Advanced Mathematics 2 - Linear Algebra

Chapter 3: Determinants and Diagonalization

Department of Mathematics
The FPT university

Chapter 3 Introduction

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Topics:

3.1 The cofactor expansion

3.2 Determinants and matrix inverses

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$$= 1(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$$

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Since D^k is easy to compute, A^k is now computable.

Diagonalization Algorithm

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- In this case D is the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the diagonal, and P is the matrix whose columns are the basic eigenvectors.

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Linear Dynamical System

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Let $P = [X_1 \quad X_2 \cdots X_n]$ be an invertible matrix such that $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let $P^{-1}V_0 = [b_1 \quad b_1 \cdots b_n]^T$. Then

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Therefore $x_k = \frac{1}{5}\left(3^{k+1} - (-2)^{k+1}\right)$.