

calculations

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July 29, 2020

kernel

using a general kernel function

$$\int_{\mathbf{x}} W[\mathbf{x} - \mathbf{r}_a(t)] = 1. \quad (1)$$

in order to rewrite

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{f_a(t)}{\rho_a(t)}. \quad (2)$$

reference density

$$\rho(\mathbf{x}) = \sum_a W(\mathbf{x} - \mathbf{r}_a). \quad (3)$$

which gives

$$\int_{\mathbf{x}} \rho(\mathbf{x}) = N. \quad (4)$$

derivatives

useful computations

$$\rho(r_a) = \sum_b W(\mathbf{r}_a - \mathbf{r}_b). \quad (5)$$

the comoving derivative is

$$\partial_{\mathbf{r}_a} \rho_b = \sum_c \partial_{\mathbf{r}_a} W_{bc} = \sum_c \mathbf{Y}_{bc} (\delta_{ba} - \delta_{ca}) = \mathbf{Y}_{ab} + \delta_{ba} \sum_c \mathbf{Y}_{ac}. \quad (6)$$

where

$$\mathbf{Y}_{bc} = -\mathbf{Y}_{cb}. \quad (7)$$

a common form of term is

$$\sum_b f_b \partial_{\mathbf{r}_a} \rho_b = f_a \sum_c \mathbf{Y}_{ac} + \sum_b f_b \mathbf{Y}_{ab} = \sum_b (f_a + f_b) \mathbf{Y}_{ab}. \quad (8)$$

which has the nice property that

$$\sum_a \sum_b f_b \partial_{\mathbf{r}_a} \rho_b = 0. \quad (9)$$

lagrangean

starting from the non-relativistic case

$$\mathcal{L} = \frac{1}{2} \rho(\mathbf{x}) \mathbf{v}(\mathbf{x})^2 - \varepsilon(\mathbf{x}). \quad (10)$$

the canonical momentum density is

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \rho \mathbf{v}. \quad (11)$$

the hamiltonian density is

$$\mathcal{H} = \mathbf{v} \frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L} = \frac{\mathbf{p}^2}{2\rho} + \varepsilon. \quad (12)$$

the discretization needs to be consistently made in both scenarios

equations of motion

in the lagrangean side, a straight forward paramaterization is

$$\begin{aligned} L(\mathbf{r}, \dot{\mathbf{r}}) &= \frac{1}{2} \int_{\mathbf{x}} [\rho(x) v(x)^2] - \int_{\mathbf{x}} \varepsilon(x) \\ &= \frac{1}{2} \sum_a \mathbf{v}_a^2 - \sum_a \frac{\varepsilon_a}{\rho_a} \end{aligned} \tag{13}$$

$$\mathbf{p} = \int_{\mathbf{x}} \rho \mathbf{v} = \sum_a v_a. \tag{14}$$

on the other side

$$\begin{aligned} H(\mathbf{r}, \mathbf{p}) &= \frac{1}{2} \int_{\mathbf{x}} \frac{p(x)^2}{\rho(x)} + \int_{\mathbf{x}} \varepsilon(x) \\ &= \frac{1}{2} \sum_a \frac{\mathbf{p}_a^2}{\rho_a^2} + \sum_a \frac{\varepsilon_a}{\rho_a} \end{aligned} \tag{15}$$

equations of motions

and the equation of motion is

$$\frac{d}{dt} \partial_{\dot{\mathbf{r}}_a} L = \partial_{\mathbf{r}_a} L. \quad (16)$$

$$\frac{d\mathbf{r}_a}{dt} = \partial_{\mathbf{p}_a} H, \quad \frac{d\mathbf{p}_a}{dt} = -\partial_{\mathbf{r}_a} H. \quad (17)$$

the total derivative of the hamiltonian is

$$\frac{d}{dt} H(\mathbf{r}, \mathbf{p}) = \sum_a \frac{d\mathbf{p}_a}{dt} \partial_{\mathbf{p}_a} H + \sum_a \frac{d\mathbf{r}_a}{dt} \partial_{\mathbf{r}_a} H. \quad (18)$$

the angular momentum is

$$\frac{d}{dt} \sum_a \mathbf{r} \times \mathbf{p} = \sum_a \frac{d\mathbf{r}_a}{dt} \times \mathbf{p}_a + \sum_a \mathbf{r}_a \times \frac{d\mathbf{p}_a}{dt}. \quad (19)$$

equations of motions

and the equation of motion is

$$\frac{d\mathbf{r}_a}{dt} = \frac{\mathbf{p}_a}{\rho_a^2}. \quad (20)$$

$$\frac{d\mathbf{p}_a}{dt} = - \sum_b (F_a + F_b) \mathbf{Y}_{ab}, \quad F_a = \frac{\partial_\rho(\varepsilon)_a}{\rho_a} - \frac{\varepsilon_a}{\rho^2} - \frac{\mathbf{p}_a^2}{\rho_a^3}. \quad (21)$$

the angular momentum is

$$\begin{aligned} \sum_a \mathbf{r} \times \frac{d\mathbf{p}}{dt} &= \sum_a \sum_b (F_a + F_b) \mathbf{r}_a \times \mathbf{Y}_{ab} \\ &= \frac{1}{2} \sum_{ab} (F_a + F_b) (\mathbf{r}_a \times \mathbf{Y}_{ab} - \mathbf{r}_b \times \mathbf{Y}_{ab}) \\ &= \frac{1}{2} \sum_{ab} (F_a + F_b) (\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{Y}_{ab} \end{aligned} \quad (22)$$

continuity

$$\int_{\mathbf{x}} \rho(x) = \sum_a 1. \quad (23)$$

$$\int_{\mathbf{x}} \rho(x) \mathbf{v}(x) = \sum_a v_a. \quad (24)$$

this gives the canonical momentum as

$$\mathbf{p}_a = \partial_{\dot{\mathbf{r}}_a} L = \dot{\mathbf{r}}_a. \quad (25)$$

and the equation of motion is

$$\frac{d\mathbf{p}_a}{dt} = \partial_{\mathbf{r}_a} L = - \sum_b (\phi_a + \phi_b) \mathbf{Y}_{ab}. \quad (26)$$

$$\phi_a = \frac{\partial_{\rho} \varepsilon_a}{\rho_a} - \frac{\varepsilon_a}{\rho_a^2} = \frac{d}{d\rho} \frac{\varepsilon}{\rho}. \quad (27)$$

reference density

the hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \frac{p(x)^2}{\rho(x)} + \varepsilon(x). \quad (28)$$

using a generic reference density

$$\phi_a = \sum_b W_{ab}. \quad (29)$$

the integral of the hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \sum_a \frac{1}{2} \frac{\mathbf{p}_a^2}{\phi_a \rho_a} + \sum_a \frac{\varepsilon_a}{\phi_a}. \quad (30)$$

from the conservation of energy

$$\sum_a \frac{1}{2} \frac{d}{dt} \frac{\mathbf{p}_a^2}{\phi_a \rho_a} = - \sum_a \frac{d}{dt} \frac{\varepsilon_a}{\phi_a}. \quad (31)$$

reference density

the velocity equation is

$$\frac{d\mathbf{r}_a}{dt} = \frac{\mathbf{p}_a}{\rho_a \phi_a}, \quad \frac{d\phi_a}{dt} = \sum_b \left(\frac{\mathbf{p}_a}{\rho_a \phi_a} - \frac{\mathbf{p}_b}{\rho_b \phi_b} \right) \cdot \mathbf{Y}_{ab}. \quad (32)$$

from the conservation of energy

$$\mathbf{p}_a \cdot \frac{d\mathbf{p}_a}{dt} = -\phi_a \rho_a \frac{\mathbf{p}_a^2}{2} \frac{d}{dt} \frac{1}{\phi_a \rho_a} - \phi_a \rho_a \frac{d}{dt} \frac{\varepsilon_a}{\phi_a}. \quad (33)$$

$$\mathbf{p}_a \cdot \frac{d\mathbf{p}_a}{dt} = \frac{\mathbf{p}_a^2}{2} \left(\frac{1}{\rho_a} \frac{d\rho_a}{dt} + \frac{1}{\phi_a} \frac{d\phi_a}{dt} \right) - \rho_a \frac{d\varepsilon_a}{dt} + \frac{\rho_a \varepsilon_a}{\phi_a} \frac{d\phi_a}{dt}. \quad (34)$$

$$\mathbf{p}_a \cdot \frac{d\mathbf{p}_a}{dt} = \frac{\mathbf{p}_a^2}{2\rho_a} \frac{d\rho_a}{dt} - \rho_a \frac{d\varepsilon_a}{dt} + \left(\frac{\mathbf{p}_a^2}{2} + \rho_a \varepsilon_a \right) \frac{1}{\phi_a} \frac{d\phi_a}{dt}. \quad (35)$$

reference density

$$\frac{d}{dt} \int_{\mathbf{x}} \rho(\mathbf{x}) = \frac{d}{dt} \sum_a \frac{\rho_a}{\phi_a} = 0. \quad (36)$$

$$\sum_a \frac{1}{\phi_a} \frac{d\rho_a}{dt} = \sum_a \frac{\rho_a}{\phi_a^2} \frac{d\phi_a}{dt}. \quad (37)$$

$$\frac{d\rho_a}{dt} = \frac{\rho_a}{\phi_a} \frac{d\phi_a}{dt}. \quad (38)$$

$$\mathbf{p}_a \cdot \frac{d\mathbf{p}_a}{dt} = -\rho_a \frac{d\varepsilon_a}{dt} + \frac{1}{\phi_a} (\mathbf{p}_a^2 + \rho_a \varepsilon_a) \frac{d\phi_a}{dt}. \quad (39)$$

$$\frac{d\varepsilon_a}{dt} = -\frac{\mathbf{p}_a}{\rho_a} \cdot \frac{d\mathbf{p}_a}{dt} + \left(\frac{\mathbf{p}_a^2}{\phi_a \rho_a} + \frac{\varepsilon_a}{\phi_a} \right) \frac{d\phi_a}{dt}. \quad (40)$$

from the hamiltonian density

$$\frac{d\varepsilon(\mathbf{x})}{dt} = -\frac{\mathbf{p}(\mathbf{x})}{\rho(\mathbf{x})} \cdot \frac{d\mathbf{p}(\mathbf{x})}{dt} + \frac{\mathbf{p}(\mathbf{x})^2}{2\rho(\mathbf{x})^2} \frac{d\rho(\mathbf{x})}{dt}. \quad (41)$$

reference density

using

$$\phi_a = \varepsilon_a. \quad (42)$$

the equations reduce to

$$\frac{d\varepsilon_a}{dt} = -\frac{\mathbf{p}_a}{\rho_a} \cdot \frac{d\mathbf{p}_a}{dt} + \left(\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + 1 \right) \frac{d\varepsilon_a}{dt}. \quad (43)$$

$$\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} \frac{d\varepsilon_a}{dt} = \frac{\mathbf{p}_a}{\rho_a} \cdot \frac{d\mathbf{p}_a}{dt}. \quad (44)$$

$$\frac{d\varepsilon_a}{dt} = \varepsilon_a \frac{\mathbf{p}_a}{\mathbf{p}_a^2} \cdot \frac{d\mathbf{p}_a}{dt}. \quad (45)$$

from the hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2} \sum_a \frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + \sum_a 1. \quad (46)$$

reference density

$$\begin{aligned}\frac{d\mathbf{p}_a}{dt} &= -\frac{1}{2} \sum_b \mathbf{p}_b^2 \partial_{\mathbf{r}_a} \frac{1}{\varepsilon_b \rho_b} \\ &= \frac{1}{2} \sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b \rho_b^2} \partial_{\mathbf{r}_a} \rho_b + \frac{1}{2} \sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b} \partial_{\mathbf{r}_a} \varepsilon_b \quad (47) \\ &= \frac{1}{2} \sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b^2} (\varepsilon_b \partial_{\varepsilon} \rho_b + \rho_b) \partial_{\mathbf{r}_a} \varepsilon_b\end{aligned}$$

relativistic case

the invariant length

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (48)$$

$$d\tau^2 = dt^2 - d\mathbf{x} \cdot d\mathbf{x}. \quad (49)$$

$$\frac{dt^2}{d\tau^2} - \frac{d\mathbf{x}}{d\tau} \cdot \frac{d\mathbf{x}}{d\tau} = u_\mu u^\mu = 1. \quad (50)$$

where

$$u_0 = \gamma = \frac{dt}{d\tau}. \quad (51)$$

$$\mathbf{u} = \gamma \mathbf{v} = \frac{d\mathbf{x}}{d\tau} = \frac{dt}{d\tau} \frac{d\mathbf{x}}{dt}. \quad (52)$$

relativistic case

from the energy-momentum tensor

$$T^{\mu\nu} = \omega u^\mu u^\nu - P g^{\mu\nu} = \varepsilon u^\mu u^\nu + P(u^\mu u^\nu - g^{\mu\nu}). \quad (53)$$

which comes from the lagrangean as

$$T^{\mu\nu} = u^\mu \partial_{u_\nu} \mathcal{L} - g^{\mu\nu} \mathcal{L}. \quad (54)$$

from that we identify that the canonical momentum is

$$\partial_{u_\nu} \mathcal{L} = \omega u^\nu. \quad (55)$$

and the lagrangean density

$$\mathcal{L} = P. \quad (56)$$

in order to make these two conclusions compatible we make use of the 4-velocity constraint in the lagrangean

$$\mathcal{L} = P + \omega(u^\mu u_\mu - 1) = -\varepsilon + \omega u^\mu u_\mu. \quad (57)$$

relativistic case

the hamiltonian expressed in terms of the canonical momentum

$$p^\mu = \omega u^\mu, \quad \mathbf{p} = \omega \gamma \mathbf{v}. \quad (58)$$

it is important to rewrite the γ in terms of p

$$\frac{\mathbf{p}^2}{\omega^2} = \gamma^2 \mathbf{v}^2 = \gamma^2 - 1, \quad \gamma^2 = \frac{\mathbf{p}^2}{\omega^2} + 1. \quad (59)$$

from that we identify that the canonical momentum is

$$T^{\mu\nu} = \frac{p^\mu p^\nu}{\omega} - P g^{\mu\nu}. \quad (60)$$

the hamiltonian density is

$$\mathcal{H} = T_{tt} = \frac{p_t^2}{\omega} - P = \frac{\mathbf{p}^2}{\omega} + \varepsilon. \quad (61)$$

energy flux

the energy flux is

$$\frac{p^t p^i}{\omega} = \gamma \mathbf{p}. \quad (62)$$

the equations of motion are

$$\partial_\mu T^{\mu\nu} = 0. \quad (63)$$

which integrates to

$$\int_{\mathbf{x}} \partial_\mu T^{\mu\nu} = \frac{d}{dt} \int_{\mathbf{x}} T^{t\nu} = 0. \quad (64)$$

the space part translates into

$$\frac{d}{dt} \int_{\mathbf{x}} \omega \gamma^2 \mathbf{v} = \frac{d}{dt} \int_{\mathbf{x}} \sqrt{\frac{\mathbf{p}^2}{\omega^2} + 1} \mathbf{p} = 0. \quad (65)$$

energy flux

the time part translates into

$$\frac{d}{dt} \int_{\mathbf{x}} (\omega \gamma^2 - P) = \frac{d}{dt} \int_{\mathbf{x}} \left(\frac{\mathbf{p}^2}{\omega} + \varepsilon \right) = 0. \quad (66)$$

by imposing that any quantity can be integrated to

$$\frac{d}{dt} \int_{\mathbf{x}} f(\mathbf{x}) = \frac{d}{dt} \sum_a \frac{f_a}{h_a}. \quad (67)$$

this allows us to define the reference density based on the hamiltonian density so that

$$h_a = \omega_a \gamma_a^2 - P_a = \frac{\mathbf{p}_a^2}{\omega_a} + \varepsilon_a. \quad (68)$$

therefore

$$h_a = \sum_b W_{ab}. \quad (69)$$

integrals

given a local density its integral

$$\mathcal{S} = \int d\tau d\mathbf{x} \mathcal{L}. \quad (70)$$

which can be converted to the common frame

$$\mathcal{S} = \int dt d\mathbf{x} \frac{d\tau}{dt} \mathcal{L} = \int dt d\mathbf{x} \mathcal{L}^*. \quad (71)$$

where

$$\frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}}. \quad (72)$$

so the lagrangean density on the common frame and the canonical momentum

$$\mathcal{L}^* = -\frac{\varepsilon}{\gamma}, \quad \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \frac{\varepsilon}{\gamma^2} \frac{\partial \gamma}{\partial \mathbf{v}}. \quad (73)$$

the gamma derivative is

$$\frac{\partial \gamma}{\partial \mathbf{v}} = \gamma^3 \mathbf{v}. \quad (74)$$

relativistic case

canonical momentum

$$\mathbf{p} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \varepsilon \gamma \mathbf{v}. \quad (75)$$

$$\frac{\mathbf{p}^2}{\varepsilon^2} = \gamma^2 \mathbf{v}^2 = \gamma^2 - 1, \quad \gamma = \sqrt{\frac{\mathbf{p}^2}{\varepsilon^2} + 1}. \quad (76)$$

the hamiltonian density in the common frame is

$$\mathcal{H}^* = \mathbf{v} \cdot \mathbf{p} - \mathcal{L}^* = \frac{\mathbf{p}^2}{\gamma \varepsilon} + \frac{\varepsilon}{\gamma}. \quad (77)$$

the hamiltonian density in the local frame is

$$\mathcal{H} = \frac{\mathbf{p}^2}{\varepsilon} + \varepsilon. \quad (78)$$

reference density

the SPH integrals are defined in the common frame so

$$\int_{\mathbf{x}} \frac{f(\mathbf{x})}{\gamma(\mathbf{x})} = \sum_a \frac{f_a}{\gamma_a \rho_a}. \quad (79)$$

so the common frame lagrangean is

$$L^* = - \sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}. \quad (80)$$

so the common frame hamiltonian is

$$H^* = \sum_a \frac{\mathbf{p}_a^2}{\gamma_a \varepsilon_a \rho_a} + \sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}. \quad (81)$$

eqs of motion

the partial derivative in respect to the canonical momentum is

$$\partial_{\mathbf{p}_a} \frac{1}{\gamma_b} = -\frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a^2} \delta_{ab}. \quad (82)$$

the partial derivative in respect to the canonical momentum is

$$\begin{aligned} \frac{d\mathbf{r}_a}{dt} &= \partial_{\mathbf{p}_a} H^* \\ &= \frac{2\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a} - \frac{\mathbf{p}_a^2 \mathbf{p}_a}{\gamma_b^3 \varepsilon_b^3 \rho_b} - \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\ &= [2\gamma_a^2 - (\gamma_a^2 - 1) - 1] \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\ &= \frac{\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a} \end{aligned} \quad (83)$$

kernel revisited

given a normalized kernel

$$\int_{\mathbf{x}} W[\mathbf{x} - \mathbf{r}_a(t)] = 1. \quad (84)$$

the relation is imposed

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{F_a(t)}{\rho[\mathbf{r}_a(t), t]}. \quad (85)$$

where F are free weights to ensure the relationship.

$$\int_{\mathbf{x}} W[\mathbf{x} - \mathbf{r}_a(t)] = \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]} = 1. \quad (86)$$

the explicit dependence on the index a has to be taken into account

$$\frac{W(0)}{\rho[\mathbf{r}_a(t), t]} + \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]} = 1. \quad (87)$$

kernel revisited

$$\rho[\mathbf{r}_a(t), t] = \frac{W(0)}{1 - \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]}}. \quad (88)$$

the usual reference density can be derived by summing the kernel integral for all fluid elements

$$\sum_a \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]} = N. \quad (89)$$

the commutation of the summations yields

$$\sum_b \frac{1}{\rho[\mathbf{r}_b(t), t]} \sum_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)] = N. \quad (90)$$

therefore

$$\rho[\mathbf{r}_b(t), t] = \sum_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]. \quad (91)$$

kernel revisited

the same procedure can be done with constant weights

$$\sum_a \varepsilon_a \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]} = \sum_a \varepsilon_a. \quad (92)$$

$$\sum_b \frac{1}{\rho[\mathbf{r}_b(t), t]} \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)] = \sum_a \varepsilon_a. \quad (93)$$

$$\rho[\mathbf{r}_b(t), t] = \frac{1}{\varepsilon_b} \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]. \quad (94)$$

$$\varepsilon_b \rho[\mathbf{r}_b(t), t] = \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]. \quad (95)$$

conserved quantity

the same procedure can be done with constant weights

$$\frac{d}{dt} \sum_a \varepsilon_a \sum_b \frac{W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]}{\rho[\mathbf{r}_b(t), t]} = \frac{d}{dt} \sum_a \varepsilon_a = 0. \quad (96)$$

$$\frac{d}{dt} \sum_b \frac{1}{\rho[\mathbf{r}_b(t), t]} \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)] = \sum_a \frac{d\varepsilon_a}{dt}. \quad (97)$$

$$\rho[\mathbf{r}_b(t), t] = \frac{1}{\varepsilon_b} \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]. \quad (98)$$

$$\varepsilon_b \rho[\mathbf{r}_b(t), t] = \sum_a \varepsilon_a W[\mathbf{r}_b(t) - \mathbf{r}_a(t)]. \quad (99)$$

attempt with fields

a

$$\rho(\mathbf{x}) = \sum_a m_a(t) W[\mathbf{x} - \mathbf{r}_a(t)]. \quad (100)$$

$$\varphi(\mathbf{x}) = \sum_a \phi_a(t) W[\mathbf{x} - \mathbf{r}_a(t)]. \quad (101)$$

so

$$\mathbf{v}(\mathbf{x}) = \sum_a \phi_a(t) \partial_{\mathbf{x}} W[\mathbf{x} - \mathbf{r}_a(t)]. \quad (102)$$

attempt with fields

a

$$\delta\rho_a = \sum_b \delta m_b W_{ab} + m_b(\delta r_a - \delta r_b) \mathbf{Y}_{ab}. \quad (103)$$

$$\delta\varphi_a = \sum_b \delta\phi_b W_{ab} + \phi_b(\delta r_a - \delta r_b) \mathbf{Y}_{ab}. \quad (104)$$

so

$$\mathcal{L}(\rho, \varphi) = \frac{1}{2} \rho \partial_i \varphi \partial_i \varphi - \varepsilon(\rho). \quad (105)$$

attempt with fields

lagrangean density using the fields

$$\mathcal{L}(\rho, \varphi) = \frac{1}{2} \rho \partial_i \varphi \partial_i \varphi - \varepsilon(\rho). \quad (106)$$

variation

$$\delta \mathcal{L}(\rho, \varphi) = \partial_\rho \mathcal{L} \delta \rho + \partial_{\partial_i \varphi} \mathcal{L} \delta(\partial_i \varphi). \quad (107)$$

the canonical momentum is

$$\delta \mathcal{L}(\rho, \varphi) = \partial_\rho \mathcal{L} \delta \rho + \hat{\varphi}_i \delta(\partial_i \varphi). \quad (108)$$

$$\delta \mathcal{L}(\rho, \varphi) = \partial_\rho \mathcal{L} \delta \rho + \delta(\hat{\varphi}_i \partial_i \varphi) - \partial_i \varphi \delta \hat{\varphi}_i. \quad (109)$$

$$\delta[\hat{\varphi}_i \partial_i \varphi - \mathcal{L}(\rho, \varphi)] = -\partial_\rho \mathcal{L} \delta \rho + \partial_i \varphi \delta \hat{\varphi}_i. \quad (110)$$

attempt with fields

hamiltonian density using the fields

$$\begin{aligned}\mathcal{H}(\rho, \varphi) &= \frac{1}{2} \rho \partial_i \varphi \partial_i \varphi + \varepsilon(\rho) \\ \mathcal{H}(\rho, \varphi) &= \frac{1}{2} \frac{\hat{\varphi}_i \hat{\varphi}_i}{\rho} + \varepsilon(\rho).\end{aligned}\tag{111}$$

the canonical momenta are

$$\delta \mathcal{H} = \delta \rho \partial_\rho \mathcal{H} + \delta \hat{\varphi}_i \partial_{\hat{\varphi}_i} \mathcal{H}.\tag{112}$$

since the only dependence are in ρ and $\hat{\varphi}$

$$\delta \rho \partial_\rho \mathcal{H} + \delta \hat{\varphi}_i \partial_{\hat{\varphi}_i} \mathcal{H} = -\partial_\rho \mathcal{L} \delta \rho + \partial_i \varphi \delta \hat{\varphi}_i.\tag{113}$$

the variations of the fields associate to

$$\partial_\rho \mathcal{H} = -\partial_\rho \mathcal{L}, \quad \partial_i \varphi = \partial_{\hat{\varphi}_i} \mathcal{H}.\tag{114}$$

attempt with fields

from the Lagrangian equations of motion

$$\partial_i \partial_{\partial_i \rho} \mathcal{L} + \partial_i \partial_{\partial_i \varphi} \mathcal{L} = -\partial_\rho \mathcal{L} - \partial_\varphi \mathcal{L}. \quad (115)$$

$$\partial_i \partial_{\partial_i \varphi} \mathcal{L} = -\partial_\rho \mathcal{L}, \quad \partial_i \hat{\varphi}_i = -\partial_\rho \mathcal{L}. \quad (116)$$

$$\partial_\rho \mathcal{H} = \partial_i \hat{\varphi}_i, \quad \partial_i \varphi = \partial_{\hat{\varphi}_i} \mathcal{H}. \quad (117)$$