calculations

Philipe Mota

CBPF

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kernel

using a general kernel function

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = 1. \tag{1}$$

in order to rewrite

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{f_a(t)}{\rho_a(t)}. \tag{2}$$

reference density

$$\rho(\mathbf{x}) = \sum_a W(\mathbf{x} - \mathbf{r}_a). \tag{3}$$

which gives

$$\int_{\mathbf{x}} \rho(\mathbf{x}) = N. \tag{4}$$

derivatives

useful computations

$$\rho(r_a) = \sum_b W(\mathbf{r}_a - \mathbf{r}_b). \tag{5}$$

the comoving derivative is

$$\partial_{\mathbf{r}_a}\rho_b = \sum_c \partial_{\mathbf{r}_a} W_{bc} = \sum_c \mathbf{Y}_{bc}(\delta_{ba} - \delta_{ca}) = \mathbf{Y}_{ab} + \delta_{ba}\sum_c \mathbf{Y}_{ac}. \tag{6}$$

where

$$\mathbf{Y}_{bc} = -\mathbf{Y}_{cb}. \tag{7}$$

a common form of term is

$$\sum_b f_b \partial_{\mathbf{r}_a}\rho_b = f_a \sum_c \mathbf{Y}_{ac} + \sum_b f_b \mathbf{Y}_{ab} = \sum_b (f_a + f_b)\mathbf{Y}_{ab}. \tag{8}$$

which has the nice property that

$$\sum_a \sum_b f_b \partial_{\mathbf{r}_a}\rho_b = 0. \tag{9}$$

lagrangean

starting from the non-relativistic case

$$\mathcal{L} = \tfrac{1}{2}\,\rho(\mathbf{x})\mathbf{v}(\mathbf{x})^2 - \varepsilon(\mathbf{x}). \tag{10}$$

the canonical momentum density is

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \rho \mathbf{v}. \tag{11}$$

the hamiltonian density is

$$\mathcal{H} = \mathbf{v}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L} = \frac{\mathbf{p}^2}{2\rho} + \varepsilon. \tag{12}$$

the discretization needs to be consistently made in both scenarios

equations of motion

in the lagrangean side, a straight forward paramaterization is

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}\int_{\mathbf{x}}\left[\rho(x)v(x)^2\right] - \int_{\mathbf{x}}\varepsilon(x) \\ = \frac{1}{2}\sum_a \mathbf{v}_a^2 - \sum_a \frac{\varepsilon_a}{\rho_a} \tag{13}$$

$$\mathbf{p}_a = \sum_b v_b W_{ab}. \tag{14}$$

on the other side

$$H(\mathbf{r},\mathbf{p}) = \frac{1}{2}\int_{\mathbf{x}}\frac{p(x)^2}{\rho(x)} + \int_{\mathbf{x}}\varepsilon(x) \\ = \frac{1}{2}\sum_a \frac{\mathbf{p}_a^2}{\rho_a^2} + \sum_a \frac{\varepsilon_a}{\rho_a} \tag{15}$$

equations of motion

$$\mathbf{p}_a = \sum_b \dot{\mathbf{r}}_b W_{ab}. \tag{16}$$

$$\begin{bmatrix} \partial_{\dot{\mathbf{r}}_a} \\ \partial_{\mathbf{r}_a} \end{bmatrix} = \begin{bmatrix} \partial_{\dot{\mathbf{r}}_a} p_b & 0 \\ \partial_{\mathbf{r}_a} p_b & \delta_{ab} \end{bmatrix} \begin{bmatrix} \partial_{\mathbf{p}_b} \\ \partial_{\mathbf{r}_b} \end{bmatrix}. \tag{17}$$

equations of motions

and the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{\mathbf{r}}_a} L = \partial_{\mathbf{r}_a} L. \tag{18}$$

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \partial_{\mathbf{p}_a} H, \quad \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\partial_{\mathbf{r}_a} H. \tag{19}$$

the total derivative of the hmailtoninan is

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{r}, \mathbf{p}) = \sum_a \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}\partial_{\mathbf{p}_a} H + \sum_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t}\partial_{\mathbf{r}_a} H. \tag{20}$$

the angular momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_a \mathbf{r} \times \mathbf{p} = \sum_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} \times \mathbf{p}_a + \sum_a \mathbf{r}_a \times \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}. \tag{21}$$

equations of motions

and the equation of motion is

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a^2}. \tag{22}$$

$$\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\sum_b (F_a + F_b)\mathbf{Y}_{ab}, \quad F_a = \frac{\partial_\rho(\varepsilon)_a}{\rho_a} - \frac{\varepsilon_a}{\rho^2} - \frac{\mathbf{p}_a^2}{\rho_a^3}. \tag{23}$$

the angular momentum is

$$\begin{aligned}
\sum_a \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \sum_a \sum_b (F_a + F_b)\mathbf{r}_a \times \mathbf{Y}_{ab} \\
&= \frac{1}{2}\sum_{ab}(F_a + F_b)(\mathbf{r}_a \times \mathbf{Y}_{ab} - \mathbf{r}_b \times \mathbf{Y}_{ab}) \qquad (24) \\
&= \frac{1}{2}\sum_{ab}(F_a + F_b)(\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{Y}_{ab}
\end{aligned}$$

continuity

$$\int_{\mathbf{x}} \rho(x) = \sum_a 1. \tag{25}$$

$$\int_{\mathbf{x}} \rho(x)\mathbf{v}(x) = \sum_a v_a. \tag{26}$$

this gives the canonical momentum as

$$\mathbf{p}_a = \partial_{\dot{\mathbf{r}}_a} L = \dot{\mathbf{r}}_a. \tag{27}$$

and the equation of motion is

$$\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \partial_{\mathbf{r}_a} L = -\sum_b (\phi_a + \phi_b)\mathbf{Y}_{ab}. \tag{28}$$

$$\phi_a = \frac{\partial_\rho \varepsilon_a}{\rho_a} - \frac{\varepsilon_a}{\rho_a^2} = \frac{\mathrm{d}}{\mathrm{d}\rho}\frac{\varepsilon}{\rho}. \tag{29}$$

reference density

the hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\frac{p(x)^2}{\rho(x)} + \varepsilon(x). \tag{30}$$

using a generic reference density

$$\phi_a = \sum_b W_{ab}. \tag{31}$$

the integral of the hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \sum_a \frac{1}{2}\frac{\mathbf{p}_a^2}{\phi_a \rho_a} + \sum_a \frac{\varepsilon_a}{\phi_a}. \tag{32}$$

from the conservation of energy

$$\sum_a \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{p}_a^2}{\phi_a \rho_a} = -\sum_a \frac{\mathrm{d}}{\mathrm{d}t}\frac{\varepsilon_a}{\phi_a}. \tag{33}$$

reference density

the velocity equation is

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a \phi_a}, \qquad \frac{\mathrm{d}\phi_a}{\mathrm{d}t} = \sum_b \left(\frac{\mathbf{p}_a}{\rho_a \phi_a} - \frac{\mathbf{p}_b}{\rho_b \phi_b} \right) \cdot \mathbf{Y}_{ab}. \tag{34}$$

from the conservation of energy

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\phi_a \rho_a \frac{\mathbf{p}_a^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\phi_a \rho_a} - \phi_a \rho_a \frac{\mathrm{d}}{\mathrm{d}t} \frac{\varepsilon_a}{\phi_a}. \tag{35}$$

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a^2}{2} \left(\frac{1}{\rho_a} \frac{\mathrm{d}\rho_a}{\mathrm{d}t} + \frac{1}{\phi_a} \frac{\mathrm{d}\phi_a}{\mathrm{d}t} \right) - \rho_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \frac{\rho_a \varepsilon_a}{\phi_a} \frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{36}$$

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a^2}{2\rho_a} \frac{\mathrm{d}\rho_a}{\mathrm{d}t} - \rho_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{2} + \rho_a \varepsilon_a \right) \frac{1}{\phi_a} \frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{37}$$

reference density

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}\rho(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \frac{\rho_a}{\phi_a} = 0. \tag{38}$$

$$\sum_a \frac{1}{\phi_a}\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \sum_a \frac{\rho_a}{\phi_a^2}\frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{39}$$

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \frac{\rho_a}{\phi_a}\frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{40}$$

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\rho_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \frac{1}{\phi_a}(\mathbf{p}_a^2 + \rho_a\varepsilon_a)\frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{41}$$

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = -\frac{\mathbf{p}_a}{\rho_a}\cdot\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{\phi_a\rho_a} + \frac{\varepsilon_a}{\phi_a}\right)\frac{\mathrm{d}\phi_a}{\mathrm{d}t}. \tag{42}$$

from the hamiltonian density

$$\frac{\mathrm{d}\varepsilon(\mathbf{x})}{\mathrm{d}t} = -\frac{\mathbf{p}(\mathbf{x})}{\rho(\mathbf{x})}\cdot\frac{\mathrm{d}\mathbf{p}(\mathbf{x})}{\mathrm{d}t} + \frac{\mathbf{p}(\mathbf{x})^2}{2\rho(\mathbf{x})^2}\frac{\mathrm{d}\rho(\mathbf{x})}{\mathrm{d}t}. \tag{43}$$

reference density

using

$$\phi_a = \varepsilon_a. \tag{44}$$

the equations reduce to

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = -\frac{\mathbf{p}_a}{\rho_a} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + 1\right)\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t}. \tag{45}$$

$$\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a}\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}. \tag{46}$$

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = \varepsilon_a \frac{\mathbf{p}_a}{\mathbf{p}_a^2} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}. \tag{47}$$

from the hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2}\sum_a \frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + \sum_a 1. \tag{48}$$

reference density

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} &= -\frac{1}{2}\sum_b \mathbf{p}_b^2 \partial_{\mathbf{r}_a} \frac{1}{\varepsilon_b \rho_b} \\
&= \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b \rho_b^2} \partial_{\mathbf{r}_a} \rho_b + \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b} \partial_{\mathbf{r}_a} \varepsilon_b \qquad (49) \\
&= \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b^2} (\varepsilon_b \partial_\varepsilon \rho_b + \rho_b) \partial_{\mathbf{r}_a} \varepsilon_b
\end{aligned}
$$

relativistic case

the invariant length

$$\mathrm{d}\tau^{\,2} = g_{\mu\nu}\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,. \tag{50}$$

$$\mathrm{d}\tau^{\,2} = \mathrm{d}t^{\,2} - \mathrm{d}\mathbf{x}\,\cdot \mathrm{d}\mathbf{x}\,. \tag{51}$$

$$\frac{\mathrm{d}t^{\,2}}{\mathrm{d}\tau^{\,2}} - \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = u_{\mu}u^{\mu} = 1. \tag{52}$$

where

$$u_0 = \gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{53}$$

$$\mathbf{u} = \gamma\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}. \tag{54}$$

relativistic case

from the energy-momentum tensor

$$T^{\mu\nu} = \omega u^\mu u^\nu - P g^{\mu\nu} = \varepsilon u^\mu u^\nu + P(u^\mu u^\nu - g^{\mu\nu}). \tag{55}$$

which comes from the lagrangean as

$$T^{\mu\nu} = u^\mu \partial_{u_\nu}\mathcal{L} - g^{\mu\nu}\mathcal{L}. \tag{56}$$

from that we identify that the canonical momentum is

$$\partial_{u_\nu}\mathcal{L} = \omega u^\nu. \tag{57}$$

and the lagrangean density

$$\mathcal{L} = P. \tag{58}$$

in order to make these two conclusions compatible we make use of the 4-velocity constraint in the lagrangean

$$\mathcal{L} = P + \omega(u^\mu u_\mu - 1) = -\varepsilon + \omega u^\mu u_\mu. \tag{59}$$

relativistic case

the hamiltonian expressed in terms of the canonical momentum

$$p^{\mu} = \omega u^{\mu}, \quad \mathbf{p} = \omega\gamma\mathbf{v}. \tag{60}$$

it is important to rewrite the γ in terms of p

$$\frac{\mathbf{p}^2}{\omega^2} = \gamma^2\mathbf{v}^2 = \gamma^2 - 1, \quad \gamma^2 = \frac{\mathbf{p}^2}{\omega^2} + 1. \tag{61}$$

from that we identify that the canonical momentum is

$$T^{\mu\nu} = \frac{p^{\mu}p^{\nu}}{\omega} - Pg^{\mu\nu}. \tag{62}$$

the hamiltonian density is

$$\mathcal{H} = T_{tt} = \frac{p_t^2}{\omega} - P = \frac{\mathbf{p}^2}{\omega} + \varepsilon. \tag{63}$$

energy flux

the energy flux is

$$\frac{p^t p^i}{\omega} = \gamma \mathbf{p}. \tag{64}$$

the equations of motion are

$$\partial_\mu T^{\mu\nu} = 0. \tag{65}$$

which integrates to

$$\int_{\mathbf{x}} \partial_\mu T^{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} T^{t\nu} = 0. \tag{66}$$

the space part translates into

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} \omega \gamma^2 \mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} \sqrt{\frac{\mathbf{p}^2}{\omega^2} + 1}\, \mathbf{p} = 0. \tag{67}$$

energy flux

the time part translates into

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}(\omega\gamma^2 - P) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}\left(\frac{\mathbf{p}^2}{\omega} + \varepsilon\right) = 0. \tag{68}$$

by imposing that any quantity can be integrated to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}} f(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \frac{f_a}{h_a}. \tag{69}$$

this allows us to define the reference density based on the hamiltonian density so that

$$h_a = \omega_a\gamma_a^2 - P_a = \frac{\mathbf{p}_a^2}{\omega_a} + \varepsilon_a. \tag{70}$$

therefore

$$h_a = \sum_b W_{ab}. \tag{71}$$

integrals

given a local density its integral

$$\mathcal{S} = \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \, \mathcal{L}. \tag{72}$$

which can be converted to the common frame

$$\mathcal{S} = \int \mathrm{d}t \, \mathrm{d}\mathbf{x} \, \frac{\mathrm{d}\tau}{\mathrm{d}t} \mathcal{L} = \int \mathrm{d}t \, \mathrm{d}\mathbf{x} \, \mathcal{L}^*. \tag{73}$$

where

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}}. \tag{74}$$

so the lagrangean density on the common frame and the canonical momentum

$$\mathcal{L}^* = -\frac{\varepsilon}{\gamma}, \quad \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \frac{\varepsilon}{\gamma^2} \frac{\partial \gamma}{\partial \mathbf{v}}. \tag{75}$$

the gamma derivative is

$$\frac{\partial \gamma}{\partial \mathbf{v}} = \gamma^3 \mathbf{v}. \tag{76}$$

relativistic case

canonical momentum

$$\mathbf{p} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \varepsilon\gamma\mathbf{v}. \tag{77}$$

$$\frac{\mathbf{p}^2}{\varepsilon^2} = \gamma^2\mathbf{v}^2 = \gamma^2 - 1, \quad \gamma = \sqrt{\frac{\mathbf{p}^2}{\varepsilon^2} + 1}\,. \tag{78}$$

the hamiltonian density in the common frame is

$$\mathcal{H}^* = \mathbf{v} \cdot \mathbf{p} - \mathcal{L}^* = \frac{\mathbf{p}^2}{\gamma\varepsilon} + \frac{\varepsilon}{\gamma}. \tag{79}$$

the hamiltonian density in the local frame is

$$\mathcal{H} = \frac{\mathbf{p}^2}{\varepsilon} + \varepsilon. \tag{80}$$

reference density

sph

the SPH integrals are defined in the common frame so

$$\int_{\mathbf{x}} \frac{f(\mathbf{x})}{\gamma(\mathbf{x})} = \sum_a \frac{f_a}{\gamma_a \rho_a}. \tag{81}$$

so the common frame lagrangean is

$$L^* = -\sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}. \tag{82}$$

so the common frame hamiltonian is

$$H^* = \sum_a \frac{\mathbf{p}_a^2}{\gamma_a \varepsilon_a \rho_a} + \sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}. \tag{83}$$

eqs of motion

the partial derivative in respect to the canonical momentum is

$$\partial_{\mathbf{p}_a} \frac{1}{\gamma_b} = -\frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a^2} \delta_{ab}. \tag{84}$$

the partial derivative in respect to the canonical momentum is

$$\begin{aligned} \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} &= \partial_{\mathbf{p}_a} H^* \\ &= \frac{2\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a} - \frac{\mathbf{p}_a^2 \mathbf{p}_a}{\gamma_b^3 \varepsilon_b^3 \rho_b} - \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\ &= \left[2\gamma_a^2 - (\gamma_a^2 - 1) - 1\right] \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\ &= \frac{\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a} \end{aligned} \tag{85}$$

kernel revisited

given a normalized kernel

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = 1. \tag{86}$$

the relation is imposed

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{F_a(t)}{\rho\left[\mathbf{r}_a(t), t\right]}. \tag{87}$$

where F are free weights to ensure the relationship.

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = \sum_b \frac{W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right]}{\rho\left[\mathbf{r}_b(t), t\right]} = 1. \tag{88}$$

the explicity dependence on the index a has to be taken into account

$$\frac{W(0)}{\rho\left[\mathbf{r}_a(t), t\right]} + \sum_b \frac{W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right]}{\rho\left[\mathbf{r}_b(t), t\right]} = 1. \tag{89}$$

kernel revisited

$$\rho\left[\mathbf{r}_a(t), t\right] = \frac{W(0)}{1 - \sum_b \frac{W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right]}{\rho\left[\mathbf{r}_b(t), t\right]}}. \tag{90}$$

the usual reference density can be derived by summing the kernel integral for all fluid elements

$$\sum_a \sum_b \frac{W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right]}{\rho\left[\mathbf{r}_b(t), t\right]} = N. \tag{91}$$

the commuatation of the summations yeilds

$$\sum_b \frac{1}{\rho\left[\mathbf{r}_b(t), t\right]} \sum_a W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right] = N. \tag{92}$$

therefore

$$\rho\left[\mathbf{r}_b(t), t\right] = \sum_a W\left[\mathbf{r}_b(t) - \mathbf{r}_a(t)\right]. \tag{93}$$

kernel revisited

the same proceedure can be done with constant weights

$$\sum_a \varepsilon_a \sum_b \frac{W\big[\mathbf{r}_b(t) - \mathbf{r}_a(t)\big]}{\rho\big[\mathbf{r}_b(t), t\big]} = \sum_a \varepsilon_a. \tag{94}$$

$$\sum_b \frac{1}{\rho\big[\mathbf{r}_b(t), t\big]} \sum_a \varepsilon_a W\big[\mathbf{r}_b(t) - \mathbf{r}_a(t)\big] = \sum_a \varepsilon_a. \tag{95}$$

$$\rho\big[\mathbf{r}_b(t), t\big] = \frac{1}{\varepsilon_b} \sum_a \varepsilon_a W\big[\mathbf{r}_b(t) - \mathbf{r}_a(t)\big]. \tag{96}$$

$$\varepsilon_b \rho\big[\mathbf{r}_b(t), t\big] = \sum_a \varepsilon_a W\big[\mathbf{r}_b(t) - \mathbf{r}_a(t)\big]. \tag{97}$$

conserved quantity

the same proceedure can be done with constant weights

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_a \varepsilon_a \sum_b \frac{W\left[\mathbf{r}_b(t)-\mathbf{r}_a(t)\right]}{\rho\left[\mathbf{r}_b(t),t\right]} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \varepsilon_a = 0. \tag{98}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_b \frac{1}{\rho\left[\mathbf{r}_b(t),t\right]}\sum_a \varepsilon_a W\left[\mathbf{r}_b(t)-\mathbf{r}_a(t)\right] = \sum_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t}. \tag{99}$$

$$\rho\left[\mathbf{r}_b(t),t\right] = \frac{1}{\varepsilon_b}\sum_a \varepsilon_a W\left[\mathbf{r}_b(t)-\mathbf{r}_a(t)\right]. \tag{100}$$

$$\varepsilon_b\rho\left[\mathbf{r}_b(t),t\right] = \sum_a \varepsilon_a W\left[\mathbf{r}_b(t)-\mathbf{r}_a(t)\right]. \tag{101}$$

attempt with fields

a

$$\rho(\mathbf{x}) = \sum_a m_a(t) W\left[\mathbf{x} - \mathbf{r}_a(t)\right]. \tag{102}$$

$$\varphi(\mathbf{x}) = \sum_a \phi_a(t) W\left[\mathbf{x} - \mathbf{r}_a(t)\right]. \tag{103}$$

so

$$\mathbf{v}(\mathbf{x}) = \sum_a \phi_a(t) \partial_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right]. \tag{104}$$

attempt with fields

a

$$\delta\rho_a = \sum_b \delta m_b W_{ab} + m_b(\delta r_a - \delta r_b)\mathbf{Y}_{ab}. \tag{105}$$

$$\delta\varphi_a = \sum_b \delta\phi_b W_{ab} + \phi_b(\delta r_a - \delta r_b)\mathbf{Y}_{ab}. \tag{106}$$

so

$$\mathcal{L}(\rho, \varphi) = \tfrac{1}{2}\,\rho\partial_i\varphi\partial_i\varphi - \varepsilon(\rho). \tag{107}$$

attempt with fields

lagrangean density using the fields

$$\mathcal{L}(\rho,\varphi) = \frac{1}{2}\,\rho\partial_i\varphi\partial_i\varphi - \varepsilon(\rho). \tag{108}$$

variation

$$\delta\mathcal{L}(\rho,\varphi) = \partial_\rho\mathcal{L}\delta\rho + \partial_{\partial_i\varphi}\mathcal{L}\delta(\partial_i\varphi). \tag{109}$$

the canonical momentum is

$$\delta\mathcal{L}(\rho,\varphi) = \partial_\rho\mathcal{L}\delta\rho + \hat{\varphi}_i\delta(\partial_i\varphi). \tag{110}$$

$$\delta\mathcal{L}(\rho,\varphi) = \partial_\rho\mathcal{L}\delta\rho + \delta(\hat{\varphi}_i\partial_i\varphi) - \partial_i\varphi\delta\hat{\varphi}_i. \tag{111}$$

$$\delta\big[\hat{\varphi}_i\partial_i\varphi - \mathcal{L}(\rho,\varphi)\big] = -\partial_\rho\mathcal{L}\delta\rho + \partial_i\varphi\delta\hat{\varphi}_i. \tag{112}$$

attempt with fields

hamiltonian density using the fields

$$\begin{aligned}\mathcal{H}(\rho,\varphi) &= \frac{1}{2}\,\rho\partial_i\varphi\partial_i\varphi + \varepsilon(\rho)\\ \mathcal{H}(\rho,\varphi) &= \frac{1}{2}\,\frac{\hat{\varphi}_i\hat{\varphi}_i}{\rho} + \varepsilon(\rho).\end{aligned} \tag{113}$$

the canonical momenta are

$$\delta\mathcal{H} = \delta\rho\partial_\rho\mathcal{H} + \delta\hat{\varphi}_i\partial_{\hat{\varphi}_i}\mathcal{H}. \tag{114}$$

since the only dependence are in ρ and $\hat{\varphi}$

$$\delta\rho\partial_\rho\mathcal{H} + \delta\hat{\varphi}_i\partial_{\hat{\varphi}_i}\mathcal{H} = -\partial_\rho\mathcal{L}\delta\rho + \partial_i\varphi\delta\hat{\varphi}_i. \tag{115}$$

the variations of the fields associate to

$$\partial_\rho\mathcal{H} = -\partial_\rho\mathcal{L}, \quad \partial_i\varphi = \partial_{\hat{\varphi}_i}\mathcal{H}. \tag{116}$$

attempt with fields

from the Lagrangian equations of motion

$$\partial_i \partial_{\partial_i \rho} \mathcal{L} + \partial_i \partial_{\partial_i \varphi} \mathcal{L} = -\partial_\rho \mathcal{L} - \partial_\varphi \mathcal{L}. \tag{117}$$

$$\partial_i \partial_{\partial_i \varphi} \mathcal{L} = -\partial_\rho \mathcal{L}, \quad \partial_i \hat{\varphi}_i = -\partial_\rho \mathcal{L}. \tag{118}$$

$$\partial_\rho \mathcal{H} = \partial_i \hat{\varphi}_i, \quad \partial_i \varphi = \partial_{\hat{\varphi}_i} \mathcal{H}. \tag{119}$$