Chapter 2. Matrix Algebra

1. Matrix Addition, Scalar Multiplication, Transposition

2. Multiplication

3. Matrix Inverse

4. Linear Transformations

2.1 Addition, Scalar Multiplication, Transpose

Definition

- An $m \times n$ matrix A is an array of m row and n column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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$$2\begin{bmatrix} -2 & 3 & 5 \\ 0 & 4 & 2 \\ 1 & -1 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 1 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 12 \\ 2 & 7 & 3 \\ -2 & -1 & 9 \end{bmatrix}$$

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Let A be $m \times n$ matrix. The transpose of A, written $A^T = [a_{ji}]$ is an $n \times m$ matrix.

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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$$\left(2A^T - 3\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\right)^T = 4A - 7\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$$

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A matrix A is symmetric if $A^T = A$ (it means that (i,j)-entry of $A =$ (j,i)-entry of A).

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Definition (Dot product)

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m$$

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Let $A_{m\times n}, B_{n\times p}$ be two matrices. The multiplication of A and B

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Find $(2, 1)$-entry of $AB - B^T A$.

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Hence $(2,1)$-entry of $AB - B^T A$ is $0 - 4 = -4$.

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Find $(2,1)$-entry of $AB - B^TA$.

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- The $(2,1)$-entry of B^TA is the dot product of the second column of B and the first column of A, is $1 * 1 + (-3) * (-2) + (-1) * 3 = 4$.

Hence $(2,1)$-entry of $AB - B^TA$ is $0 - 4 = -4$.

Example

Find the second entry of the first row of the matrix A that satisfies

$$A^T - \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

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Find the second entry of the first row of the matrix A that satisfies

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$$A^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ 5 & 10 \end{bmatrix}$$

Hence the second entry of the first row of the matrix A is 1.

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A system of m linear equations and n variables:

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The system of linear equations

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$$\begin{aligned} x - 2z &= 1 \\ y - 2t &= 0 \\ -x + 2z &= 0 \\ -y + 2t &= 1 \end{aligned}$$

This system has no solution. Hence, A is not invertible.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then* A *is invertible if* $det(A) = ad - bc \neq 0$ *and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

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Find $(2,1)$-entry of the matrix A such that

$$(2A^{-1}+I)^{-1}=\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

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Matrix inverse algorithm

If A is an invertible matrix, there exists a sequence of elementary row operations that

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Example

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-1/2R_2+R_3]{1/2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & -1/2 & -1 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow[3R_3+R_2]{-2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & -6 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

2.6 Linear transformation

Definition

Let A be $m \times n$ matrix. The transformation $T_A : \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$T_A(x) = Ax$$

is called the matrix transformation induced by A.

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Consider a transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^3 \text{ given by}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 3y \\ 5y \\ -x + 6y \end{bmatrix}.$$

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A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies the following two conditions for all vectors x and y in $\mathbb{R}^n$ and all scalars $a \in \mathbb{R}$:

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Example

$T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x, x + y)$ is a linear transformation. Indeed,

$$(1)\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = T\left(\begin{bmatrix} x + x' \\ y + y' \end{bmatrix}\right) = \begin{bmatrix} x + x' \\ x + x' + y + y' \end{bmatrix}$$

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Answer: Find a, b: $\begin{bmatrix} 0 \\ 5 \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Solve this system : $\begin{cases} a = 1 \\ b = 2 \end{cases}$.

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If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \ldots$:

$$T(a_1x_1 + a_2x_2 + \cdots + a_kx_k) = a_1T(x_1) + a_2T(x_2) + \cdots + a_kT(x_k),$$

for all scalars a_i and all vectors x_i in $\mathbb{R}^n$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and assume that

$$T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \text{ and } T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}. \text{ Find } T\begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

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Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a transformation. Then T is linear if and only if it is a matrix transformation induced by an $m \times n$ matrix A, given in terms of its columns by

$$A = \begin{bmatrix} T(e_1) & T(e_2) \cdots T(e_n) \end{bmatrix}$$

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Consider a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ x + 4y \end{bmatrix}$.

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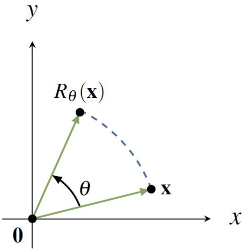
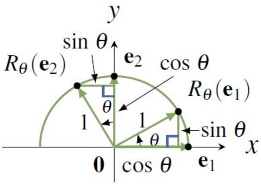
Rotation





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$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denote counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ.

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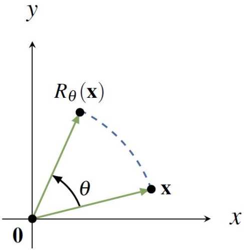

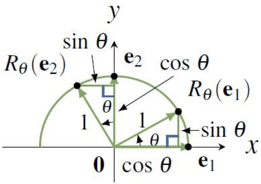

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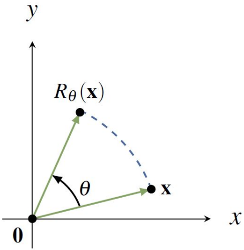

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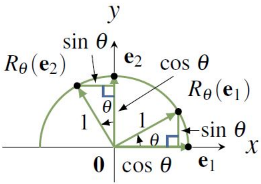

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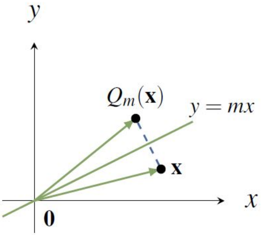

The line through the origin with slope m has equation $y = mx$, and we let $Q_m\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$.

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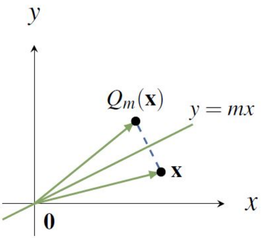

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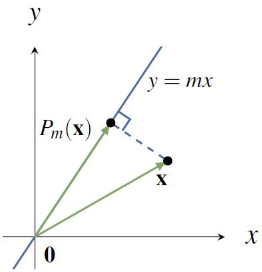

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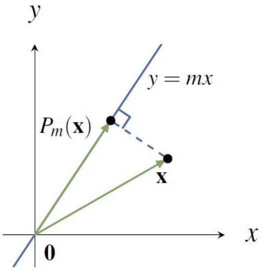

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Composition

Given two “linked” transformations

$$\begin{array}{ccccc} \mathbb{R}^k & \xrightarrow{T} & \mathbb{R}^n & \xrightarrow{S} & \mathbb{R}^m \\ x & \mapsto & T(x) & \mapsto & S[T(x)] \end{array}$$

The composite of S and T:

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Let $\mathbb{R}^k \xrightarrow[A_{n\times k}]{T} \mathbb{R}^n \xrightarrow[B_{m\times n}]{S} \mathbb{R}^m$ be linear transformations, and let A and B be the matrices of T and S respectively. Then $S \circ T$ is linear with matrix BA.

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Example

Let T and S be linear transformations given by:

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ -x + 2y \end{bmatrix}$$

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Answer:

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation induced by $A = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}\right]$. Find $(T \circ T)\left[\begin{array}{c} -1 \\ 2 \end{array}\right]$.

Answer: We have

$$\begin{aligned}(T \circ T)\begin{bmatrix}-1\\2\end{bmatrix} &= A^2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\\ &= \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-5\\2\end{bmatrix}.\end{aligned}$$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation induced by $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Find $(T \circ T)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Answer: We have

$$\begin{aligned} (T \circ T)\begin{bmatrix} -1 \\ 2 \end{bmatrix} &= A^2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}. \end{aligned}$$

Exercises

Section 2.1 : $1ad, 2aef, 3acd, 4, 14, 15, 19$ (page $45, 46$)

Section 2.3 : $1adef, 2, 3, 4, 6, 7, 12, 22$ (page $76 - 79$)

Section 2.4 : $1bd, 2aeg, 3acd, 4, 5, 6, 9, 12$ (page $91 - 93$)

Section 2.6 : $1, 3, 7, 8, 9, 12$ (page $115, 116$)