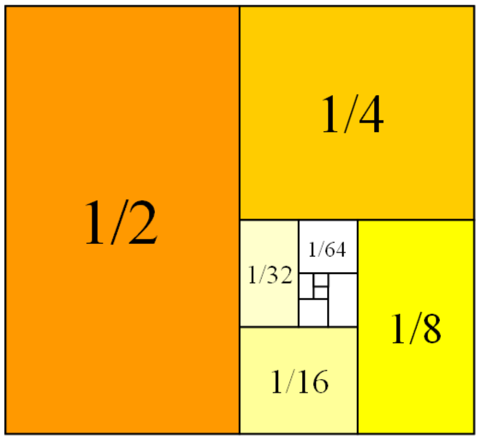


Chapter 3: DERIVATIVES

Department of Mathematics, FPT University

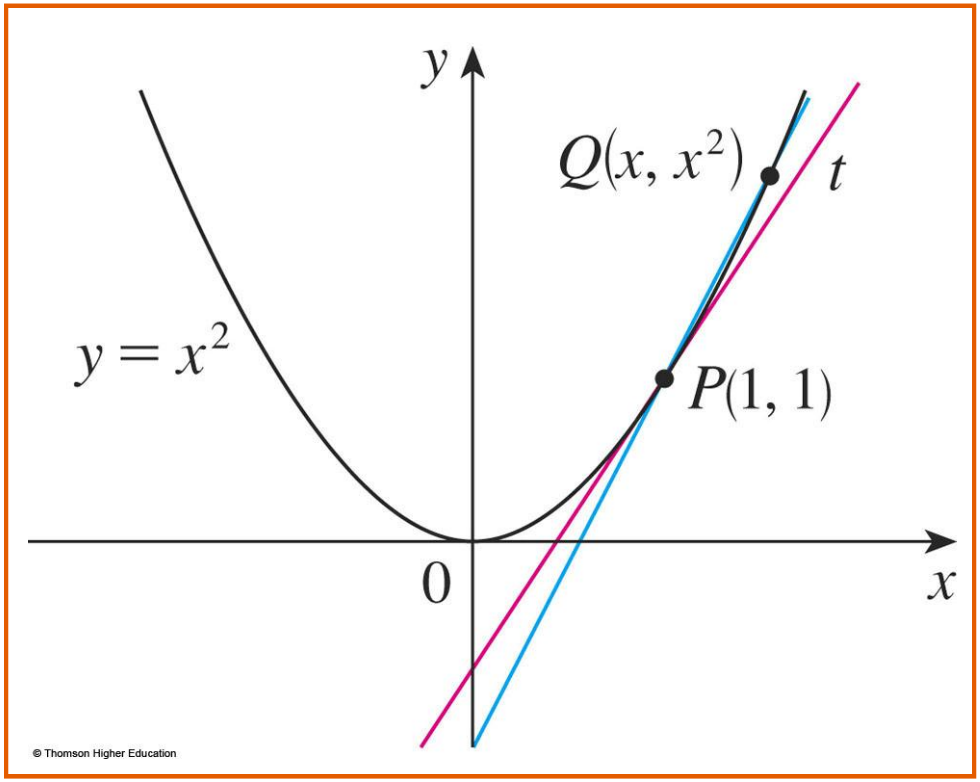
3.1

Defining the Derivatives

THE TANGENT PROBLEM

The slope of the tangent line is said to be the limit of the slopes of the secant lines.

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$



TANGENTS

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P **with slope**

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

THE VELOCITY PROBLEM

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

$$s(t) = \frac{1}{2} g t^2$$

The instantaneous velocity **at time *t***

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

The **derivative** of a function *f* at a number *a*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- A function f(x) is said to be differentiable at a if f ′(a) exists.
- f is differentiable on D (open set) if it is differentiable at every point in D.

3.2

The Derivative as a Function

Derivative of f at a: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Replace a by a variable x, we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For each x, $f'(x)$ is defined. Hence, f' is a function.

When $y = f(x)$ then $f'(x)$ can be written as

$$\frac{dy}{dx}; \quad y'; \quad \frac{d}{dx}(f(x))$$

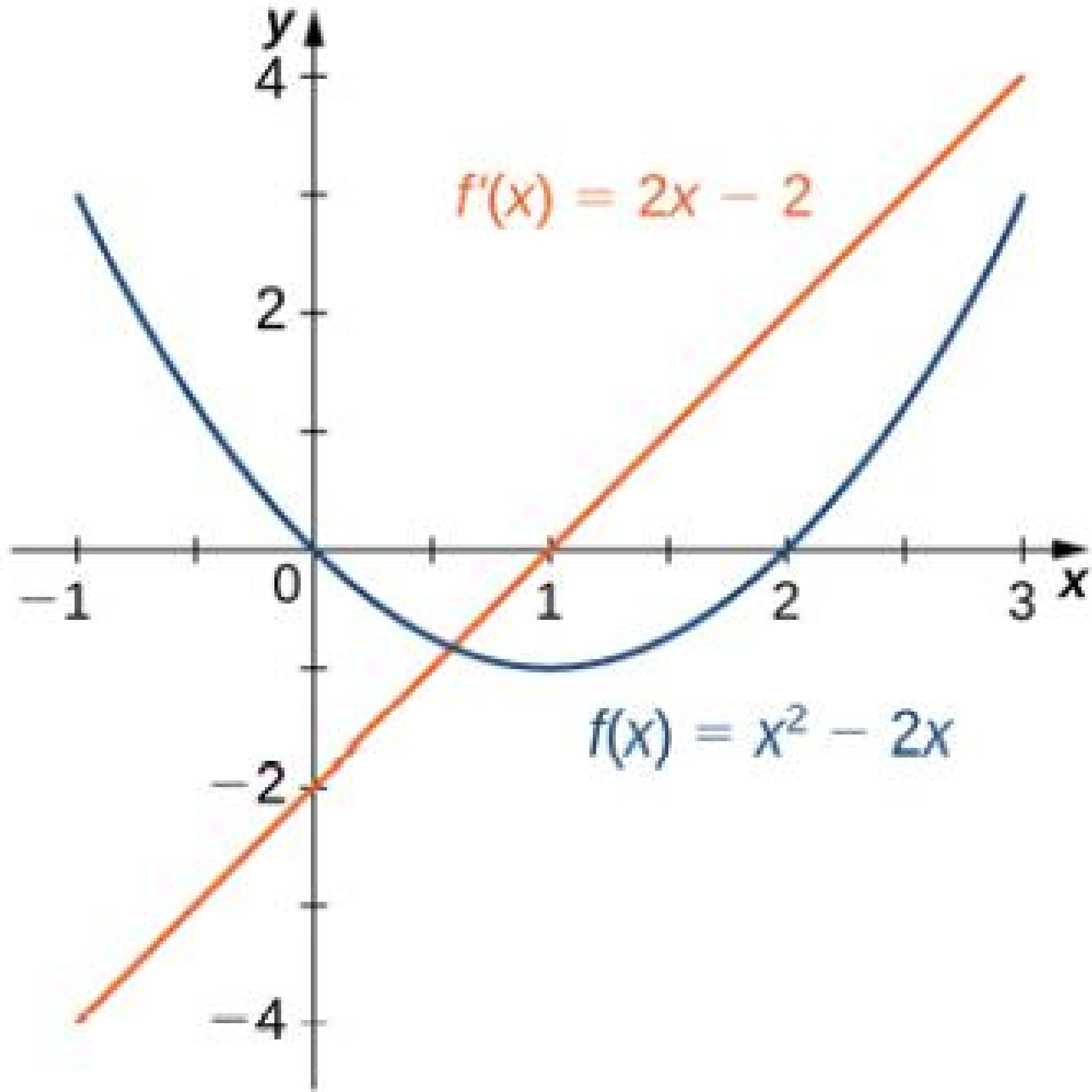
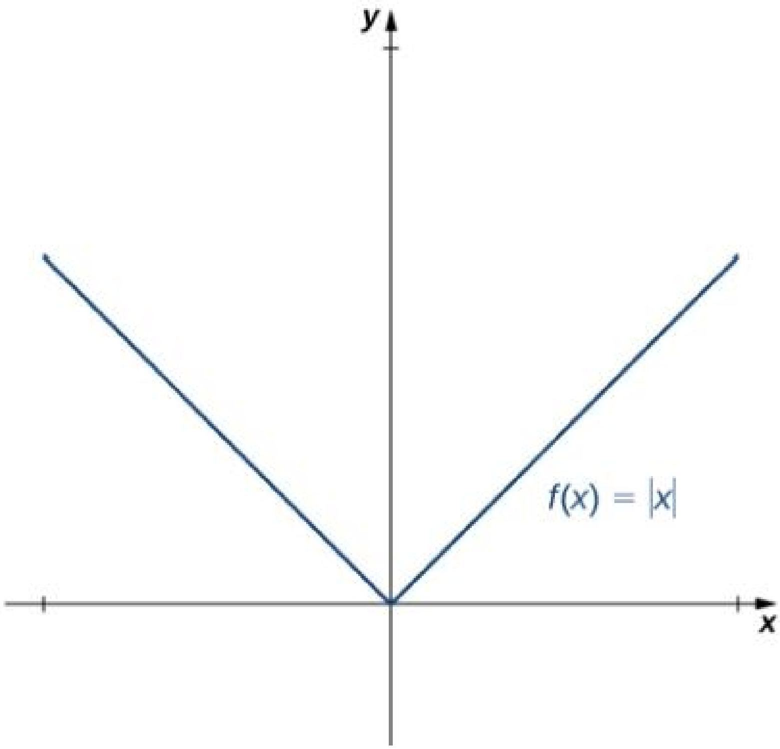


Figure 3.13 The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Differentiability $\Rightarrow$ Continuity

f is differentiable at a, then f is continuous at a.

Continuity $\not\Rightarrow$ Differentiability



HIGHER DERIVATIVES

If f is a differentiable function, its derivative f' is also a function.

So, f' may have a derivative of its own, denoted by $\left(f'\right)' = f''$

(the second derivative of f)

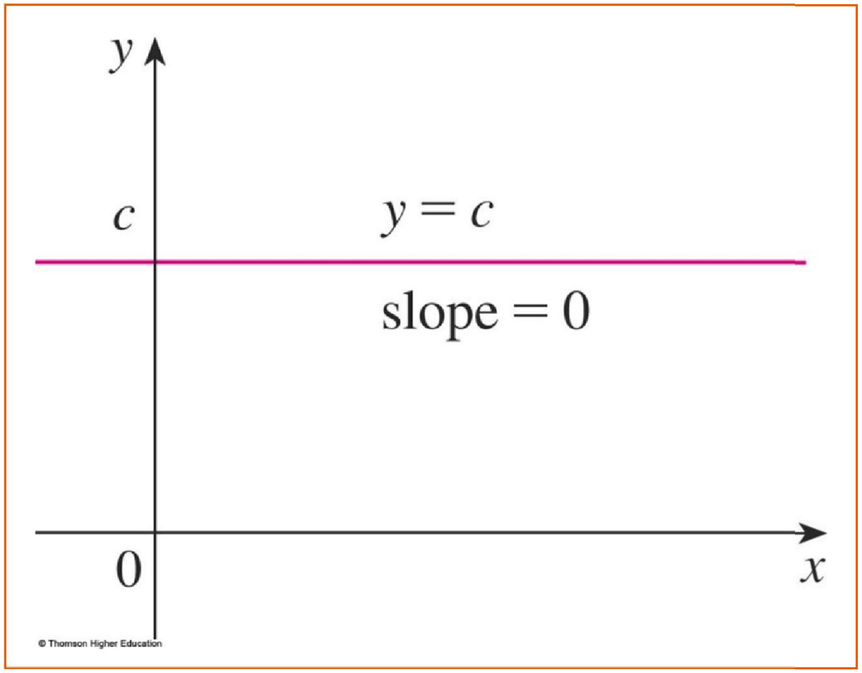
The process can be continued, resulting in $f''', \ldots, f^{(n)}$.

When $y = f(x)$, then $f^{(n)}$ can be written as $y^{(n)}$ or $\dfrac{d^n y}{dx^n}$

3.3 DIFFERENTIATION RULES

- If f is a constant function, i.e. $f(x) = c \ \forall \ x$,

  then $f'(x) = \frac{d}{dx}(c) = 0$



- If $f(x) = x^n \ (n \in \mathbb{Z})$, then

  $f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}$

Find the derivative of $f(x) = x^7; g(x) = \frac{1}{x^5}$

3.3 DIFFERENTIATION RULES

$$(f+g)' = f' + g' \qquad (f-g)' = f' - g' \qquad (cf)' = c\,f'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2}$$

a) Find the derivative of $f(x) = 2x^5 - 7x + 5$; $k(x) = \frac{5x-1}{4x+3}$

b) Find the equation of the line tangent to the graph of $f(x) = x^4 - 2x - 1$ at $x = 1$. Use the point-slope form.

c) Find the values of x for which the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line y = 2x + 3.

3.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Find the derivative of

- $f(x) = \sin x \,.\, \cos x$
- $g(x) = \dfrac{x}{\cos x}$
- $h(x) = 2\tan x - 3\cot x$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

3.9 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
3. $\frac{d}{dx}(b^x) = b^x \ln b$
4. $\frac{d}{dx}(log_b x) = \frac{1}{x\ln b}$

TANGENT AND NORMAL LINES

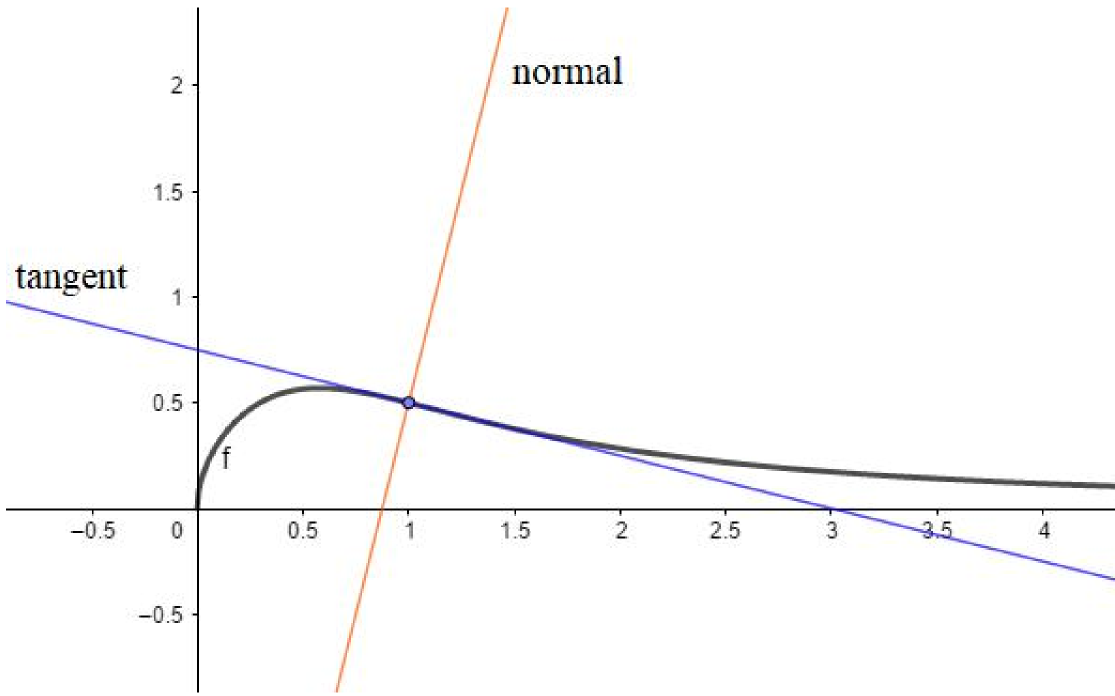
Example: Find equations of the **tangent line** and **normal line** to the curve

$$y = \sqrt{x}/(1+x^2)$$

at the point (1, ½).

Tangent line:

$$y = -\frac{1}{4}x + \frac{3}{4}$$



3.4

Derivatives as Rates of change

DERIVATIVES

Definition

Let $s(t)$ be a function giving the position of an object at time t.

The velocity of the object at time t is given by $v(t) = s'(t)$.

The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

The position of a particle moving along a coordinate axis is given by

$$s(t) = t^3 - 9t^2 + 24t + 4, t \geq 0.$$

a. Find $v(t)$

b. At what time(s) is the particle at rest?

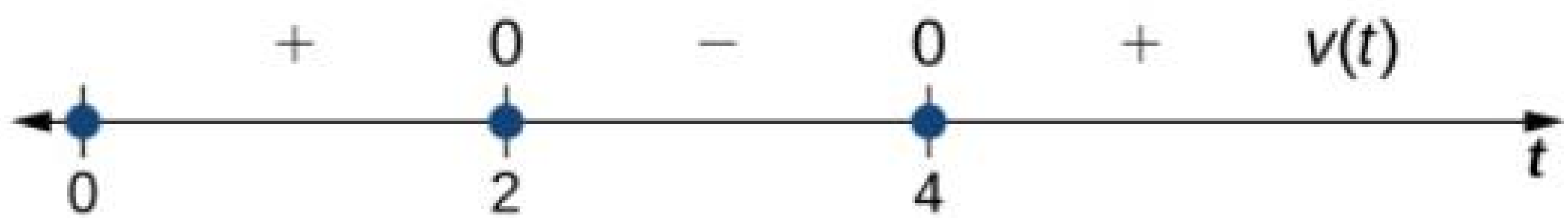
c. On what time intervals is the particle moving from left to right? From right to left?

Solution

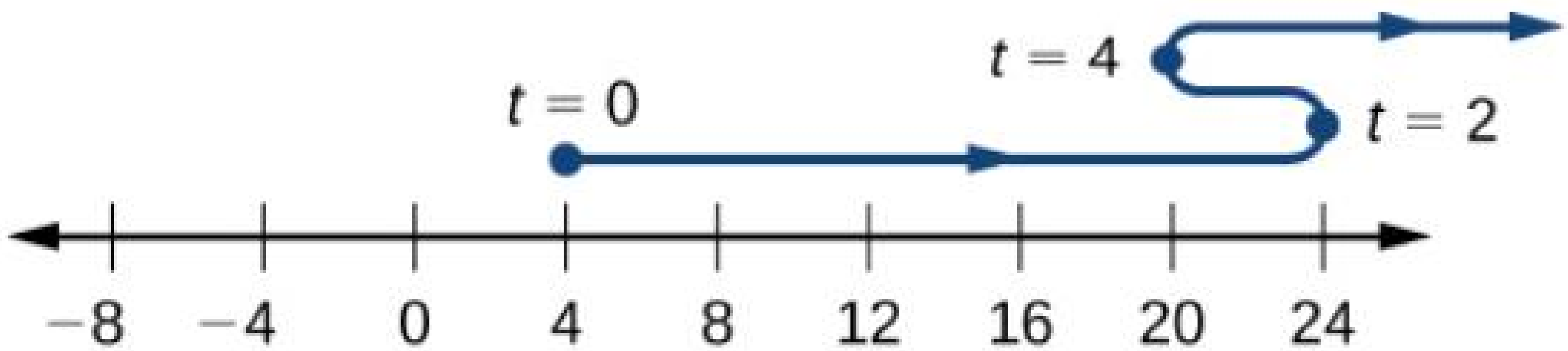
a. $v(t) = s'(t) = 3t^2 - 18t + 24$

b. Take rest when $v(t) = 0 \Leftrightarrow t = 2 \; or \; t = 4.$

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$



s(0) = 4, s(2) = 24, and s(4) = 20



Population change

The population of a city is tripling (gấp 3 lần) every 5 years. If its current population is 10000, what will be its approximate population 2 years from now?

Let $P(t)$ be the population (in thousands) t years from now.

Then $P(0) = 10$ and $P(5) = 30$

Estimate $P'(0) \approx \frac{P(5)-P(0)}{5-0} = \frac{30-10}{5} = 4$

Estimate $P(2) \approx P(0) + 2.P'(0) = 10 + 2.4 = 18$

In 2 years, the population will be 18000.

RATES OF CHANGE

Let $D(t)$ be the US national debt at time t. The table gives approximate values of this function by providing end-of-year estimates, in billions of dollars, from 1980 to 2000.

Interpret and estimate the value of $D'(1990)$.

The derivative $D'(1990)$ means the rate of change of D with respect to t when t =1990, that is, the rate of increase of the national debt in 1990.

t	$D(t)$
1980	930.2
1985	1945.9
1990	3233.3
1995	4974.0
2000	5674.2

RATES OF CHANGE

$$D'(1990) = \lim_{t \to 1990} \frac{D(t) - D(1990)}{t - 1990}$$

So, we compute values of the difference quotient as follows.

t	$\dfrac{D(t) - D(1990)}{t - 1990}$
1980	230.31
1985	257.48
1995	348.14
2000	244.09

t	$D(t)$
1980	930.2
1985	1945.9
1990	3233.3
1995	4974.0
2000	5674.2

RATES OF CHANGE

Example

We estimate that the rate of increase of the national debt in 1990 was the **average of these two numbers,**

namely $D'(1990) \approx 303$ billion dollars per year.

t	$\dfrac{D(t) - D(1990)}{t - 1990}$
1980	230.31
1985	257.48
1995	348.14
2000	244.09

3.6

The chain rule

THE CHAIN RULE

If g is differentiable at x and f is differentiable at $g(x)$, the composite function $F = f \circ g$ is differentiable at x and F' is given by the product:

$$\boxed{(f \circ g)'(x) = f'(g(x)) \bullet g'(x)}$$

Note: If $h(x) = ((g(x))^n$, then

$$h'(x) = n\big(g(x)\big)^{n-1} . g'(x)$$

Let f(x)=g(sin3x). Find f' in terms of g'.	
A	3cos3xg'(x)
B	3cos3xg'(sin3x)
C	cos3xg'(sin3x)

Suppose $h(x)=f(g(x))$ and $f(2)=3$, $g(2)=1$, $g'(2)=1$, $f'(2)=2$, $f'(1)=5$. Find $h'(2)$.	
A	1
B	2
C	3
D	4
E	5

a) Simlify: $\frac{f(x+h)-f(x)}{h}$ $\quad for \quad f(x) = -3x^2$

b) $y = x^7\, f(x) \rightarrow y' =?$

c) $z^3 = x^2 + 6y^2 - 6, \quad x = 3, \quad y = 2, \quad \frac{dx}{dt} = 4, \frac{dy}{dt} = 3$

$\rightarrow \frac{dz}{dt} =?$

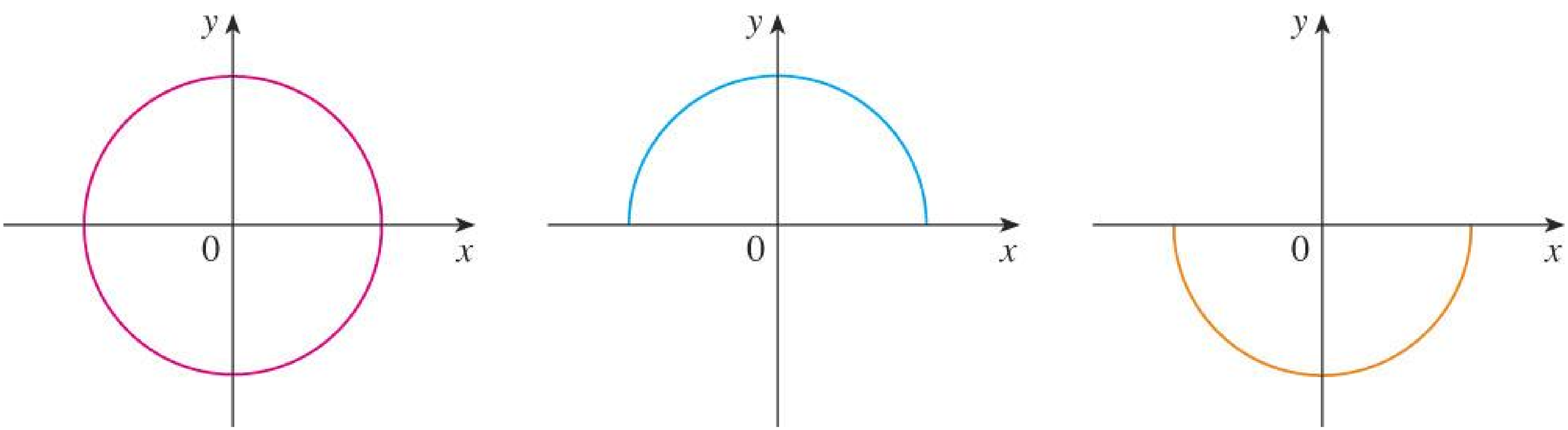
3.6

Implicit Differentiation

Example (Implicit function):

The graphs of *f(x) (figure b)* and *g(x) (figure c)* are the upper and lower semicircles of the circle

$$x^2 + y^2 = 25.$$



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

IMPLICIT DIFFERENTIATION METHOD

It is not always easy to find the formula of a function which is defined implicitly by an equation.

For example, $x^4 + 3x^2 - 3xy = 1$,

$$x^5y - y^3(x+1) + 2yx + y^2 = 5, \ldots$$

$\Rightarrow$ Use the implicit differentiation method to find the derivative of an implicitly defined function.

Example

a. If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$

b. Find an equation of the tangent to the circle

$x^2 + y^2 = 25$ at the point (3, 4).

SOLUTION

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Since y is a function of x and using the Chain Rule, we obtain

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0$$

Then

$$\frac{dy}{dx} = -\frac{x}{y}$$

At the point (3, 4) we have $x = 3$ and $y = 4$.

So,

$$\frac{dy}{dx} = -\frac{3}{4}$$

Thus, the equation of the tangent to the circle at (3, 4)

$$y - 4 = -\frac{3}{4}(x - 3)$$

or

$$3x + 4y = 25$$

IMPLICIT DIFFERENTIATION

Exercise: Find y" if $x^4 + y^4 = 16$.

Solution: Differentiating the equation implicitly with respect to *x*, we get

$$4x^3 + 4y^3y' = 0$$

Hence,

$$y' = -\frac{x^3}{y^3}$$

To find *y*, we differentiate this expression for *y*' using the Quotient Rule and remembering that *y* is a function of *x*.

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3(d/dx)(x^3) - x^3(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2 y')}{y^6}$$

If we now substitute $y' = -\dfrac{x^3}{y^3}$ into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

However, the values of x and y must satisfy the original equation $x^4 + y^4 = 16$.

So, the answer simplifies to:

$$y'' = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}$$

Exercises:

- 197-201 (p.285)
- 301-305, 317, 322 (p.317)
- 331-345 (p.331)