Lecture 9

Some Advanced Topics in Linear Algebra

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Agenda

Today's Setup

1. Vector Spaces

2. The Eigenvalue Problem

3. Similarity and Diagonalization

4. Symmetric Matrices and Orthogonal Diagonalization

5. Quadratic Forms

Vector spaces

- A collection of vectors on which we apply certain mathematical operations forms a vector space.
- A vector is a special kind of matrix with one dimension.
  We write a column vector as

$$v = \begin{pmatrix} \nu_1 \\ \nu_2 \\ . \\ . \\ \nu_n \end{pmatrix}$$

  and a row vector as

$$v^T = \begin{pmatrix} \nu_1 & \nu_1 & . & . & \nu_n \end{pmatrix}$$

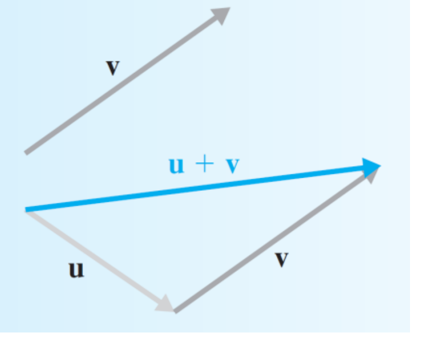
Basic vector operations

In ***vector addition***, if $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$, then their ***sum*** $\mathbf{u} + \mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

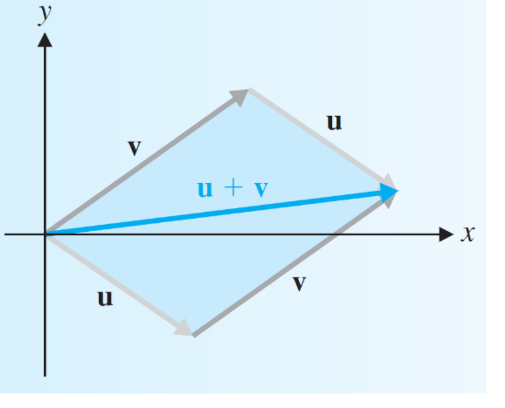
The Head-to-Tail Rule

Given vectors **u** and **v** in $\mathbb{R}^2$, translate **v** so that its tail coincides with the head of **u**. The ***sum*** $\mathbf{u} + \mathbf{v}$ of **u** and **v** is the vector from the tail of **u** to the head of **v**.



The Parallelogram Rule

Given vectors **u** and **v** ir $\mathbb{R}^2$, (in standard position), their ***sum*** $\mathbf{u} + \mathbf{v}$ is the vector in standard position along the diagonal of the parallelogram determined by **u** and **v**.



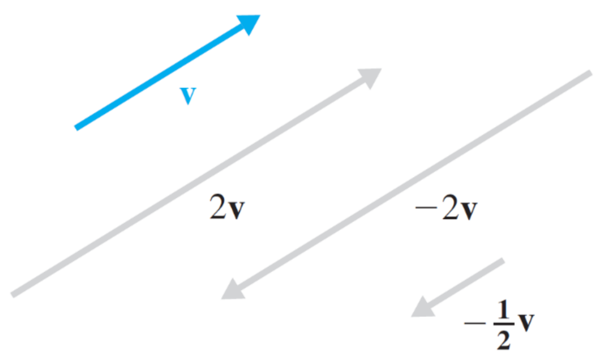
Basic vector operations

Scalar multiplication. Given a vector **v** and a real number *c*, the ***scalar multiple*** *c***v** is the vector obtained by multiplying each component of **v** by *c*. In general,

$$c\mathbf{v} = c\,[v_1, v_2] = [cv_1, cv_2]$$

Geometrically, *c***v** is a “scaled” version of **v**.

Observe that *c***v** has the same direction as **v** if $c > 0$ and the opposite direction if $c < 0$. We also see that *c***v** is $|c|$ times as long as **v.** As Figure shows, when translation of vectors is taken into account, two vectors are scalar multiples of each other if and only if they are ***parallel***.



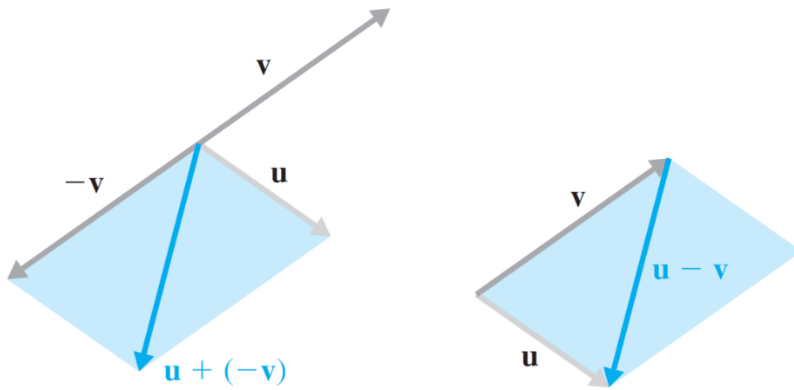
Basic vector operations

A special case of a scalar multiple is $(-1)\mathbf{v}$, which is written as $-\mathbf{v}$ and is called the ***negative of*** **v.**

Vector subtraction: The ***difference*** of **u** and **v** is the vector $\mathbf{u} - \mathbf{v}$ defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Figure shows that $\mathbf{u} - \mathbf{v}$ corresponds to the "other" diagonal of the parallelogram determined by **u** and **v**.



Linear Combinations

- Definition: Two vectors in R^2 , v and w are linearly independent if the scalars λ_1, λ_2 satisfying

$$\lambda_1 v + \lambda_2 w = \underset{\substack{\downarrow \\ \textit{null} \\ \textit{vector}}}{0}$$

are zero, where 0 is the null vector. If the λ's are non-zero, w and v are said to be linearly dependent.

Linear Combinations

Example

Establish whether the following vectors in $\mathbb{R}^2$ are linearly independent:

(i) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution

In case (i),

$$\lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Rightarrow\; \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 1$$

Therefore $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.
In case (ii),

$$\lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only solution to this equation is $\lambda_1 = \lambda_2 = 0$, and the vectors are linearly independent.

Inner product

- Inner Product: The product of two vectors of the same dimension n, v and w defined as:

$$v^T w = w^T v = \sum_{i=1}^{n} \omega_i \nu_i$$

is known as the inner product of v and w.

(Note that vw^T and wv^T is a $n \times n$ matrix and is known as the outer product.)

Example: Compute $\mathbf{u}^{\mathbf{T}} \cdot \mathbf{v}$ when $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$.

Solution:

$\mathbf{u}^{\mathbf{T}} \cdot \mathbf{v} = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1$

Length

- Length of a vector: The length of a vector v is given as

$$||v|| = \sqrt{\nu_1^2 + \nu_2^2 + \ldots + \nu_n^2}$$

  The length of a vector is also known as Euclidean norm.

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}$$

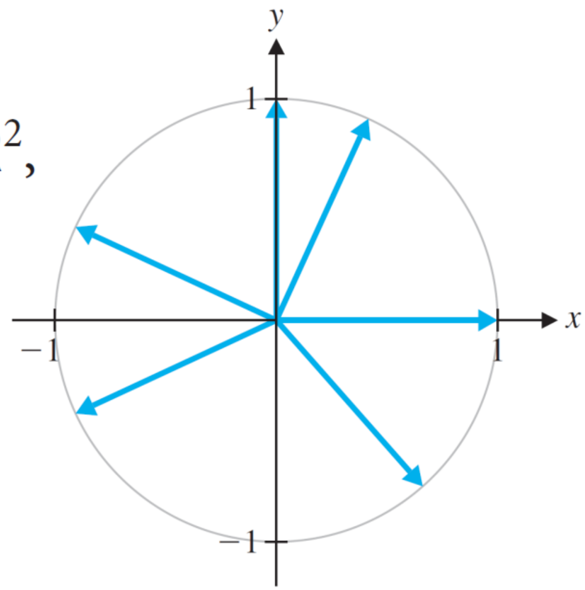
- Example 10.1: Find length of vector

$$v = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

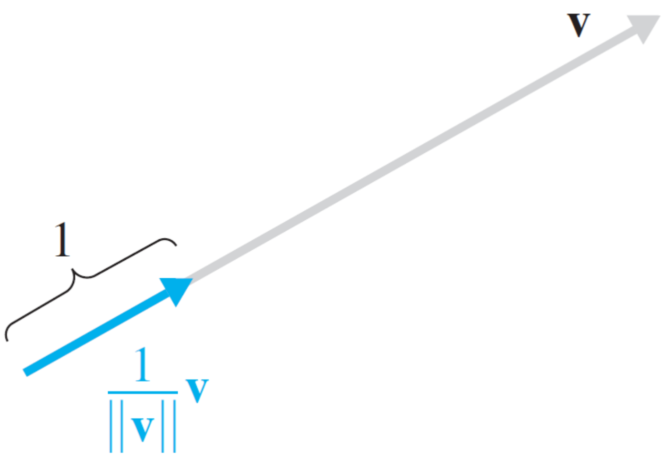
$$||v|| = \sqrt{1+4+4} = \sqrt{9} = 3$$

Length

A vector of length 1 is called a ***unit vector***. In $\mathbb{R}^2$, the set of all unit vectors can be identified with the *unit circle*, the circle of radius 1 centered at the origin.



Finding a unit vector in the same direction is often referred to as ***normalizing*** a vector.



Normalizing a vector

Vector spaces

- Theorem 10.2: Any vector in R^2 can be expanded as a linear combination of two independent vectors in R^2.

- Basis of a Vector Space: If v and w are vectors in R^n, then
  (i) v $\pm$ w is a vector in R^n as well, and
  (ii) if v is a vector in R^n, then λv is a vector in R^n.
  We say that R^n is closed under the operations of addition and scalar multiplication.

- Definition: A set of of vectors for which addition and scalar multiplication can be defined and which is closed under those operations is called a vector space.
  For example R^2, R^3 and R^n are vector spaces.

Vector spaces

- Example: Let

$$\Gamma = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} s.t \;\; xy \geqslant 0 \right\}$$

Find vectors u and v such that u+v is not in Γ.
e.g. Γ is not a vector space.
Let $u = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Both belong to Γ but

$$u + v = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

does not belong to Γ. Hence, Γ is <u>not</u> a vector space.

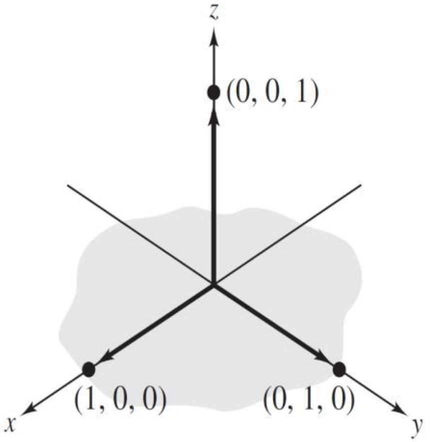
Basis for a Vector spaces

- Definition: A basis is a set of linearly independent vectors that generates all vectors in the space.

- Example: Consider $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a basis vectors for R^2.

  Then $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ can be expressed as a linear combination of e_1 and e_2 (be generated by e_1 and e_2)

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



- **Example**: Show that the set below is a basis for R^3.

  $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Solution:

S spans R^3. Furthermore, S is linearly independent because the vector equation $c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$ has only the trivial solution $c_1 = c_2 = c_3 = 0$. So, S is a basis for R^3.

Orthogonal Vectors

- Vector Orthogonality: The vectors v and w are orthogonal if and only if $v^T w = \cos 90° = 0$;
  A set of orthogonal vectors that are also of unit length that constitute a basis are known as orthogonal basis.

- Example: Vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are all orthogonal vectors of unit length and constitute a basis for R^3.

Example

In $\mathbb{R}^3$, $u = [1, 1, -2]$ and $v = [3, 1, 2]$ are orthogonal, since $u^T \cdot v = 3 + 1 - 4 = 0$.

Dimension of a Vector Space

- Definition: The number of vectors that belong to the basis of a finite vector space is known as the dimension of the space.

Example: Find the dimension of each subspace R^3.
of

$W = \{(d, c - d, c)$: c and d are real numbers$\}$

Solution:

By writing the representative vector $(d, c - d, c)$ as

$$(d, c - d, c) = (0, c, c) + (d, -d, 0) = c(0, 1, 1) + d(1, -1, 0)$$

you can see that W is spanned by the set

$S = \{(0, 1, 1), (1, -1, 0)\}$.

You can show that this set is linearly independent. So, S is a basis for W, and W is a two-dimensional subspace of R^3.

Rank of a Matrix

- Definition 10.6: The maximum number of linearly independent columns equals the number of linearly independent rows. This number is known as the rank of the matrix.

Example: To find the rank of the matrix *A* below, convert to a matrix *B* in row-echelon form as shown.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The matrix *B* has three nonzero rows, so the rank of *A* is 3.

Rank of a Matrix

- Definition 10.7: An $n \times n$ matrix A is nonsingular if and only if

$$\begin{array}{rl} (i) & det(A) \neq 0 \\ (ii) & A^{-1} \; exists \\ (iii) & rank(A) \; = \; n \end{array}$$

- Example: Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 4 \end{pmatrix}$$

Clearly column 4 is twice column 1, hence these two are linearly dependent. Columns 1, 2, 3 and columns 2, 3, 4 constitute two different sets of three linearly independent columns of A.
The rank is 3.

Summary of Equivalent Conditions for Square Matrices

If A is an $n \times n$ matrix, then the conditions below are equivalent.

1. A is invertible.

2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.

3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

4. A is row-equivalent to I_n.

5. $A \neq 0$

6. $\mathrm{Rank}(A) = n$

7. The n row vectors of A are linearly independent.

8. The n column vectors of A are linearly independent

1. Vector Spaces
2. The Eigenvalue Problem
3. Similarity and Diagonalization
4. Symmetric Matrices and Orthogonal Diagonalization
5. Quadratic Forms

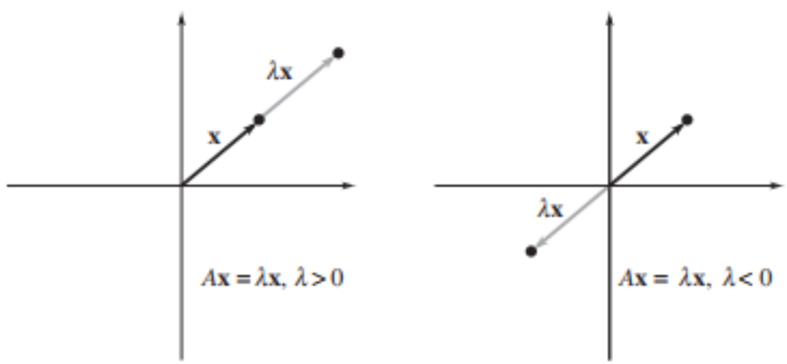
The Eigenvalue problem

Its central question is "when A is an $n \times n$ matrix, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?"

Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is an **eigenvalue** of A when there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is an **eigenvector** of A corresponding to λ.



The Eigenvalue problem

Example – Verifying Eigenvectors and Eigenvalues

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and that $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

The Eigenvalue problem

Example – Verifying Eigenvectors and Eigenvalues - Solution

Multiplying $\mathbf{x}_1$ on the left by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\text{Eigenvalue}}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Eigenvector}}.$$

So, $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$.

Example 1 – Solution (2 of 2)

Similarly, multiplying $\mathbf{x}_2$ on the left by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Introduction to Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an ***eigenvalue*** or **characteristic root,** or **latent root** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ **(1)**.

Such a vector **x** is called an ***eigenvector*****,** or **characteristic vector,** or **latent vector,** of A corresponding to λ.

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the ***eigenspace*** of λ and is denoted by E_λ.

Introduction to Eigenvalues and Eigenvectors

From (1) we have

$$(A - \lambda\mathcal{I})\cancel{q}\,x = 0$$

For a non-trivial solution where x $q \neq 0$, we need that $(A - \lambda\mathcal{I})$ to be singular. That implies that

$$det(A - \lambda\mathcal{I}) = 0 \tag{2}$$

The above equation (2) is known as the characteristic equation or characteristic polynomial and its roots are the eigenvalues.

Introduction to Eigenvalues and Eigenvectors

Example (1 of 5)

Find all of the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Solution:

The preceding remarks show that we must find all solutions λ of the equation $\det(A - \lambda I) = 0$. Since

$$\det(A - \lambda I) = \det\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (3-\lambda)(3-\lambda) - 1 = \lambda^2 - 6\lambda + 8$$

Introduction to Eigenvalues and Eigenvectors

Example (2 of 5)

We need to solve the quadratic equation $\lambda^2 - 6\lambda + 8 = 0$.

The solutions to this equation are easily found to be $\lambda = 4$ and $\lambda = 2$. These are therefore the eigenvalues of *A*.

To find the eigenvectors corresponding to the eigenvalue $\lambda = 4$, we compute the null space of $A - 4I$. We find

$$[A - 4I \mid \mathbf{0}] = \left[\begin{array}{rr|r} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Example (3 of 5)

from which it follows that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$ if and only if $x_1 - x_2 = 0$ or $x_1 = x_2$.

Hence, the eigenspace $E_4 = \left\{ \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$

Similarly, for $\lambda = 2$, we have

$$[A - 2I \mid 0] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Introduction to Eigenvalues and Eigenvectors

Example (4 of 5)

so $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 2$ if and only

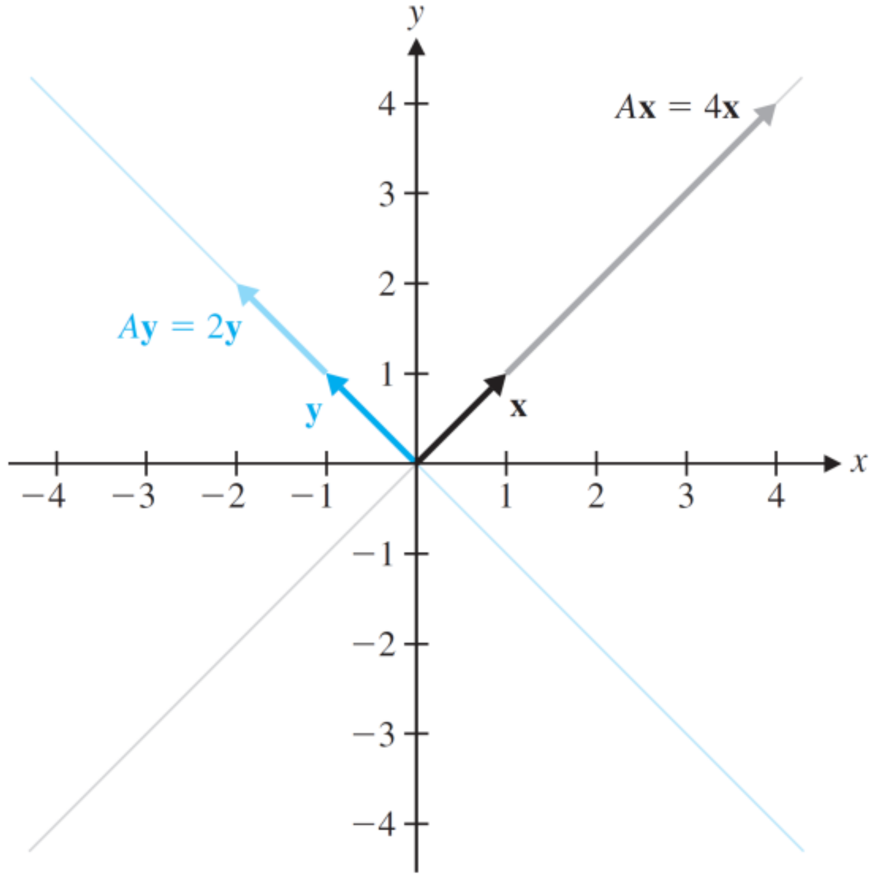
if $y_1 + y_2 = 0$ or $y_1 = -y_2$. Thus, the eigenspace

$$E_2 = \left\{ \begin{bmatrix} -y_2 \\ y_2 \end{bmatrix} \right\} = \left\{ y_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right).$$

Introduction to Eigenvalues and Eigenvectors

Example (5 of 5)

Figure shows graphically how the eigenvectors of A are transformed when multiplied by A: an eigenvector **x** in the eigenspace E_4 is transformed into 4**x**, and an eigenvector **y** in the eigenspace E_2 is transformed into 2**y**.



How *A* transforms eigenvectors

Eigenvalues and Eigenvectors of *n* × *n* Matrices

The eigenvalues of a square matrix *A* are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

When we expand $\det(A - \lambda I)$, we get a polynomial in λ, called the ***characteristic polynomial*** of *A*. The equation $\det(A - \lambda I) = 0$ is called the ***characteristic equation*** of *A*.

Eigenvalues and Eigenvectors of *n* × *n* Matrices

Find the eigenvalues and eigenvectors (eigenspaces) of a matrix.

Let *A* be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$ of *A*.
2. Find the eigenvalues of *A* by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.
3. For each eigenvalue λ, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. This can require row reducing an $n \times n$ matrix. The reduced row-echelon form must have at least one row of zeros.

Example

Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solution:

We follow the procedure outlined previously. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix}$$

Example – Solution (1 of 5)

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 4\lambda + 5) - (-2)$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

To find the eigenvalues, we need to solve the characteristic equation $\det(A - \lambda I) = 0$ for λ. The characteristic polynomial factors as $-(\lambda - 1)^2(\lambda - 2)$. Thus, the characteristic equation is $-(\lambda - 1)^2(\lambda - 2) = 0$, which clearly has solutions $\lambda = 1$ and $\lambda = 2$.

Example – Solution (2 of 5)

Since $\lambda = 1$ is a multiple root and $\lambda = 2$ is a simple root, let us label them $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

To find the eigenvectors corresponding to $\lambda_1 = \lambda_2 = 1$, we find the null space of

$$A - 1I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 4 - 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

Example – Solution (3 of 5)

Row reduction produces

$$[A - I \mid 0] = \left[\begin{array}{rrr|r} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the eigenspace E_1 if and only if $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$.

Example – Solution (4 of 5)

Setting the free variable $x_3 = t$, we see that $x_1 = t$ and $x_2 = t$, from which it follows that

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

To find the eigenvectors corresponding to $\lambda_3 = 2$, we find the null space of $A - 2I$ by row reduction:

$$[A - 2I \mid 0] = \left[\begin{array}{rrr|r} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Eigenvalues and Eigenvectors of *n* × *n* Matrices

Example – Solution (5 of 5)

So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the eigenspace E_2 if and only if $x_1 = \frac{1}{4}x_3$ and $x_2 = \frac{1}{2}x_3$. Setting the free variable $x_3 = t$, we have

$$E_2 = \left\{ \begin{bmatrix} \frac{1}{4}t \\ \frac{1}{2}t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right)$$

where we have cleared denominators in the basis by multiplying through by the least common denominator 4.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Let us define the ***algebraic multiplicity*** of an eigenvalue to be its multiplicity as a root of the characteristic equation. Thus, $\lambda = 1$ has algebraic multiplicity 2 and $\lambda = 2$ has algebraic multiplicity 1.

Let us define the ***geometric multiplicity*** of an eigenvalue λ to be dim E_λ, the dimension of its corresponding eigenspace.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Example

Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution:

The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

Example – Solution (1 of 4)

$$= -\lambda(\lambda^2 + 2\lambda) = -\lambda^2(\lambda + 2)$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$. Thus, the eigenvalue 0 has algebraic multiplicity 2 and the eigenvalue −2 has algebraic multiplicity 1.

Example – Solution (2 of 4)

For $\lambda_1 = \lambda_2 = 0$, we compute

$$[A - 0I \,|\, 0] = [A \,|\, 0] = \left[\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which it follows that an eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in E_0

satisfies $x_1 = x_3$. Therefore, both x_2 and x_3 are free.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Example – Solution (3 of 4)

Setting $x_2 = s$ and $x_3 = t$, we have

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

For $\lambda_3 = -2$,

$$[A - (-2)I \mid 0] = [A + 2I \mid 0] = \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $x_3 = t$ is free and $x_1 = -x_3 = -t$ and $x_2 = 3x_3 = 3t$.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Example – Solution (4 of 4)

Consequently,

$$E_{-2} = \left\{ \begin{bmatrix} -t \\ 3t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right)$$

It follows that $\lambda_1 = \lambda_2 = 0$ has geometric multiplicity 2 and $\lambda_3 = -2$ has geometric multiplicity 1.

Note that the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem

A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A.

Eigenvalues and Eigenvectors of $n \times n$ Matrices

The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

f. $\operatorname{rank}(A) = n$

g. $\operatorname{nullity}(A) = 0$

h. The column vectors of A are linearly independent.

Eigenvalues and Eigenvectors of *n* × *n* Matrices

i. The column vectors of *A* span $\mathbb{R}^n$.

j. The column vectors of *A* form a basis for $\mathbb{R}^n$.

k. The row vectors of *A* are linearly independent.

l. The row vectors of *A* span $\mathbb{R}^n$.

m. The row vectors of *A* form a basis for $\mathbb{R}^n$.

n. $\det A \neq 0$

o. 0 is not an eigenvalue of *A*.

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Eigenvalues and Eigenvectors of $n \times n$ Matrices

Theorem

Let A be a square matrix with eigenvalue λ and corresponding eigenvector **x**.

a. For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector **x**.

b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector **x**.

c. If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector **x**.

Eigenvalues and Eigenvectors of $n \times n$ Matrices Example (1 of 3)

Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; then what we want to find is $A^{10}\mathbf{x}$. The eigenvalues of *A* are $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Eigenvalues and Eigenvectors of $n \times n$ Matrices Example (2 of 3)

That is,

$$A\mathbf{v}_1 = -\mathbf{v}_1 \text{ and } A\mathbf{v}_2 = 2\mathbf{v}_2$$

Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for $\mathbb{R}^2$, we can write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Indeed, as is easily checked, $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.

Therefore, using Theorem above, we have

$$A^{10}\mathbf{x} = A^{10}(3\mathbf{v}_1 + 2\mathbf{v}_2) = 3(A^{10}\mathbf{v}_1) + 2(A^{10}\mathbf{v}_2)$$

$$= 3(\lambda_1^{10})\mathbf{v}_1 + 2(\lambda_2^{10})\mathbf{v}_2$$

Eigenvalues and Eigenvectors of *n* × *n* Matrices Example (3 of 3)

$$= 3(-1)^{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2(2^{10})\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{12} \end{bmatrix} = \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}$$

This is certainly a lot easier than computing A^{10} first; in fact, there are no matrix multiplications at all!

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Theorem

Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors—say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

1. Vector Spaces
2. The Eigenvalue Problem
3. $\boxed{\text{Similarity and Diagonalization}}$
4. Symmetric Matrices and Orthogonal Diagonalization
5. Quadratic Forms

Similar Matrices (1 of 2)

Definition

Let A and B be $n \times n$ matrices. We say that A ***is similar to*** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

Similar Matrices (2 of 2)

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \sim B$,

since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus, $AP = PB$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Diagonalization (1 of 4)

Once we obtain the eigenvalues of a matrix A as solutions to the characteristic equation, we proceed to obtain the corresponding eigenvectors. This leads us to a very important result whereby matrix A is transformed to a diagonal matrix. This result is known as the **spectral decomposition** of a square matrix. A is called a **spectral decomposition** of A because it breaks up A into pieces determined by the spectrum (eigenvalues) of A.

Definition of a Diagonalizable Matrix

An $n \times n$ matrix A is **diagonalizable** when A is similar to a diagonal matrix Λ. That is, A is diagonalizable when there exists an invertible matrix Q such that $Q^{-1}AQ = \Lambda$

or $A = Q\Lambda Q^{-1}$

Diagonalization (2 of 4)

Theorem Similar Matrices Have the Same Eigenvalues

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Theorem Condition for Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Diagonalization (3 of 4)

Steps for Diagonalizing a Square Matrix

Let A be an $n \times n$ matrix.

Step 1. **Find the eigenvalues of *A***

Step 2. **Find linearly independent eigenvectors of *A*.**

Find n linearly independent eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ for A (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

Step 3. **Construct *Q* from the vectors in step 2.**

Q be the $n \times n$ matrix whose consist of these eigenvectors. That is, $Q = [\mathbf{q}_1\ \mathbf{q}_2 \ldots \mathbf{q}_n]$.

Diagonalization (4 of 4)

Steps for Diagonalizing a Square Matrix

Step 4. **Construct Λ from the corresponding eigenvalues**

The diagonal matrix $\Lambda = Q^{-1}AQ$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal. Note that the order of the eigenvectors used to form Q will determine the order in which the eigenvalues appear on the main diagonal of Λ

Diagonalization

Example (1 of 7) – Diagonalizing a Matrix

Show that the matrix *A* is diagonalizable.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Then find a matrix *Q*, and a diagonal matrix Λ such that

$Q^{-1}AQ$

Diagonalization

Example (2 of 7) – Diagonalizing a Matrix

Solution

- Step 1. **Find the eigenvalues of *A*.**
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$

$$= -(\lambda - 1)(\lambda + 2)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Diagonalization

Example (3 of 7) – Diagonalizing a Matrix

Solution

- Step 2. **Find three linearly independent eigenvectors of *A*.**
  - Three vectors are needed because *A* is a 3×3 matrix.
  - This is a critical step.
  - If it fails, then *A* cannot be diagonalized.

Diagonalization

Example (4 of 7) – Diagonalizing a Matrix

Solution

Step 2 (cont.) **Find three linearly independent eigenvectors of *A*.**

- Basis for $\lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- Basis for $\lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ independent set.

Diagonalization

Example (5 of 7) – Diagonalizing a Matrix

Solution

- Step 3. **Construct *Q* from the vectors in step 2**

  The order of the vectors is unimportant.

  Using the order chosen in step 2, form

$$Q = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Step 4. **Construct Λ from the corresponding eigenvalues.**

  In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of *Q*.

Diagonalization

Example (6 of 7) – Diagonalizing a Matrix

Solution: Step 4 (cont.)

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- To avoid computing Q^{-1} simply verify that AQ = Q Λ.

This is equivalent to $\Lambda = Q^{-1}AQ$ when Q is invertible. (However, be sure that Q is invertible)

Diagonalization

Example (7 of 7) – Diagonalizing a Matrix

Solution: Step 4 (cont.)

Compute

$$AQ = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$Q\Lambda = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

1. Vector Spaces
2. The Eigenvalue Problem
3. Similarity and Diagonalization
4. $\boxed{\text{Symmetric Matrices and Orthogonal Diagonalization}}$
5. Quadratic Forms

Symmetric matrix

- A **symmetric matrix** is a matrix *A* such that $A^T = A$
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal ($\boldsymbol{a_{ij} = a_{ji}}$ ***for all*** $\boldsymbol{i \neq j}$)

The matrices *A* and *B* are symmetric, but the matrix *C* is not.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Symmetric Matrices and non symmetric Matrices

Non symmetric matrices have properties that are not exhibited by symmetric matrices, as listed below.

1. A non symmetric matrix may not be diagonalizable.

2. A non symmetric matrix can have eigenvalues that are not real.

3. For a non symmetric matrix, the number of linearly independent eigenvectors corresponding to an eigenvalue can be less than the multiplicity of the eigenvalue.

Symmetric Matrices

Theorem 7.7 Properties of Symmetric Matrices

If A is an $n \times n$ symmetric matrix, then the properties listed below are true.

1. A is diagonalizable.

2. All eigenvalues of A are real.

3. If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

Orthogonal matrix (1 of 6)

A matrix Q that has the property that

$$Q^T Q = Q Q^T = I$$

is known as an **orthogonal matrix.** An orthogonal matrix is a matrix for which its inverse equals its transpose

$$Q^{-1} = Q^T$$

Theorem: For the eigen value problem in equation, where A is a **symmetric matrix**, the eigen vectors that correspond to distinct eigenvalues are pairwise orthogonal and if put into a matrix, they form an **orthogonal matrix**

Orthogonal matrix (2 of 6)

- Example: For $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ where $\lambda_1 = 5$, $\lambda_2 = 0$ we have

  that corresponding to $\lambda_1 = 5$ we obtain

$$(A - 5\mathcal{I}) = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{rcl} -q_1 + 2q_2 & = & 0 \\ 2q_1 - 4q_2 & = & 0 \end{array}\right\} \quad q_1 = 2q_2$$

Orthogonal matrix (3 of 6)

Since we have an infinite number of eigenvectors satisfying the above, we choose q_1 , q_2 that also satisfy $q_1^2 + q_2^2 = 1$.

This condition is known as the **Euclidean distance condition** or **normalization**

In other words, we choose an eigenvector of unit length.

Hence, $4q_2^2 + q_2^2 = 1 \implies q_2^2 = \frac{1}{5} \implies q_2^2 = \pm\frac{1}{\sqrt{5}}$

Choosing $q_2^2 = \frac{1}{\sqrt{5}}$ we have $q_1 = \frac{2}{\sqrt{5}}$.

Therefore, $q^1 = \begin{pmatrix} 2/\sqrt{5} \\ \\ 1/\sqrt{5} \end{pmatrix}$

Orthogonal matrix (4 of 6)

Similarly, corresponding to $\lambda_2 = 0$ we obtain

$$q^2 = \begin{pmatrix} 1/\sqrt{5} \\ \\ -2/\sqrt{5} \end{pmatrix}$$

Note that $q^{1T} q^2 = 0$ since $\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \\ \frac{-2}{\sqrt{5}} \end{pmatrix} = \frac{2}{5} - \frac{2}{5} = 0.$

Therefore, $\quad Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$ is an orthogonal matrix.

Orthogonal matrix (5 of 6)

- Theorem 10.4: If an eigenvalue λ is repeated r times, there will be r orthogonal vectors corresponding to this root.

- Theorem 10.5: Let the $n \times n$ symmetric matrix A have eigenvalues $\lambda_1, \lambda_2, \ldots\lambda_n$ possibly not all distinct. Then there will be a set of n orthogonal eigenvectors $q_1, q_2, \ldots q_n$ such that

$$q_i^T q_j = 0 \qquad i \neq j \qquad i, j = 1, 2, \ldots n.$$

THEOREM Property of Orthogonal Matrices

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.

Orthogonal Diagonalization

- An $n \times n$ symmetric A is said to be **orthogonally diagonalizable**

if there are an orthogonal matrix Q

(with $(Q^{-1} = Q^{T}$ *)*

and a diagonal matrix Λ such that

$$Q^{-1}AQ = Q^{T}AQ = \Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Orthogonal Diagonalization (1 of 3)

THEOREM Fundamental Theorem of Symmetric Matrices

Let A be an $n \times n$ symmetric matrix. Then A is orthogonally diagonalizable (and has real eigenvalues) if and only if A is symmetric.

Orthogonal Diagonalization (2 of 3)

Orthogonal Diagonalization of a Symmetric Matrix

Let A be an $n \times n$ symmetric matrix.

1. Find all eigenvalues of A and determine the multiplicity of each.

2. For *each* eigenvalue of multiplicity 1, find a unit eigenvector. (Find any eigenvector and then normalize it.)

3. For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, then orthonormalization process.

Orthogonal Diagonalization (3 of 3)

Orthogonal Diagonalization of a Symmetric Matrix

4. The results of Steps 2 and 3 produce an orthonormal set of *n* eigenvectors.

Use these eigenvectors to form the columns of *Q*. The matrix $Q^{-1}AQ = Q^{T}AQ = \Lambda$ will be diagonal. (The main diagonal entries of Λ are the eigenvalues of *A*.)

Example – Orthogonal Diagonalization

Find a matrix *Q* that orthogonally diagonalizes

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution

1. The characteristic polynomial of *A* is $(A - \lambda I) = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda) - 4 \\ &= 4 - \lambda - 4\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 5\lambda = 0 \end{aligned}$$

The roots are $\lambda_1 = 5$ and $\lambda_2 = 0$.

Example – Solution (1 of 4)

2&3. For each eigenvalue, find an eigenvector, normalize these eigenvectors to produce an *orthonormal* basis.

- Example: For $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ where $\lambda_1 = 5$, $\lambda_2 = 0$ we have that corresponding to $\lambda_1 = 5$ we obtain

$$(A - 5\mathcal{I}) \quad = \quad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{rcl} -q_1 + 2q_2 & = & 0 \\ 2q_1 - 4q_2 & = & 0 \end{array}\right\} \quad q_1 = 2q_2$$

Example – Solution (2 of 4)

Since we have an infinite number of eigenvectors satisfying the above, we choose q_1 , q_2 that also satisfy $q_1^2 + q_2^2 = 1$.

In other words, we choose an eigenvector of unit length.

Hence, $4q_2^2 + q_2^2 = 1 \implies q_2^2 = \frac{1}{5} \implies q_2^2 = \pm\frac{1}{\sqrt{5}}$

Choosing $q_2^2 = \frac{1}{\sqrt{5}}$ we have $q_1 = \frac{2}{\sqrt{5}}$.

Therefore, $q^1 = \begin{pmatrix} 2/\sqrt{5} \\ \\ 1/\sqrt{5} \end{pmatrix}$

Example – Solution (3 of 4)

Similarly, corresponding to $\lambda_2 = 0$ we obtain

$$q^2 = \begin{pmatrix} 1/\sqrt{5} \\ \\ -2/\sqrt{5} \end{pmatrix}$$

Note that $q^{1T} q^2 = 0$ since $\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \\ \frac{-2}{\sqrt{5}} \end{pmatrix} = \frac{2}{5} - \frac{2}{5} = 0.$

Therefore, $\quad Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$ is an orthogonal matrix.

Example – Solution (4 of 4)

4. Verify that *Q* orthogonally diagonalizes *A* by finding

$$Q^T A Q = \Lambda$$

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigen value and eigen vector of symmetric matrix

- Theorem 10.8: The sum of the eigenvalues for a symmetric matrix A is equal to the sum of the elements of the main diagonal, the trace of A.

Proof Since $Q^T A Q = \Lambda$, using the properties of traces (see chapter 8), we have that

$$\begin{aligned}\operatorname{trace}(\Lambda) &= \operatorname{trace}(Q^T A Q)\\ &= \operatorname{trace}(A Q Q^T)\\ &= \operatorname{trace}(A)\end{aligned}$$

The above implies that

$$\boxed{\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}}$$

Eigen value and eigen vector of symmetric matrix

- Theorem 10.9: The product of the eigenvalues of a symmetric matrix equals the determinant of the matrix:

$$det(A) = \lambda_1 \lambda_2 ... \lambda_n$$

- Theorem 10.10: The eigenvalues of A^2 are the squares of the eigenvalues of A, but the eigenvectors of both matrices are the same.

Proof Premultiplying by A gives

$$A^2\mathbf{q} = \lambda A\mathbf{q} = \lambda^2\mathbf{q}$$

- Theorem 10.11: The eigenvalues of A^n are the same as the eigenvalues of A raised to the n-th power, but the eigenvector of both A and A^n are the same.

Eigen value and eigen vector of symmetric matrix

Example (1 of 5)

Compute A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution:

We found that this matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Eigen value and eigen vector of symmetric matrix

Example (2 of 5)

It follows that *A* is diagonalizable and $P^{-1}AP = D$, where

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Solving for *A*, we have $A = PDP^{-1}$ and, by Theorem 10.11 $A^n = PD^nP^{-1}$ for all $n \geq 1$.

Eigen value and eigen vector of symmetric matrix

Example (3 of 5)

Since

$$D^n = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

We have

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Eigen value and eigen vector of symmetric matrix

Example (4 of 5)

$$= \begin{bmatrix} \dfrac{2(-1)^n + 2^n}{3} & \dfrac{(-1)^{n+1} + 2^n}{3} \\ \dfrac{2(-1)^{n+1} + 2^{n+1}}{3} & \dfrac{(-1)^{n+2} + 2^{n+1}}{3} \end{bmatrix}$$

Since we were only asked for A^{10}, this is more than we needed.

Eigen value and eigen vector of symmetric matrix

Example (5 of 5)

But now we can simply set $n = 10$ to find

$$A^{10} = \begin{bmatrix} \dfrac{2(-1)^{10} + 2^{10}}{3} & \dfrac{(-1)^{11} + 2^{10}}{3} \\ \dfrac{2(-1)^{11} + 2^{11}}{3} & \dfrac{(-1)^{12} + 2^{11}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$$

Eigen value and eigen vector of symmetric matrix

- Theorem 10.12: The eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, but the vectors of both matrices A and A^{-1} are the same.

Proof

Premultiply $A\mathbf{q} = \lambda\mathbf{q}$ by A^{-1} to obtain

$$\mathbf{q} = A^{-1}\lambda\mathbf{q} \Rightarrow A^{-1}\mathbf{q} = \left(\frac{1}{\lambda}\right)\mathbf{q}$$

- Theorem 10.13: The eigenvalues of an idempotent matrix are either zero or one.

Proof

By theorem 10.10, $A^2\mathbf{q} = \lambda^2\mathbf{q}$. Since A is idempotent, $A^2 = A$. Then $A\mathbf{q} = A^2\mathbf{q} = \lambda^2\mathbf{q}$. Therefore $A\mathbf{q} = \lambda^2\mathbf{q}$ and

$$A^2\mathbf{q} = \lambda\mathbf{q} \quad \Rightarrow \quad \lambda^2\mathbf{q} - \lambda\mathbf{q} = (\lambda^2 - \lambda)\mathbf{q} = \mathbf{0} \quad \Rightarrow \quad \lambda(\lambda - 1)\mathbf{q} = \mathbf{0}$$

Since $\mathbf{q} \neq \mathbf{0}$, the above establishes the result.

Agenda

Today's Setup

1. Vector Spaces
2. The Eigenvalue Problem
3. Similarity and Diagonalization
4. Symmetric Matrices and Orthogonal Diagonalization
5. Quadratic Forms

Quadratic Forms

- Definition: Given an $n \times n$ matrix A and an $n \times 1$ vector x, we call the scalar expression

$$q(x) = x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

a quadratic form.

If A is not symmetric, then we can always define a symmetric matrix A^* that yields the same quadratic form as A.

The matrix *A* is called the **matrix of the quadratic form**.

Quadratic Forms

- **Example 1:** Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices.

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Quadratic Forms

- **Solution:**

a. $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2.$

b. There are two −2 entries in *A*.

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -\mathbf{2} \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - \mathbf{2}x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - \mathbf{2}x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - \mathbf{2}x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

Finding the Matrix of the Quadratic Form

The expression

$$ax^2 + bxy + cy^2 \qquad \text{Quadratic form}$$

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

- **Example:** Make a change of variable that transforms the quadratic form $Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$ into a quadratic form with no cross-product term.

- **Solution:** The matrix of the given quadratic form is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

Classifying Quadratic Forms

- Definition:
  - If $q(x) = x^T Ax > 0$ for all $x \neq 0$, then $q(x)$ is said to be positive definite and A is said to be a positive definite matrix.
  - If $q(x) = x^T Ax \geqslant 0$ for all $x \neq 0$, then $q(x)$ is said to be positive semidefinite , and A is a positive semidefinite matrix.
  - If $q(x) = x^T Ax < 0$ for all $x \neq 0$, then $q(x)$ is said to be negative definite and A is said to be a negative definite matrix.
  - If $q(x) = x^T Ax \leqslant 0$ for all $x \neq 0$, then $q(x)$ is said to be negative semidefinite , and A is a negative semidefinite matrix.
  - If $q(x)$ is positive for some x and negative for some other x , it is said to be indefinite.

Classifying Quadratic Forms

Positive and Negative Definiteness

Idea

Think of positive and negative definiteness as a way one applies to matrices the idea of "positive" and "negative".

- In the one-variable case, $Q(x) = ax^2$ and definiteness follows the sign of a.
  - Obviously, there are lots of indefinite matrices when $n > 1$.
- Diagonal matrices also help with intuition. When $\mathbf{A}$ is diagonal:

$$Q(\mathbf{x}) = \mathbf{x}^t \mathbf{A} \mathbf{x} = \sum_{i=1}^{n} a_{ii} x_i^2 .$$

therefore the quadratic form is:

  - positive definite if and only if $a_{ii} > 0$ for all i, positive semi definite if and only if $a_{ii} \geq 0$ for all i
  - negative definite if and only if $a_{ii} < 0$ for all i, negative semi definite if and only if $a_{ii} \leq 0$ for all i, and
  - indefinite if $\mathbf{A}$ has both negative and positive diagonal entries.

Classifying Quadratic Forms

- Definition: Given a $n \times n$ matrix A, we define a principal submatrix of order k ($1 \leqslant k \leqslant n$) to be a submatrix that is obtained by removing n-k rows and columns of A.

- The leading principal submatrix of order k is the submatrix obtained by removing the last n-k rows and columns.

- Example: For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For $k = 1$, the principal submatrices are a_{11}, a_{22}, a_{33} with a_{11} being the leading principal submatrix.

Classifying Quadratic Forms

- For $k = 2$ the principal submatrices are

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}; \quad \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

  with the first of these being the leading principal submatrix.

- Theorem 10.15: A necessary and sufficient condition for a real symmetric matrix A to be positive definite is that <u>all</u> its eigenvalues are positive.

- Theorem 10.16: A necessary and sufficient condition for a real symmetric matrix A to be positive semidefinite is that its eigenvalues be greater or equal to zero.

Classifying Quadratic Forms

- Theorem 10.17: A necessary and sufficient condition for a real symmetric matrix A to be negative definite is that all its eigenvalues be negative.

- Theorem 10.18: A necessary and sufficient condition for a real symmetric matrix A to be negative semidefinite is that its eigenvalues be less <u>or</u> equal to zero.

- Theorem 10.19: A necessary and sufficient condition for a real symmetric matrix A to be positive definite is that the determinant of every leading principal submatrix be positive. The leading principal submatrices of A are a set of submatrices:

$$A_1 = a_{11}, \quad A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots. A_n = A$$

Classifying Quadratic Forms

- Theorem 10.20: A necessary and sufficient condition for a real symmetric matrix A to be positive semidefinite and not positive definite is that some of its principal minors be zero and the rest positive.

- Theorem 10.21: A necessary and sufficient condition for a real symmetric matrix A to be negative definite is that its principal minors alternate in sign starting with negative.

- Theorem 10.22: A necessary and sufficient condition for a real symmetric matrix A to be negative semidefinite and not negative definite is that some of its principal minors be zero and the rest alternate in sign starting with negative.

- Theorem 10.23: If A is real symmetric and positive definite, one can find a nonsingular matrix P such that

$$A = PP^T$$

Summary definiteness of a matrix

1. ***Check definiteness of a quadratic form using the eigenvalues of A.***

Theorem

The quadratic form $Q(\mathbf{x}) = \mathbf{x}^t\mathbf{A}\mathbf{x}$ *is*

1. *positive definite if* $\lambda_i > 0$ *for all* i.
2. *positive semi definite if* $\lambda_i \geq 0$ *for all* i.
3. *negative definite if* $\lambda_i < 0$ *for all* i.
4. *negative semi definite if* $\lambda_i \leq 0$ *for all* i.
5. *indefinite if there exists* j *and* k *such that* $\lambda_j > 0 > \lambda_k$.

Summary definiteness of a matrix

2. ***Check definiteness of a quadratic form using (leading) principal minors of A.***

A matrix is

1. *Positive definite* if and only if the determinant of every leading principal submatrix be positive.
2. *Positive semidefinite* if and only if some of its principal minors be zero and the rest positive.
3. *Negative definite* if and only if its principal minors alternate in sign starting with negative (or its odd principal minors are negative and its even principal minors are positive).
4. *Negative semidefinite* if and only if some of its principal minors be zero and the rest alternate in sign starting with negative.
5. *Indefinite* if one of its k^{th} order principal minors is negative for an even k or if there are two odd principal minors that have diferent signs.

Definiteness of a Matrix
Example

Is the following matrix Positive Definite?

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Find the determinants of all possible $k \times k$ upper sub-matrices.

$$|2| = 2$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4$$

All determinants are Positive $\rightarrow$ Positive Definite

Definiteness of a Matrix
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All determinants are Positive $\rightarrow$ Positive Definite

Definiteness of a Matrix
Example

- Example: What is the "definiteness" of A?

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

There is an eigenvalue $\lambda = -1$ (repeated twice).
Therefore, by Theorem 10.17, $q(x)$ is negative definite.
Also,

$$q(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -x_1^2 - x_2^2 < 0$$

Furthermore, principal minors are -1 and 1, alternating in sign starting with a negative value.

Definiteness of a Matrix
Example

EXAMPLE Is $Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite?

SOLUTION Because of all the plus signs, this form "looks" positive definite. But the matrix of the form is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

and the eigenvalues of A turn out to be 5, 2, and -1. So Q is an indefinite quadratic form, not positive definite.

Agenda

Wrap up

1. Vector Spaces
2. The Eigenvalue Problem
3. Similarity and Diagonalization
4. Symmetric Matrices and Orthogonal Diagonalization
5. Quadratic Forms