \mathcal{S}$  do
16:    $\pi(s) \leftarrow \arg \max_a Q(s, a)$ 
17: end for
```

---

# Temporal difference

- Consider the tuple  $h_t = (s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$  obtained by an agent using a policy  $\pi$  to interact with an environment
- Let  $Q$  denote an estimate of the action value function  $Q^\pi$
- The temporal difference for  $(s_t, a_t)$  based on  $h_t$  and  $Q$  is given by

$$r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)$$

- In other words, the difference between the immediate reward plus the (estimated) expected return from the next state and the (estimated) expected return for the current state

## Sarsa control

- An algorithm bootstraps if it improves the estimate of the value of a state based on estimates of the values of other states
- Sarsa control is similar to Monte Carlo control, but it bootstraps based on temporal differences
- Sarsa control is comparatively more sample efficient, since it does not rely on the return that follows  $(s_t, a_t)$  after a single episode
- Given the tuple  $h_t$  and the estimate  $Q$ , Sarsa control updates its estimate of  $Q(s_t, a_t)$  using

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r_{t+1} + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)],$$

where  $\alpha$  is the so-called learning rate

# Sarsa control

---

## Algorithm 4 Sarsa control algorithm

---

**Input:** set of states  $\mathcal{S}$ , number of episodes  $N$ , learning rate  $\alpha$ , probability of random action  $\epsilon$ , discount factor  $\gamma$ .

**Output:** deterministic policy  $\pi$ , optimal when  $N \rightarrow \infty$  and  $\alpha$  decays appropriately.

```
1: for each  $(s, a) \in \mathcal{S} \times \mathcal{A}$  do
2:    $Q(s, a) \leftarrow 0$ 
3: end for
4: for each  $i$  in  $\{1, \dots, N\}$  do
5:    $s \leftarrow$  initial state for episode  $i$ 
6:   Select action  $a$  for state  $s$  according to an  $\epsilon$ -greedy policy based on  $Q$ .
7:   while state  $s$  is not terminal do
8:      $r \leftarrow$  observed reward for action  $a$  at state  $s$ 
9:      $s' \leftarrow$  observed next state for action  $a$  at state  $s$ 
10:    Select action  $a'$  for state  $s'$  according to an  $\epsilon$ -greedy policy based on  $Q$ .
11:     $Q(s, a) \leftarrow Q(s, a) + \alpha[r + \gamma Q(s', a') - Q(s, a)]$ 
12:     $s \leftarrow s'$ 
13:     $a \leftarrow a'$ 
14:   end while
15: end for
16: for each state  $s \in \mathcal{S}$  do
17:    $\pi(s) \leftarrow \arg \max_a Q(s, a)$ 
18: end for
```

# Q-learning

- An algorithm is off-policy if it learns about a policy that is different from the policy that it uses to act in the environment
- Q-learning learns about a greedy policy while acting using an  $\epsilon$ -greedy policy
- Q-learning control is similar to Sarsa control: both algorithms bootstrap based on temporal differences
- Given the tuple  $h_t$  and the estimate  $Q$ , Q-learning control updates its estimate of  $Q(s_t, a_t)$  using

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha[r_{t+1} + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t)]$$

where  $\alpha$  is the so-called learning rate

# Q-learning control

---

## Algorithm 5 Q-learning control algorithm

---

**Input:** set of states  $\mathcal{S}$ , number of episodes  $N$ , learning rate  $\alpha$ , probability of random action  $\epsilon$ , discount factor  $\gamma$ .

**Output:** deterministic policy  $\pi$ , optimal when  $N \rightarrow \infty$  and  $\alpha$  decays appropriately.

```
1: for each  $(s, a) \in \mathcal{S} \times \mathcal{A}$  do
2:    $Q(s, a) \leftarrow 0$ 
3: end for
4: for each  $i$  in  $\{1, \dots, N\}$  do
5:    $s \leftarrow$  initial state for episode  $i$ 
6:   while state  $s$  is not terminal do
7:     Select action  $a$  for state  $s$  according to an  $\epsilon$ -greedy policy based on  $Q$ .
8:      $r \leftarrow$  observed reward for action  $a$  at state  $s$ 
9:      $s' \leftarrow$  observed next state for action  $a$  at state  $s$ 
10:     $Q(s, a) \leftarrow Q(s, a) + \alpha[r + \gamma \max_{a'} Q(s', a') - Q(s, a)]$ 
11:     $s \leftarrow s'$ 
12:   end while
13: end for
14: for each state  $s \in \mathcal{S}$  do
15:    $\pi(s) \leftarrow \arg \max_a Q(s, a)$ 
16: end for
```

---

# Exercises

- 1 Implement an interactive version of the environment described in p. 9. This implementation must provide an initial state and respond to actions by transitioning between states and providing rewards.
- 2 Implement Monte Carlo control, Sarsa control, and Q-learning control <sup>5</sup>
- 3 Compare the results of these implementations to the results obtained by your implementations of policy iteration and value iteration
- 4 (\*) Implement the “Cliff World” environment and compare your results to those of [Greydanus and Olah, 2019]

---

<sup>5</sup>You may want to study <https://github.com/paulorauber/rl>



## Additional reading

- Reinforcement learning: an introduction [Sutton and Barto, 2018]
- Algorithms for reinforcement learning [Szepesvári, 2010]
- Notes on reinforcement learning [Rauber, 2015]
- UCL course on reinforcement learning [Silver, 2015]
- Deep reinforcement learning [Levine, 2018]
- Stanford course on reinforcement learning [Ng, 2008]
- Deep reinforcement learning bootcamp [Abbeel et al., 2017]
- Stanford course on reinforcement learning [Brunskill, 2019]

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