All calculations here are performed automatically SymPy

1 kernel

The Kernel is determined by the desired property of the density integral form,

$$\int f(x)\,dx = \sum_{i=0}^{N} F_i V_i \tag{1}$$

this is needed so that the expression of the integral of the density does not depend on x explicitly.The quantities F and V are not defined yet, but we suppose that F is related to the function and V is notThe general form of the interpolation using a general kernel function G is

$$f(x) = \sum_{i=0}^{N} G(x - r_i) F_i \tag{2}$$

The relations (2) and (1) bind the integral of the kernel function,

$$\int G(x - r_i)\,dx = V_i \tag{3}$$

Therefore we can write the general kernel as a normalized kernel W times the fluid element volume V_i

$$G(x - r_i) = W(x - r_i) V_i \tag{4}$$

Taking the interpolation computed on r_j,

$$f(r_i) = \sum_{j=0}^{N} W\left(r_i - r_j\right) F_j V_j \tag{5}$$

and summing it with the kernel gives

$$\sum_{i\,=0}^{N} W(x - r_i) f(r_i) = \sum_{j=0}^{N} W\left(x - r_j\right) \sum_{i=0}^{N} W\left(-r_i + r_j\right) F_i V_i \tag{6}$$

which is satisfied if

$$W(x - r_i) f(r_i) = F_i V_i \sum_{j=0}^{N} W\left(x - r_j\right) W\left(-r_i + r_j\right) \tag{7}$$

the integral of the previous relation is

$$f(r_i) = F_i V_i \sum_{j=0}^{N} W\left(r_i - r_j\right) \tag{8}$$

substituting back to (2),

$$f(x) = \sum_{i=0}^{N} \frac{W(x - r_i) f(r_i)}{\sum_{j=0}^{N} W\big(r_i - r_j\big)} \tag{9}$$

for the special case that $f(x) = 1$ the integral of the rhs is the total system volume, which can be written as the sum of all the volumes of the fluid elements

$$\sum_{i\ =0}^{N} V_i = \sum_{i=0}^{N} \frac{1}{\sum_{j=0}^{N} W\big(r_i - r_j\big)} \tag{10}$$

It can be shown that the only way to compute the element volume with a non-recurrent formula

$$\sum_{i\ =0}^{N} \frac{G(x - r_i)}{V_i} = \sum_{i=0}^{N} W(x - r_i) \tag{11}$$

This is the fluid element density $\rho = V^{-1}$

$$\rho(x) = \sum_{i=0}^{N} W(x - r_i) \tag{12}$$

Therefore the expression of the fluid element density ultimately arises from the requirement that the integral of the density does not depend on the kernel (and that is the only explicit definition)

2 Kernel

The Kernel function is defined by the following properties. Its integral is

$$1 \tag{13}$$

Its derivative in respect to x

$$\frac{\partial}{\partial x} W\left(\frac{x - r_i(t)}{h}\right) = \frac{W'\left(\frac{x - r_i(t)}{h}\right)}{h} \tag{14}$$

Its derivative in respect to $r(t, j)$

$$\frac{\partial}{\partial r_j(t)} W\left(\frac{\big|x - r_i(t)\big|}{h}\right) = -\frac{W'\left(\frac{x - r_i(t)}{h}\right)\delta_{ij}}{h} \tag{15}$$

3 Interpolation

the interpolation reference is

$$\rho(x) = \sum_{i=0}^{N} W\left(\frac{x - r_i(t)}{h}\right)\nu_i \tag{16}$$

and its derivative is

$$\frac{d}{dx}\rho(x) = \frac{\sum_{i=0}^{N} W'\left(\frac{x-r_i(t)}{h}\right)\nu_i}{h} \tag{17}$$

on the other hand, when taking the variational approach one gets

$$\frac{d}{dr_j(t)}\rho\big(r_k(t)\big) = \frac{-W'\left(\frac{-r_j(t)+r_k(t)}{h}\right)\nu_j + \delta_{jk}\sum_{i=0}^{N} W'\left(\frac{-r_i(t)+r_k(t)}{h}\right)\nu_i}{h} \tag{18}$$

and its derivative is

$$\frac{d}{dt}\rho(x) = -\frac{\sum_{i=0}^{N} W'\left(\frac{x-r_i(t)}{h}\right)dot_r(t,i)\nu_i}{h} \tag{19}$$

and its derivative is

$$\frac{d}{dt}\rho\big(r_k(t)\big) = \frac{\sum_{i=0}^{N}\big(-dot_r(t,i) + dot_r(t,k)\big)\,W'\left(\frac{-r_i(t)+r_k(t)}{h}\right)\nu_i}{h} \tag{20}$$

4 Monaghan recipe

The Lagrangean density is

$$\mathcal{L}(x,t) = -\epsilon\big(\rho(x)\big)\rho(x) + \frac{\rho(x)v^2(x)}{2} \tag{21}$$

Using the Monaghan recipe it transcribes into with the transformation of $v(x)$ into $v(t,i)$

$$\mathcal{L}(x,t) = \sum_{i=0}^{N}\left(\frac{{dot_r}^2(t,i)}{2} - \epsilon\Big(\rho\big(r_i(t)\big)\Big)\right)W\left(\frac{x - r_i(t)}{h}\right)\nu_i \tag{22}$$

The integrated Lagrangean density is

$$L(x,t) = \sum_{i=0}^{N}\frac{\left(\frac{{dot_r}^2(t,i)\rho(r_i(t))}{2} - \epsilon\Big(\rho\big(r_i(t)\big)\Big)\rho\big(r_i(t)\big)\right)\nu_i}{\rho\big(r_i(t)\big)} \tag{23}$$

The variation in respect to r

$$\frac{\partial}{\partial r_j(t)} L(x,t) \tag{24}$$

$$= \sum_{i=0}^{N} \left(- \frac{\left(\frac{dot_r{}^2\left(t,i\right)\rho\left(r_i(t)\right)}{2} - \epsilon\left(\rho\left(r_i(t)\right)\right)\rho\left(r_i(t)\right) \right) \nu_i \sum_{i_1=0}^{N} \frac{\left(\delta_{ij} - \delta_{i_1 j}\right) W'\left(\frac{r_i(t) - r_{i_1}(t)}{h}\right)\nu_{i_1}}{h}}{\rho^2\left(r_i(t)\right)} \right.$$

$$+ \frac{\Bigg(\frac{dot_r{}^2\left(t,i\right) \sum_{i_1=0}^{N} \frac{\left(\delta_{ij} - \delta_{i_1 j}\right) W'\left(\frac{r_i(t) - r_{i_1}(t)}{h}\right)\nu_{i_1}}{h}}{2} - \epsilon\left(\rho\left(r_i(t)\right)\right) \sum_{i_1=0}^{N} \frac{\left(\delta_{ij} - \delta_{i_1 j}\right) W'\left(\frac{r_i(t) - r_{i_1}(t)}{h}\right)\nu_{i_1}}{h} - \epsilon_{\rho}\Big(\rho\big(r_i(}{\rho\left(r_i(t)\right)}$$

The variation in respect to v

$$\frac{\partial}{\partial dot_r(t,j)} L(x,t) = dot_r(t,j)\nu_j \tag{25}$$

and its time derivative

$$\ddot{r}(t,j)\nu_j \tag{26}$$

Finally the Euler-Lagrange relation

$$0 = -\ddot{r}(t,j)\nu_j + \frac{\sum_{i=0}^{N} \left(W'\left(\frac{r_i(t) - r_j(t)}{h}\right)\nu_j - \delta_{ij} \sum_{i_1=0}^{N} W'\left(\frac{r_i(t) - r_{i_1}(t)}{h}\right)\nu_{i_1} \right) \epsilon_{\rho}\left(\rho\left(r_i(t)\right)\right)\nu_i}{h} \tag{27}$$

$$0 = -\begin{cases} \ddot{r}(t,j)\nu_j & \text{for } j \geq 0 \\ 0 & \text{otherwise} \end{cases} + \sum_{i=0}^{N} -\epsilon_{\rho}\left(\rho\left(r_i(t)\right)\right)\nu_i \sum_{i_1=0}^{N} \frac{W'\left(\frac{r_i(t)}{h} - \frac{r_{i_1}(t)}{h}\right)\nu_{i_1}\delta_{ij}}{h} + \sum_{i=0}^{N} \begin{cases} \frac{W'\left(\frac{r_i(t)}{h} - \frac{r_j(t)}{h}\right)\epsilon_{\rho}\left(\rho\left(r_i(t)\right)\right)\nu_i\nu_j}{h} & \text{for } j \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

5 Beyond Monaghan

In Monaghan's recipe, the fluid element velocity is set to be the energy velocity by hand. We will attempt to derive the relation between these two velocities out of the variational principle directly

$$\begin{aligned}\mathcal{L}(x,t) = &-\epsilon\big(\rho(x)\big)\rho(x) + \lambda(x)\rho(x)\frac{d}{dx}v(x) + \frac{\rho(x)v^2(x)}{2} \\ &+ \frac{\lambda(x)\sum_{i=0}^{N}\big(-dot_r(t,i) + v(x)\big)\,W'\Big(\frac{x-r_i(t)}{h}\Big)\nu_i}{h}\end{aligned} \tag{29}$$

Differential with respect to $v(x)$