Algebra I

My Mathematics Notes

Phædrus

Linear Algebra

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Contents

Chapter 1

Introduction

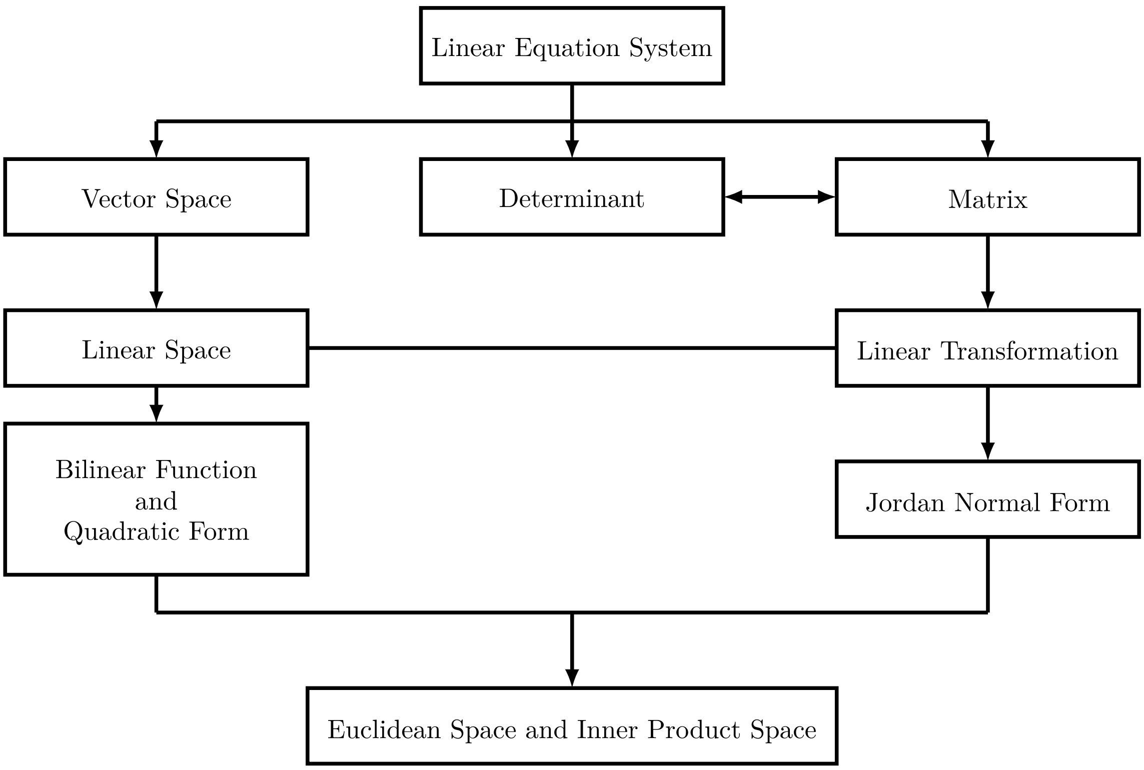
1.1 What is Algebra? & The structure of this note

Generally speaking, Algebra is a subject about operations. However, to discuss Algebra in a more detailed way, we have to first talk about some history of Algebra.

	815 AD	The first book of Algebra appeared.
Era of Elementary Algebra Focusing on: Operations, $\mathbb{C}$, Solving equations	$\updownarrow$	1637: Fermat's Last Theorem
	1832	
Era of Modern Algebra Focusing on: Mathematical Structures and their Morphisms	$\updownarrow$	1994: Fermat's Last Theorem proved by Wiles
	Now	

To help further understanding, the map of this note is now given.

Part 1: Linear Algebra



Part 2: Theory of Polynomials

$$\mathbb{Z} \longrightarrow \text{Polynomial Ring in One Variable} \longrightarrow \text{Polynomial Ring in Several Variables}$$

Part 3: Tensor Product and Exterior Algebra

$$\text{Affine Space} \longrightarrow \text{Projective Space} \longrightarrow \text{Tensor Product} \longrightarrow \text{Exterior Algebra}$$

Finally, how to correctly treat matrix is worth discussing. In peronal view, maxtrix is an important tool but it should not dominant this note. For most of the topic, we should use linear space and linear transformation to understand Algebra.

1.2 Several knowledge as preparation

1.2.1 $\mathbb{C}$

Please consult the note: <under construction>

1.2.2 The field of numbers

Here we should clarify the object we are going to study for mathematics require rigorness. Therefore, we need the following definition:

Definition 1.2.1 (The Field of Numbers)
Let K be a set and $K \subseteq \mathbb{C}$. If $\exists a \in K \implies a \neq 0$, $\forall x, y \implies x \pm y, xy \in K$ and $\forall x, y \in K, y \neq 0 \implies x/y \in K$. Then K is a field of numbers.

Several common fields of numbers are $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. Please note that the set all integers is not a field but since we use it very often, we have a symbol $\mathbb{Z}$ for it.
Also, please aware that there are much more fields of numbers than just $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$. For example:

Lemma 1.2.1 The set of all Gaussian rationals $\mathbb{Q}(\mathrm{i}) := \{a+b\mathrm{i} \in \mathbb{C} \mid a, b \in \mathbb{Q}\}$ is a field of number.

Proof. $\forall x = a + b\mathrm{i}, y = c + d\mathrm{i} \in \mathbb{Q}(\mathrm{i})$:

- $x \pm y = (a + b\mathrm{i}) \pm (c + d\mathrm{i}) = (a \pm c) + (b \pm d)\mathrm{i} \in \mathbb{Q}(\mathrm{i})$
- $xy = (a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i} \in \mathbb{Q}(\mathrm{i})$
- $\dfrac{x}{y} = \dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{ac + bd}{c^2 + d^2}\,\mathrm{i} \in \mathbb{Q}(\mathrm{i})$

Therefore, the set of all Gaussian rationals $\mathbb{Q}(\mathrm{i}) := \{a + b\mathrm{i} \in \mathbb{C} \mid a, b \in \mathbb{Q}\}$ is a field of number. $\square$

Lemma 1.2.2 Let K be any field of numbers, then $\mathbb{Q} \subseteq K$

Proof. We first prove that $\mathbb{N} \subseteq K$, we use induction. In the base case where $n = 0$ and $n = 1$. Since K is a field fo numbers, $\exists a \in K$ and $a \neq 0$. Then $0 = a - a \in K$ and $1 = a/a \in K$. Now suppose inductively that $n \in K$, then as to $n + 1$, $n + 1 \in K$. This closes the induction, and thus for all natural number n, $n \in K$, *i.e.,* $\mathbb{N} \subseteq K$.
We then demonstrate that $\mathbb{Z} \subseteq K$. $\forall x \in \mathbb{Z}, \exists a, b \in \mathbb{N} \implies x = a - b \in K$. Therefore, $\mathbb{Z} \subseteq K$.
We finally show that $\mathbb{Q} \subseteq K$. $\forall x \in \mathbb{Q}, \exists a, b \in \mathbb{Z} \implies x = a/b \in K$. Therefore, $\mathbb{Q} \subseteq K$. $\square$

1.2.3 Fundamentals of set theory

Please consult the note: *Analysis I—Logic, Sets,* $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{Q}$.

1.2.4 Σ and Π

In order to reduce the unnecessary writings, we here introduce the following symbols:

Definition 1.2.2 ($\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$) $\forall i, j \in \mathbb{N}_+$

$$\sum_{1 \leq i \leq n} a_i = \sum_{i=1}^{n} a_i := a_1 + a_2 + \cdots + a_n$$

$$\prod_{1 \leq i \leq n} a_i = \prod_{i=1}^{n} a_i := a_1 a_2 \cdots a_n$$

Lemma 1.2.3 (Properties of $\boldsymbol{\Sigma}$) $\forall \lambda \in \mathbb{C}$, $\forall i, j \in \mathbb{N}_+$

$$\sum_i \lambda a_i = \lambda \sum_i a_1$$

$$\sum_i (a_i + b_i) = \sum_i a_i + \sum_i b_i$$

$$\sum_i \sum_i a_{ij} = \sum_j \sum_i a_{ij}$$

1.2.5 Fundamentals of logic

Please consult the note: *Analysis I—Logic, Sets,* $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{Q}$.

Part I

Linear Algebra

Chapter 2

Vector Space and Matrix

under construction

Chapter 3

Determinant

under construction

Chapter 4

Linear Space & Linear Transformation

under construction

Chapter 5

Bilinear Function & Quadratic Form

under construction

Part II

Polynomial Theory

Part III

Tensor Product & Exterior Algebra