Homework 2

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1 Proof multivariate Gaussian distribution normalization.

Solution:
First, we have the PDF of the Gaussian Distribution is:

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{\frac{D}{2}} \mid \Sigma \mid^{\frac{1}{2}} e^{\frac{-1}{2}} (x-\mu)^T \Sigma^{-}1(x-\Sigma)}$$

The Gaussian Distribution is normalize:

$$\Leftrightarrow \int_{-\infty}^{+\infty} p(x \mid \mu, \sigma^2) = 1$$

Where μ is a D-dimensional mean vector
Σ is a D x D co-variance matrix
$\mid \Sigma \mid$ denotes the determinant of Σ
Set:

$$\Delta^2 = \frac{-1}{2}(x-\mu)^T \Sigma^{-}1(x-\mu)$$

$$= \frac{-1}{2} x^T \Sigma^{-}1x + x^T \Sigma^{-}1\mu + constant$$

Consider eigenvalues and eigenvectors of Σ we have:

$$\Sigma \mu_i = \lambda_i \mu_i, i = 1, ..., D$$

Because Σ is a real, symmetric matrix
$\rightarrow$ its eigenvalues will be real and its eigenvectors form an orthonormal set.
Proof:
    its eigenvalues will be real
Example:

$$\begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{pmatrix}$$

$\rightarrow$ The equation to find the eigenvalues is :

$$(\sigma_1^2 - \lambda)(\sigma_2^2 - \lambda) - (\sigma_{1,2})^2 = 0$$

$$\Leftrightarrow (\sigma_1^2 - \lambda)(\sigma_2^2 - \lambda) = (\sigma_{1,2})^2$$

$\rightarrow$ With $\lambda = \lambda_1$:

$$\begin{pmatrix} \sigma_1^2 - \lambda_1 & cov(\sigma_{1,2}) \\ cov(\sigma_{2,1}) & \sigma_2^2 - \lambda_1 \end{pmatrix}$$

$$(\sigma_1^2 - \lambda_1)x_1 + (\sigma_{1,2})x_2 = 0(1)$$

$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_1)x_2 = 0(2)$$

From (1) we have:

$$x_1 = \frac{-ycov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1}$$

$$x_1 = x_2$$

So the eigenvector in this case is:

$$\begin{pmatrix} \frac{\frac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_1}}{1} \end{pmatrix}$$

With $\lambda = \lambda_2$:

$$\begin{pmatrix} \sigma_1^2 - \lambda_2 & cov(\sigma_{1,2}) \\ cov(\sigma_{1,2}) & \sigma_2^2 - \lambda_2 \end{pmatrix}$$

$$(\sigma_1^2 - \lambda_2)x_1 + (\sigma_{1,2})x_2 = 0(3)$$
$$(\sigma_{1,2})x_1 + (\sigma_2^2 - \lambda_2)x_2 = 0(4)$$

From (3) we have:

$$x_1 = \frac{-ycov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}$$
$$\rightarrow x_1 = x_2$$

So the eigenvector in this case is:

$$\begin{pmatrix} \frac{\frac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}}{1} \end{pmatrix}$$

And:

$$\begin{pmatrix} \frac{\frac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}}{1} \end{pmatrix}^T \begin{pmatrix} \frac{\frac{-cov(\sigma_{1,2})}{\sigma_1^2 - \lambda_2}}{1} \end{pmatrix} = 1$$

2 Calculate marginal normal distribution.

3 Calculate conditional normal distribution.