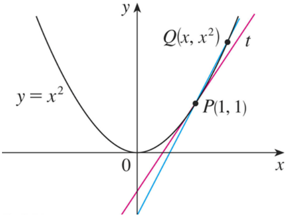
# LIMITS

Department of Mathematics, FPT University

Hanoi
2021

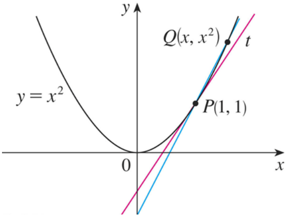
# Table of Contents

How to find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$?




We know that the slope of the secant line $PQ$ is $m_{PQ} = \dfrac{x^2 - 1}{x - 1}$.

How to find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$?




We know that the slope of the secant line $PQ$ is $m_{PQ} = \dfrac{x^2 - 1}{x - 1}$.

## Investigate the example of a falling ball

Suppose that a ball is dropped from upper observation deck of the CN Tower in Toronto, $450m$ above the ground. Find the velocity of the ball after 5 seconds.

If the distance fallen after $t$ seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation

$$s(t) = 4.9t^2.$$

## Investigate the example of a falling ball

Suppose that a ball is dropped from upper observation deck of the CN Tower in Toronto, $450m$ above the ground. Find the velocity of the ball after 5 seconds.

If the distance fallen after $t$ seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the following equation

$$s(t) = 4.9t^2.$$

$$average = \frac{change\ in\ position}{time\ elapsed}$$

$$= \frac{s(5.1) - s(5)}{0.1} = 49.49\ m/s$$

Thus, the (instantaneous) velocity after $5s$ is: $v = 49\ m/s$.

| Time interval | Average velocity (m/s) |
|---|---|
| $5 \leqslant t \leqslant 6$ | 53.9 |
| $5 \leqslant t \leqslant 5.1$ | 49.49 |
| $5 \leqslant t \leqslant 5.05$ | 49.245 |
| $5 \leqslant t \leqslant 5.01$ | 49.049 |
| $5 \leqslant t \leqslant 5.001$ | 49.0049 |

$$average = \frac{change\ in\ position}{time\ elapsed}$$
$$= \frac{s(5.1) - s(5)}{0.1} = 49.49\ m/s$$

Thus, the (instantaneous) velocity after $5s$ is: $v = 49\ m/s$.

| Time interval | Average velocity (m/s) |
|---|---|
| $5 \leqslant t \leqslant 6$ | 53.9 |
| $5 \leqslant t \leqslant 5.1$ | 49.49 |
| $5 \leqslant t \leqslant 5.05$ | 49.245 |
| $5 \leqslant t \leqslant 5.01$ | 49.049 |
| $5 \leqslant t \leqslant 5.001$ | 49.0049 |

$$average = \frac{change\ in\ position}{time\ elapsed}$$
$$= \frac{s(5.1) - s(5)}{0.1} = 49.49\ m/s$$

Thus, the (instantaneous) velocity after $5s$ is: $v = 49\ m/s$.

| Time interval | Average velocity (m/s) |
|---|---|
| $5 \leqslant t \leqslant 6$ | 53.9 |
| $5 \leqslant t \leqslant 5.1$ | 49.49 |
| $5 \leqslant t \leqslant 5.05$ | 49.245 |
| $5 \leqslant t \leqslant 5.01$ | 49.049 |
| $5 \leqslant t \leqslant 5.001$ | 49.0049 |

We begin by attempting to solve the are problem:
Find the area of the region $S$ that lies under the curve $y = f(x)$ from $a$ to $b$.

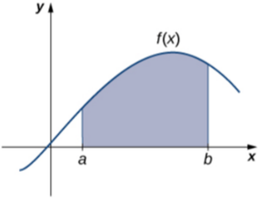



**Figure 2.8** The Area Problem: How do we find the area of the shaded region?

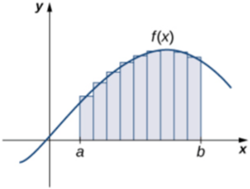
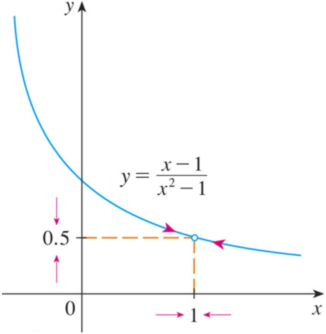


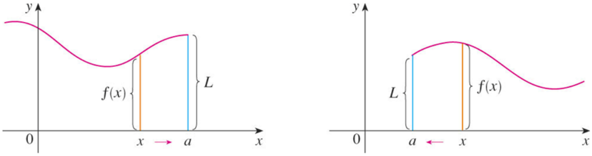

**Figure 2.9** The area of the region under the curve is approximated by summing the areas of thin rectangles.

# Table of Contents

In general,
we write $\boxed{\lim_{x \to a} f(x) = L}$ if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ to be sufficiently close to $a$ but not equal to $a$.



We write $\boxed{\lim_{x \to a^-} f(x) = L}$ if we can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ to be sufficiently close to $a$ and $x$ less than $a$.




(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

Similarly, "the right-hand limit of f(x) as x approaches a is equal to L" and we write $\boxed{\lim_{x\to a^+} f(x) = L}$

### Example

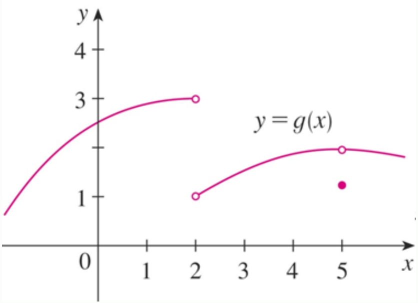
$\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$

$\lim_{x\to 2} g(x) = ?$

$\lim_{x\to 5^-} g(x) = ?$

$\lim_{x\to 5^+} g(x) = ?$

$\lim_{x\to 5} g(x) = ?$



Similarly, "the right-hand limit of f(x) as x approaches a is equal to L" and we write $\boxed{\lim_{x \to a^+} f(x) = L}$

## Example

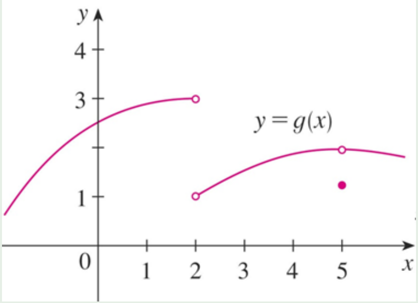
$$\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$$

$$\lim_{x \to 2} g(x) = ?$$

$$\lim_{x \to 5^-} g(x) = ?$$

$$\lim_{x \to 5^+} g(x) = ?$$

$$\lim_{x \to 5} g(x) = ?$$



Similarly, "the right-hand limit of f(x) as x approaches a is equal to L" and we write $\boxed{\lim_{x \to a^+} f(x) = L}$

**Example**

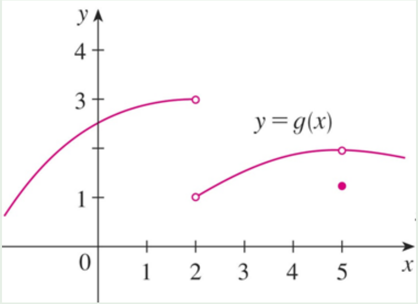
$$\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$$

$$\lim_{x \to 2} g(x) = ?$$

$$\lim_{x \to 5^-} g(x) = ?$$

$$\lim_{x \to 5^+} g(x) = ?$$

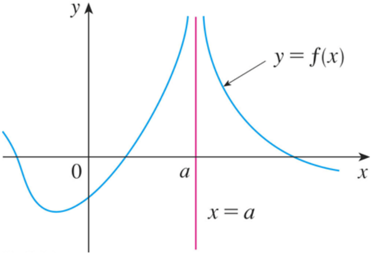
$$\lim_{x \to 5} g(x) = ?$$



Let $f$ be a function defined on both sides of $a$, except possibly at $a$ itself. Then,

$$\boxed{\lim_{x \to a} f(x) = \infty}$$

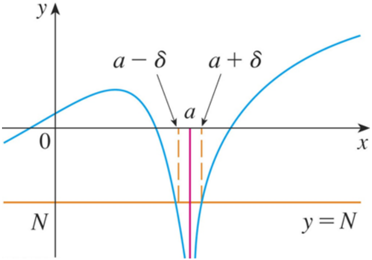
means that the values of $f(x)$
can be made arbitrarily large by taking
$x$ sufficiently close to $a$, but not equal to $a$.



Let f be defined on both
sides of a, except possibly at a itself. Then,

$$\boxed{\lim_{x \to a} f(x) = -\infty}$$

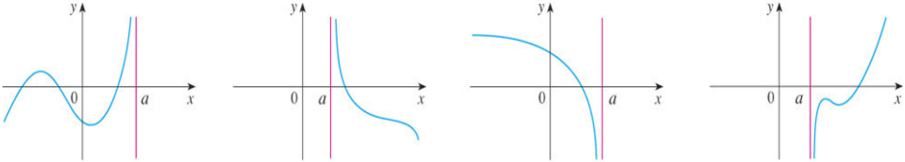
means that the values of $f(x)$ can
be made arbitrarily large negative by taking
$x$ sufficiently close to $a$, but not equal to $a$.



Similar definitions can be given for the one-sided limits:

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



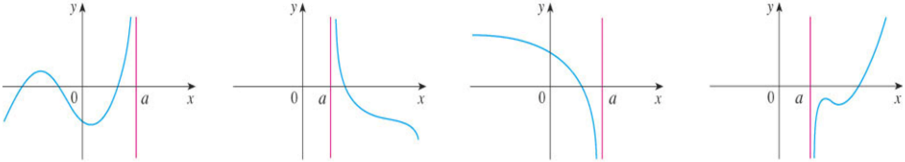

(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

# Infinite Limits

## Definition

$x = a$ is called ***the vertical asymptote*** of $f(x)$ if we have one of the following:

$$\boxed{\begin{array}{ll} \lim\limits_{x\to a^-} f(x) = \infty & \qquad \lim\limits_{x\to a^+} f(x) = \infty \\ \lim\limits_{x\to a^-} f(x) = -\infty & \qquad \lim\limits_{x\to a^+} f(x) = -\infty \end{array}}$$




(a) $\lim\limits_{x\to a^-} f(x) = \infty$ (b) $\lim\limits_{x\to a^+} f(x) = \infty$ (c) $\lim\limits_{x\to a^-} f(x) = -\infty$ (d) $\lim\limits_{x\to a^+} f(x) = -\infty$

# Table of Contents

# The Limit Laws

Suppose that $c$ is a constant and the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Then

1. $\lim\limits_{x\to a} \left[f(x) \pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a} \left[cf(x)\right] = c \lim\limits_{x\to a} f(x)$
4. $\lim\limits_{x\to a} \left[f(x) \cdot g(x)\right] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$
5. $\lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$

Using the limit laws, we have

6. $\lim\limits_{x\to a} \left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$
7. $\lim\limits_{x\to a} c = c$
8. $\lim\limits_{x\to a} x = a$
9. $\lim\limits_{x\to a} x^n = a^n$
10. $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

Suppose that $c$ is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}\left[f(x) \cdot g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Using the limit laws, we have

6. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

**Theorem**

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

## Theorem

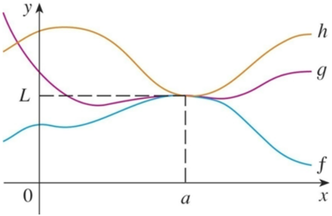
Let $f(x)$, $g(x)$ and $h(x)$ be defined for all $x \neq a$ over an open interval containing $a$. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing $a$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where $L$ is a real number, then $\lim_{x \to a} g(x) = L$.



## Example

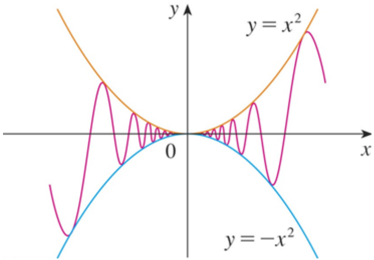
Show that $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

***Solution.***

Note that we cannot use $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = \lim\limits_{x\to 0} x^2 \cdot \lim\limits_{x\to 0} \sin\frac{1}{x} = 0$ since $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.

However, since $-1 \le \sin\frac{1}{x} \le 1$, we have

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$




Taking $f(x) = -x^2$ and $h(x) = x^2$ in the Squeeze Theorem, we obtain: $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

## Example

Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$

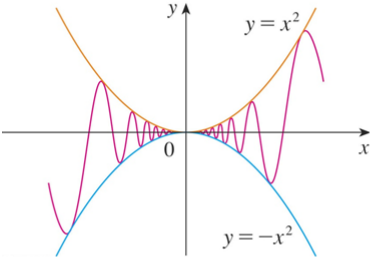
***Solution.***

Note that we cannot use $\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x} = 0$ since $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

However, since $-1 \le \sin\frac{1}{x} \le 1$, we have

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

Taking $f(x) = -x^2$ and $h(x) = x^2$ in the Squeeze Theorem, we obtain: $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$



**Example**

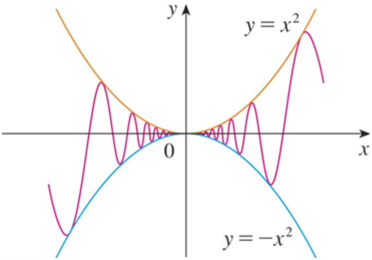
Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

***Solution.***

Note that we cannot use $\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x} = 0$ since $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

However, since $-1 \le \sin \frac{1}{x} \le 1$, we have

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$




Taking $f(x) = -x^2$ and $h(x) = x^2$ in the Squeeze Theorem, we obtain: $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

## Example

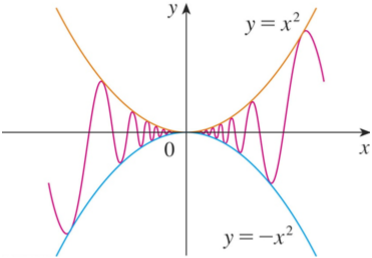
Show that $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0.$

***Solution.***

Note that we cannot use $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = \lim\limits_{x\to 0} x^2 \cdot \lim\limits_{x\to 0} \sin\dfrac{1}{x} = 0$ since $\lim\limits_{x\to 0} \sin\dfrac{1}{x}$ does not exist.

However, since $-1 \le \sin\dfrac{1}{x} \le 1$, we have

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$




Taking $f(x) = -x^2$ and $h(x) = x^2$ in the Squeeze Theorem, we obtain: $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0.$

## Quiz questions

Choose one correct answer (TRUE or FALSE) for the following statements.

1. If $\lim_{x\to 3} f(x) = 0$ and $\lim_{x\to 3} g(x) = 0$ then $\dfrac{\lim_{x\to 3} f(x)}{\lim_{x\to 3} g(x)}$ does not exist.

2. If $\lim_{x\to a} \left[f(x)g(x)\right]$ exists, then the limit must be $f(a)g(a)$.

## Quiz questions

Choose one correct answer (TRUE or FALSE) for the following statements.

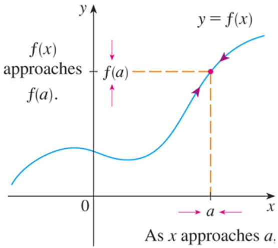
1. If $\lim\limits_{x\to 3} f(x) = 0$ and $\lim\limits_{x\to 3} g(x) = 0$ then $\dfrac{\lim\limits_{x\to 3} f(x)}{\lim\limits_{x\to 3} g(x)}$ does not exist.
2. If $\lim\limits_{x\to a} \left[f(x)g(x)\right]$ exists, then the limit must be $f(a)g(a)$.

## Table of Contents

## Definition

A function $f$ is ***continuous at a point*** $a$ if and only if the following three conditions are satisfied

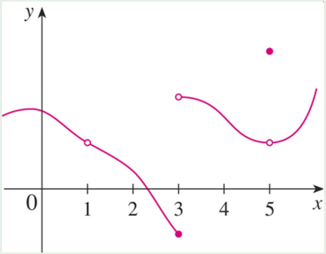
1. $f(a)$ is defined;
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.




A function is ***discontinuous at a point*** $a$ if it fails to be continuous at $a$.

## Example

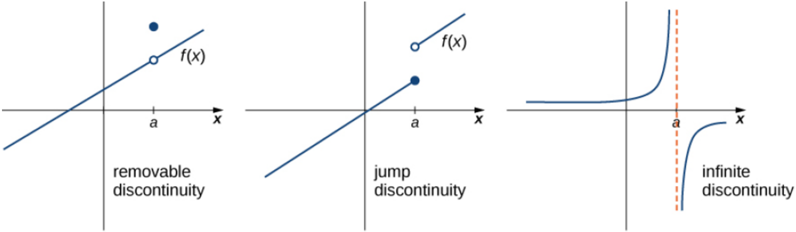
The figure shows the graph of a function $f$.
At which numbers is $f$ discontinuous? Why?



## Definition

If $f(x)$ is discontinuous at $a$, then

1. $f$ has a ***removable discontinuity*** at a if $\lim_{x\to a} f(x)$ exists. (Note: When we state that $\lim_{x\to a} f(x)$ exists, we mean that $\lim_{x\to a} f(x) = L$, where $L$ is a real number.)
2. $f$ has a ***jump discontinuity*** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. (Note: When we state that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. $f$ has an ***infinite discontinuity*** at $a$ if $\lim_{x\to a^-} f(x) = \pm\infty$ and or $\lim_{x\to a^+} f(x) = \pm\infty$.



## Example

Classify discontinuous points of the following functions

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$.
2. $g(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$.
3. $h(x) = \dfrac{x + 2}{x + 1}$.

**Example**

Classify discontinuous points of the following functions

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$.
2. $g(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$.
3. $h(x) = \dfrac{x + 2}{x + 1}$.

## Example

Classify discontinuous points of the following functions

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$.
2. $g(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$.
3. $h(x) = \dfrac{x + 2}{x + 1}$.

**Definition**

A function $f(x)$ is said to be ***continuous from the right*** at $a$ if $\lim_{x\to a^+} f(x) = f(a)$.
A function $f(x)$ is said to be ***continuous from the left*** at $a$ if $\lim_{x\to a^-} f(x) = f(a)$.

**Definition**

A function $f$ is ***continuous on an interval*** if it is continuous at every number in the interval.

If $f$ is defined only on one side of an endpoint of the interval, we understand "continuous at the endpoint" to mean "continuous from the right" or "continuous from the left."

## Definition

A function $f(x)$ is said to be ***continuous from the right*** at $a$ if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be ***continuous from the left*** at $a$ if $\lim_{x \to a^-} f(x) = f(a)$.

## Definition

A function $f$ is ***continuous on an interval*** if it is continuous at every number in the interval.

If $f$ is defined only on one side of an endpoint of the interval, we understand "continuous at the endpoint" to mean "continuous from the right" or "continuous from the left."

## Definition

A function $f(x)$ is said to be ***continuous from the right*** at $a$ if $\lim\limits_{x\to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be ***continuous from the left*** at $a$ if $\lim\limits_{x\to a^-} f(x) = f(a)$.

## Definition

A function $f$ is ***continuous on an interval*** if it is continuous at every number in the interval.

If $f$ is defined only on one side of an endpoint of the interval, we understand "continuous at the endpoint" to mean "continuous from the right" or "continuous from the left."

## Theorem

If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f - g$
3. $cf,\ cg$
4. $fg$
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

## Remarks

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions

## Theorem

If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f - g$
3. $cf$, $cg$
4. $fg$
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

## Remarks

The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions

## Composite Function Theorem

If $f(x)$ is continuous at $L$ and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f\big[g(x)\big] = f\Big(\lim_{x\to a} g(x)\Big) = f(L).$$

## Theorem

If $x$ is close to $a$, then $g(x)$ is close to $L$; and, since $f$ is continuous at $L$, if $g(x)$is close to $L$, then $f(g(x))$ is close to $f(L)$.

This theorem is often expressed informally by saying “a continuos function of a continuous function is a continuous function”.

## The Intermediate Value Theorem

Let $f$ be continuous over a closed, bounded interval $[a, b]$. If $z$ is any real number between $f(a)$ and $f(b)$, then there is a number $c$ in $[a, b]$ satisfying $f(c) = z$ in Figure 1.

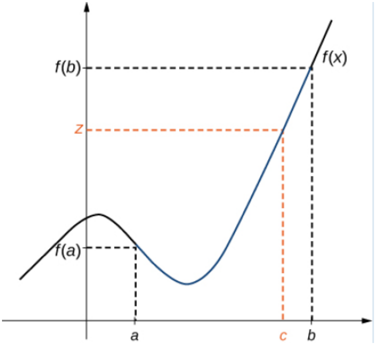



Figure 1:

**Example**

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

***Solution.***

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. This function is continuous over $[1, 2]$.

We are looking for a solution of the given equation that is, a number $c$ between 1 and 2 such that $f(c) = 0$.

We have $f(1) = -1 < 0$ and $f(2) = 12 > 0$.

Therefore, by Intermediate Value Theorem there exists a number $c \in [a, b]$ such that $f(c) = 0$.

**Example**

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

***Solution.***
Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. This function is continuous over $[1, 2]$.
We are looking for a solution of the given equation that is, a number $c$ between 1 and 2 such that $f(c) = 0$.
We have $f(1) = -1 < 0$ and $f(2) = 12 > 0$.
Therefore, by Intermediate Value Theorem there exists a number $c \in [a, b]$ such that $f(c) = 0$.

## Definition

Let $f$ be a function defined for every $x > a$. Then $\lim_{x \to \infty} f(x) = L$ means that

$$\forall \epsilon > 0,\ \exists M > 0 \text{ if } x > M \text{ then } |f(x) - L| < \epsilon.$$

## Definition

The line $y = L$ is called the ***horizontal asymptote*** of $f(x)$ if we have one of the following:

1. $\lim_{x \to +\infty} f(x) = L;$
2. $\lim_{x \to -\infty} f(x) = L.$

### Definition

Let $f$ be a function defined for every $x > a$. Then $\lim\limits_{x \to \infty} f(x) = L$ means that

$$\forall \epsilon > 0,\ \exists M > 0 \text{ if } x > M \text{ then } |f(x) - L| < \epsilon.$$

### Definition

The line $y = L$ is called the ***horizontal asymptote*** of $f(x)$ if we have one of the following:

1. $\lim\limits_{x \to +\infty} f(x) = L$;
2. $\lim\limits_{x \to -\infty} f(x) = L$.

1) If $f(1) > 0$ and $f(3) < 0$ then there exists a number $c$ between 1 and 3 such that $f(c) = 0$.

A. True

B. False

2) Which is the equation expressing the fact that "$f$ is continuous at 2"?

A. $\lim_{x\to 2} f(x) = 2$

B. $\lim_{x\to\infty} f(x) = f(2)$

C. $\lim_{x\to 2} f(x) = 0$

D. $\lim_{x\to 2} f(x) = f(2)$

3) Let $f(x) = \dfrac{x^3 - 1}{x^3 + x^2 - 2}$. The horizontal asymptote of $f(x)$ is

A. $y = 1$

B. $y = -1$

C. $y = 0$

D. None of them

4) $\lim_{x\to\infty} \cos x = ?$

A. Infinity

B. $-1$

C. $1$

D. Does not exist

# Quiz questions

1) If $f(1) > 0$ and $f(3) < 0$ then there exists a number $c$ between 1 and 3 such that $f(c) = 0$.

A. True

B. False

2) Which is the equation expressing the fact that "$f$ is continuous at 2"?

A. $\lim\limits_{x \to 2} f(x) = 2$

B. $\lim\limits_{x \to \infty} f(x) = f(2)$

C. $\lim\limits_{x \to 2} f(x) = 0$

D. $\lim\limits_{x \to 2} f(x) = f(2)$

3) Let $f(x) = \dfrac{x^3 - 1}{x^3 + x^2 - 2}$. The horizontal asymptote of $f(x)$ is

A. $y = 1$

B. $y = -1$

C. $y = 0$

D. None of them

4) $\lim\limits_{x \to \infty} \cos x = ?$

A. Infinity

B. $-1$

C. $1$

D. Does not exist

1) If $f(1) > 0$ and $f(3) < 0$ then there exists a number $c$ between 1 and 3 such that $f(c) = 0$.

A. True

B. False

2) Which is the equation expressing the fact that "$f$ is continuous at 2"?

A. $\lim_{x\to 2} f(x) = 2$

B. $\lim_{x\to\infty} f(x) = f(2)$

C. $\lim_{x\to 2} f(x) = 0$

D. $\lim_{x\to 2} f(x) = f(2)$

3) Let $f(x) = \dfrac{x^3 - 1}{x^3 + x^2 - 2}$. The horizontal asymptote of $f(x)$ is

A. $y = 1$

B. $y = -1$

C. $y = 0$

D. None of them

4) $\lim_{x\to\infty} \cos x = ?$

A. Infinity

B. $-1$

C. $1$

D. Does not exist

1) If $f(1) > 0$ and $f(3) < 0$ then there exists a number $c$ between 1 and 3 such that $f(c) = 0$.

A. True

B. False

2) Which is the equation expressing the fact that "$f$ is continuous at 2"?

A. $\lim_{x\to 2} f(x) = 2$

B. $\lim_{x\to\infty} f(x) = f(2)$

C. $\lim_{x\to 2} f(x) = 0$

D. $\lim_{x\to 2} f(x) = f(2)$

3) Let $f(x) = \dfrac{x^3 - 1}{x^3 + x^2 - 2}$. The horizontal asymptote of $f(x)$ is

A. $y = 1$

B. $y = -1$

C. $y = 0$

D. None of them

4) $\lim_{x\to\infty} \cos x = ?$

A. Infinity

B. $-1$

C. $1$

D. Does not exist

1) If $f(1) > 0$ and $f(3) < 0$ then there exists a number $c$ between 1 and 3 such that $f(c) = 0$.

A. True

B. False

2) Which is the equation expressing the fact that "$f$ is continuous at 2"?

A. $\lim\limits_{x \to 2} f(x) = 2$

B. $\lim\limits_{x \to \infty} f(x) = f(2)$

C. $\lim\limits_{x \to 2} f(x) = 0$

D. $\lim\limits_{x \to 2} f(x) = f(2)$

3) Let $f(x) = \dfrac{x^3 - 1}{x^3 + x^2 - 2}$. The horizontal asymptote of $f(x)$ is

A. $y = 1$

B. $y = -1$

C. $y = 0$

D. None of them

4) $\lim\limits_{x \to \infty} \cos x = ?$

A. Infinity

B. $-1$

C. 1

D. Does not exist

THANK YOU!