

### Solution 1: SVM – Kernel Trick

The polynomial kernel is defined as

$$k(x, \tilde{x}) = (x^T \tilde{x} + b)^d.$$

Furthermore, assume  $x \in \mathbb{R}^2$  and  $d = 2$ .

- (a) Derive the explicit feature map  $\phi$  taking into account that the following equation holds:

$$k(x, \tilde{x}) = \langle \phi(x), \phi(\tilde{x}) \rangle$$

**Solution:**

$$\begin{aligned} k(x, \tilde{x}) &= \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + b \right)^2 \\ &= (x_1 \tilde{x}_1 + x_2 \tilde{x}_2 + b)^2 \\ &= (x_1 \tilde{x}_1 + x_2 \tilde{x}_2)^2 + 2(x_1 \tilde{x}_1 + x_2 \tilde{x}_2)b + b^2 \\ &= x_1^2 \tilde{x}_1^2 + 2x_1 \tilde{x}_1 x_2 \tilde{x}_2 + x_2^2 \tilde{x}_2^2 + 2bx_1 \tilde{x}_1 + 2bx_2 \tilde{x}_2 + b^2 \\ &= \left\langle \begin{pmatrix} x_1^2 \\ \sqrt{2}x_1 x_2 \\ x_2^2 \\ \sqrt{2}bx_1 \\ \sqrt{2}bx_2 \\ b \end{pmatrix}, \begin{pmatrix} \tilde{x}_1^2 \\ \sqrt{2}\tilde{x}_1 \tilde{x}_2 \\ \tilde{x}_2^2 \\ \sqrt{2}b\tilde{x}_1 \\ \sqrt{2}b\tilde{x}_2 \\ b \end{pmatrix} \right\rangle \\ &= \langle \phi(x), \phi(\tilde{x}) \rangle \end{aligned}$$

- (b) Describe the main differences between the kernel method and the explicit feature map.

**Solution:**

Using the kernel method reduces the computational costs of computing the scalar product in the higher-dimensional features space after calculating the feature map.