calculations

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kernel

using a general kernel function

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = 1.$$

in order to rewrite

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{f_a(t)}{\rho_a(t)}.$$

reference density

$$\rho(\mathbf{x}) = \sum_a W(\mathbf{x} - \mathbf{r}_a).$$

which gives

$$\int_{\mathbf{x}} \rho(\mathbf{x}) = N.$$

derivatives

useful computations

$$\rho(r_a) = \sum_b W(\mathbf{r}_a - \mathbf{r}_b).$$

the comoving derivative is

$$\partial_{\mathbf{r}_a} \rho_b = \sum_c \partial_{\mathbf{r}_a} W_{bc} = \sum_c \mathbf{Y}_{bc}(\delta_{ba} - \delta_{ca}) = \mathbf{Y}_{ab} + \delta_{ba} \sum_c \mathbf{Y}_{ac}.$$

where

$$\mathbf{Y}_{bc} = -\mathbf{Y}_{cb}.$$

a common form of term is

$$\sum_b f_b \partial_{\mathbf{r}_a} \rho_b = f_a \sum_c \mathbf{Y}_{ac} + \sum_b f_b \mathbf{Y}_{ab} = \sum_b (f_a + f_b) \mathbf{Y}_{ab}.$$

which has the nice property that

$$\sum_a \sum_b f_b \partial_{\mathbf{r}_a} \rho_b = 0.$$

lagrangean

starting from the non-relativistic case

$$\mathcal{L} = \tfrac{1}{2}\,\rho(\mathbf{x})\mathbf{v}(\mathbf{x})^2 - \varepsilon(\mathbf{x}).$$

the canonical momentum density is

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \rho \mathbf{v}.$$

the hamiltonian density is

$$\mathcal{H} = \mathbf{v}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L} = \frac{\mathbf{p}^2}{2\rho} + \varepsilon.$$

the discretization needs to be consistently made in both scenarios

equations of motion

in the lagrangean side, a straight forward paramaterization is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \int_{\mathbf{x}} \left[\rho(x) v(x)^2\right] - \int_{\mathbf{x}} \varepsilon(x)$$

$$= \frac{1}{2} \sum_a \mathbf{v}_a^2 - \sum_a \frac{\varepsilon_a}{\rho_a}$$

$$\mathbf{p} = \int_{\mathbf{x}} \rho \mathbf{v} = \sum_a v_a.$$

on the other side

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2} \int_{\mathbf{x}} \frac{p(x)^2}{\rho(x)} + \int_{\mathbf{x}} \varepsilon(x)$$

$$= \frac{1}{2} \sum_a \frac{\mathbf{p}_a^2}{\rho_a^2} + \sum_a \frac{\varepsilon_a}{\rho_a}$$

equations of motions

and the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{\mathbf{r}}_a} L = \partial_{\mathbf{r}_a} L.$$

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \partial_{\mathbf{p}_a} H, \quad \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\partial_{\mathbf{r}_a} H.$$

the total derivative of the hmailtoninan is

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\mathbf{r}, \mathbf{p}) = \sum_a \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}\partial_{\mathbf{p}_a} H + \sum_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t}\partial_{\mathbf{r}_a} H.$$

the angular momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_a \mathbf{r} \times \mathbf{p} = \sum_a \frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} \times \mathbf{p}_a + \sum_a \mathbf{r}_a \times \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}.$$

equations of motions

and the equation of motion is

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a^2}.$$

$$\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\sum_b (F_a + F_b)\mathbf{Y}_{ab}, \quad F_a = \frac{\partial_\rho(\varepsilon)_a}{\rho_a} - \frac{\varepsilon_a}{\rho^2} - \frac{\mathbf{p}_a^2}{\rho_a^3}.$$

the angular momentum is

$$\begin{aligned}
\sum_a \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \sum_a \sum_b (F_a + F_b)\mathbf{r}_a \times \mathbf{Y}_{ab} \\
&= \frac{1}{2} \sum_{ab} (F_a + F_b)(\mathbf{r}_a \times \mathbf{Y}_{ab} - \mathbf{r}_b \times \mathbf{Y}_{ab}) \\
&= \frac{1}{2} \sum_{ab} (F_a + F_b)(\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{Y}_{ab}
\end{aligned}$$

continuity

$$\int_{\mathbf{x}} \rho(x) = \sum_a 1.$$

$$\int_{\mathbf{x}} \rho(x)\mathbf{v}(x) = \sum_a v_a.$$

this gives the canonical momentum as

$$\mathbf{p}_a = \partial_{\dot{\mathbf{r}}_a} L = \dot{\mathbf{r}}_a.$$

and the equation of motion is

$$\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \partial_{\mathbf{r}_a} L = -\sum_b (\varphi_a + \varphi_b)\mathbf{Y}_{ab}.$$

$$\varphi_a = \frac{\partial_\rho \varepsilon_a}{\rho_a} - \frac{\varepsilon_a}{\rho_a^2} = \frac{\mathrm{d}}{\mathrm{d}\rho}\frac{\varepsilon}{\rho}.$$

reference density

the hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\,\frac{p(x)^2}{\rho(x)} + \varepsilon(x).$$

using a generic reference density

$$\varphi_a = \sum_b W_{ab}.$$

the integral of the hamiltonian is

$$H(\mathbf{r},\mathbf{p}) = \sum_a \frac{1}{2}\,\frac{\mathbf{p}_a^2}{\varphi_a \rho_a} + \sum_a \frac{\varepsilon_a}{\varphi_a}.$$

from the conservation of energy

$$\sum_a \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{p}_a^2}{\varphi_a \rho_a} = -\sum_a \frac{\mathrm{d}}{\mathrm{d}t}\frac{\varepsilon_a}{\varphi_a}.$$

reference density

the velocity equation is

$$\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a\varphi_a}, \qquad \frac{\mathrm{d}\varphi_a}{\mathrm{d}t} = \sum_b \left(\frac{\mathbf{p}_a}{\rho_a\varphi_a} - \frac{\mathbf{p}_b}{\rho_b\varphi_b} \right) \cdot \mathbf{Y}_{ab}.$$

from the conservation of energy

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\varphi_a\rho_a \frac{\mathbf{p}_a^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\varphi_a\rho_a} - \varphi_a\rho_a \frac{\mathrm{d}}{\mathrm{d}t} \frac{\varepsilon_a}{\varphi_a}.$$

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a^2}{2} \left(\frac{1}{\rho_a} \frac{\mathrm{d}\rho_a}{\mathrm{d}t} + \frac{1}{\varphi_a} \frac{\mathrm{d}\varphi_a}{\mathrm{d}t} \right) - \rho_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \frac{\rho_a\varepsilon_a}{\varphi_a} \frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

$$\mathbf{p}_a \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = \frac{\mathbf{p}_a^2}{2\rho_a} \frac{\mathrm{d}\rho_a}{\mathrm{d}t} - \rho_a \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{2} + \rho_a\varepsilon_a \right) \frac{1}{\varphi_a} \frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

reference density

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}\rho(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \frac{\rho_a}{\varphi_a} = 0.$$

$$\sum_a \frac{1}{\varphi_a}\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \sum_a \frac{\rho_a}{\varphi_a^2}\frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \frac{\rho_a}{\varphi_a}\frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

$$\mathbf{p}_a\cdot\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = -\rho_a\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} + \frac{1}{\varphi_a}(\mathbf{p}_a^2+\rho_a\varepsilon_a)\frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = -\frac{\mathbf{p}_a}{\rho_a}\cdot\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{\varphi_a\rho_a}+\frac{\varepsilon_a}{\varphi_a}\right)\frac{\mathrm{d}\varphi_a}{\mathrm{d}t}.$$

from the hamiltonian density

$$\frac{\mathrm{d}\varepsilon(\mathbf{x})}{\mathrm{d}t} = -\frac{\mathbf{p}(\mathbf{x})}{\rho(\mathbf{x})}\cdot\frac{\mathrm{d}\mathbf{p}(\mathbf{x})}{\mathrm{d}t} + \frac{\mathbf{p}(\mathbf{x})^2}{2\rho(\mathbf{x})^2}\frac{\mathrm{d}\rho(\mathbf{x})}{\mathrm{d}t}.$$

reference density

using

$$\varphi_a = \varepsilon_a.$$

the equations reduce to

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = -\frac{\mathbf{p}_a}{\rho_a} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} + \left(\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + 1 \right) \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t}.$$

$$\frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} \frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = \frac{\mathbf{p}_a}{\rho_a} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}.$$

$$\frac{\mathrm{d}\varepsilon_a}{\mathrm{d}t} = \varepsilon_a \frac{\mathbf{p}_a}{\mathbf{p}_a^2} \cdot \frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}.$$

from the hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2} \sum_a \frac{\mathbf{p}_a^2}{\varepsilon_a \rho_a} + \sum_a 1.$$

reference density

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} &= -\frac{1}{2}\sum_b \mathbf{p}_b^2 \partial_{\mathbf{r}_a} \frac{1}{\varepsilon_b \rho_b} \\
&= \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b \rho_b^2} \partial_{\mathbf{r}_a} \rho_b + \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b} \partial_{\mathbf{r}_a} \varepsilon_b \\
&= \frac{1}{2}\sum_b \mathbf{p}_b^2 \frac{1}{\varepsilon_b^2 \rho_b^2} (\varepsilon_b \partial_\varepsilon \rho_b + \rho_b) \partial_{\mathbf{r}_a} \varepsilon_b
\end{aligned}
$$

relativistic case

the invariant length

$$\mathrm{d}\tau^{\,2} = g_{\mu\nu}\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,.$$

$$\mathrm{d}\tau^{\,2} = \mathrm{d}t^{\,2} - \mathrm{d}\mathbf{x}\,\cdot\mathrm{d}\mathbf{x}\,.$$

$$\frac{\mathrm{d}t^{\,2}}{\mathrm{d}\tau^{\,2}} - \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = u_{\mu}u^{\mu} = 1.$$

where

$$u_0 = \gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

$$\mathbf{u} = \gamma\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}.$$

relativistic case

from the energy-momentum tensor

$$T^{\mu\nu} = \omega u^\mu u^\nu - P g^{\mu\nu} = \varepsilon u^\mu u^\nu + P(u^\mu u^\nu - g^{\mu\nu}).$$

which comes from the lagrangean as

$$T^{\mu\nu} = u^\mu \partial_{u_\nu} \mathcal{L} - g^{\mu\nu} \mathcal{L}.$$

from that we identify that the canonical momentum is

$$\partial_{u_\nu} \mathcal{L} = \omega u^\nu.$$

and the lagrangean density

$$\mathcal{L} = P.$$

in order to make these two conclusions compatible we make use of the 4-velocity constraint in the lagrangean

$$\mathcal{L} = P + \omega(u^\mu u_\mu - 1) = -\varepsilon + \omega u^\mu u_\mu.$$

relativistic case

the hamiltonian expressed in terms of the canonical momentum

$$p^\mu = \omega u^\mu, \quad \mathbf{p} = \omega\gamma\mathbf{v}.$$

it is important to rewrite the γ in terms of p

$$\frac{\mathbf{p}^2}{\omega^2} = \gamma^2\mathbf{v}^2 = \gamma^2 - 1, \quad \gamma^2 = \frac{\mathbf{p}^2}{\omega^2} + 1.$$

from that we identify that the canonical momentum is

$$T^{\mu\nu} = \frac{p^\mu p^\nu}{\omega} - Pg^{\mu\nu}.$$

the hamiltonian density is

$$\mathcal{H} = T_{tt} = \frac{p_t^2}{\omega} - P = \frac{\mathbf{p}^2}{\omega} + \varepsilon.$$

energy flux

the energy flux is

$$\frac{p^t p^i}{\omega} = \gamma \mathbf{p}.$$

the equations of motion are

$$\partial_\mu T^{\mu\nu} = 0.$$

which integrates to

$$\int_{\mathbf{x}} \partial_\mu T^{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} T^{t\nu} = 0.$$

the space part translates into

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} \omega \gamma^2 \mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x}} \sqrt{\frac{\mathbf{p}^2}{\omega^2} + 1}\, \mathbf{p} = 0.$$

energy flux

the time part translates into

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}(\omega\gamma^2 - P) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}}\left(\frac{\mathbf{p}^2}{\omega} + \varepsilon\right) = 0.$$

by imposing that any quantity can be integrated to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{x}} f(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_a \frac{f_a}{h_a}.$$

this allows us to define the reference density based on the hamiltonian density so that

$$h_a = \omega_a\gamma_a^2 - P_a = \frac{\mathbf{p}_a^2}{\omega_a} + \varepsilon_a.$$

therefore

$$h_a = \sum_b W_{ab}.$$

integrals

given a local density its integral

$$\mathcal{S} = \int \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \, \mathcal{L}.$$

which can be converted to the common frame

$$\mathcal{S} = \int \mathrm{d}t \, \mathrm{d}\mathbf{x} \, \frac{\mathrm{d}\tau}{\mathrm{d}t} \mathcal{L} = \int \mathrm{d}t \, \mathrm{d}\mathbf{x} \, \mathcal{L}^*.$$

where

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}}.$$

so the lagrangean density on the common frame and the canonical momentum

$$\mathcal{L}^* = -\frac{\varepsilon}{\gamma}, \quad \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \frac{\varepsilon}{\gamma^2} \frac{\partial \gamma}{\partial \mathbf{v}}.$$

the gamma derivative is

$$\frac{\partial \gamma}{\partial \mathbf{v}} = \gamma^3 \mathbf{v}.$$

relativistic case

canonical momentum

$$\mathbf{p} = \frac{\partial \mathcal{L}^*}{\partial \mathbf{v}} = \varepsilon\gamma\mathbf{v}.$$

$$\frac{\mathbf{p}^2}{\varepsilon^2} = \gamma^2\mathbf{v}^2 = \gamma^2 - 1, \quad \gamma = \sqrt{\frac{\mathbf{p}^2}{\varepsilon^2} + 1}\,.$$

the hamiltonian density in the common frame is

$$\mathcal{H}^* = \mathbf{v} \cdot \mathbf{p} - \mathcal{L}^* = \frac{\mathbf{p}^2}{\gamma\varepsilon} + \frac{\varepsilon}{\gamma}.$$

the hamiltonian density in the local frame is

$$\mathcal{H} = \frac{\mathbf{p}^2}{\varepsilon} + \varepsilon.$$

reference density

sph

the SPH integrals are defined in the common frame so

$$\int_{\mathbf{x}} \frac{f(\mathbf{x})}{\gamma(\mathbf{x})} = \sum_a \frac{f_a}{\gamma_a \rho_a}.$$

so the common frame lagrangean is

$$L^* = -\sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}.$$

so the common frame hamiltonian is

$$H^* = \sum_a \frac{\mathbf{p}_a^2}{\gamma_a \varepsilon_a \rho_a} + \sum_a \frac{\varepsilon_a}{\gamma_a \rho_a}.$$

eqs of motion

the partial derivative in respect to the canonical momentum is

$$\partial_{\mathbf{p}_a} \frac{1}{\gamma_b} = -\frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a^2} \delta_{ab}.$$

the partial derivative in respect to the canonical momentum is

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{r}_a}{\mathrm{d}t} &= \partial_{\mathbf{p}_a} H^* \\
&= \frac{2\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a} - \frac{\mathbf{p}_a^2 \mathbf{p}_a}{\gamma_b^3 \varepsilon_b^3 \rho_b} - \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\
&= \left[2\gamma_a^2 - (\gamma_a^2 - 1) - 1\right] \frac{\mathbf{p}_a}{\gamma_a^3 \varepsilon_a \rho_a} \\
&= \frac{\mathbf{p}_a}{\gamma_a \varepsilon_a \rho_a}
\end{aligned}$$

kernel revisited

given a normalized kernel

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = 1.$$

the relation is imposed

$$\int_{\mathbf{x}} f(\mathbf{x}, t) = \sum_a \frac{f_a(t)}{\rho_a(t)}.$$

therefore,

$$\int_{\mathbf{x}} W\left[\mathbf{x} - \mathbf{r}_a(t)\right] = \sum_b \frac{W(\mathbf{r}_b - \mathbf{r}_a)}{\rho_b}.$$

$$\sum_a \sum_b \frac{W(\mathbf{r}_b - \mathbf{r}_a)}{\rho_b} = N.$$

$$\sum_b \frac{1}{\rho_b} \sum_a W(\mathbf{r}_b - \mathbf{r}_a) = N.$$

kernel revisited

$$\sum_a \varepsilon_a \sum_b \frac{W(\mathbf{r}_b - \mathbf{r}_a)}{\rho_b} = \sum_a \varepsilon_a.$$

$$\sum_b \frac{1}{\rho_b} \sum_a \varepsilon_a W(\mathbf{r}_b - \mathbf{r}_a) = \sum_a \varepsilon_a.$$

$$\rho_b = \frac{1}{\varepsilon_b} \sum_a \varepsilon_a W(\mathbf{r}_b - \mathbf{r}_a).$$