

# Advanced Mathematics 2 - Linear Algebra

## Chapter 4: Vector Geometry

Department of Mathematics  
The FPT university

# Chapter 4 Introduction

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**Topics:**

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## Topics:

### 4.1 Vectors and lines

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4.2 Projections and Planes

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- 4.2 Projections and Planes
- 4.3 The cross product
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- 4.5 An applications to computer graphics



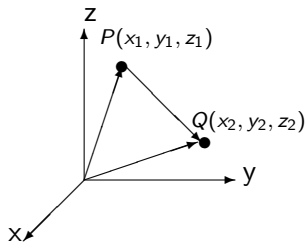
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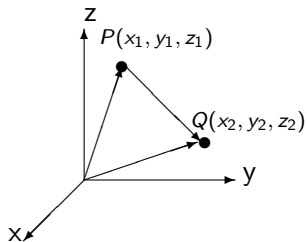
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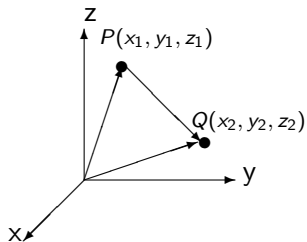
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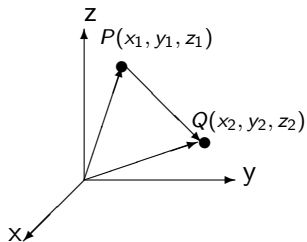


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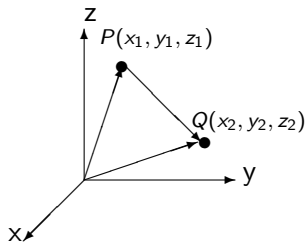
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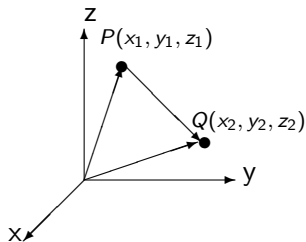
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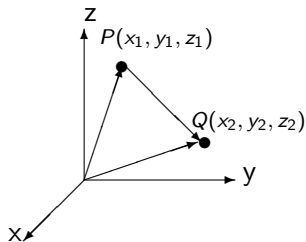
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The parallelogram law:

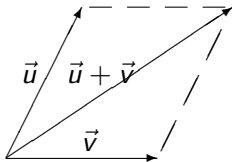
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Let  $\vec{u} = [u_1 \ u_2 \ u_3]^T$ ,  $\vec{v} = [v_1 \ v_2 \ v_3]^T$ . The **dot product** of  $\vec{u}$  and  $\vec{v}$ , is

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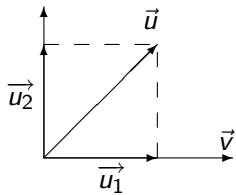
### Properties

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$
- Let  $\theta$  be the angle between  $\vec{u}$  and  $\vec{v}$ , then  $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$
- $\vec{u}$  and  $\vec{v}$  are orthogonal if and only if  $\vec{u} \cdot \vec{v} = 0$

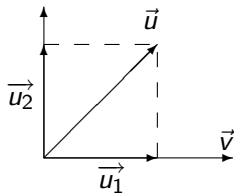
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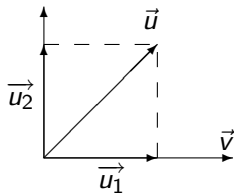
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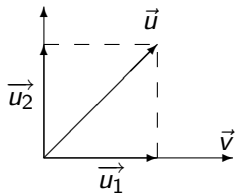
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**Example.** Find the projection of the vector  $\vec{u} = [1 \ 2 \ 3]^T$  on the vector  $\vec{v} = [1 \ -1 \ 0]^T$ . Also find the orthogonal component.

# Equation of planes

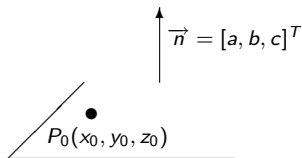
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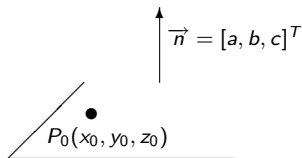
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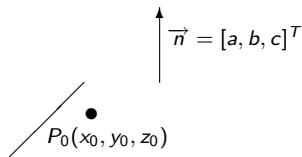
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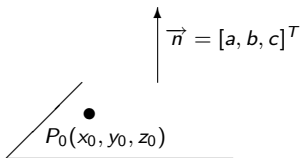


Equation

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**Example.** Write an equation for the plane passing through  $P(1, 2, 1)$  and perpendicular to the line

$$x = 2 + t, y = -1 + 3t, z = -2 - t.$$



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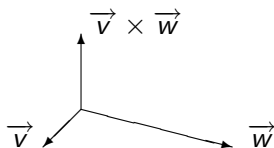
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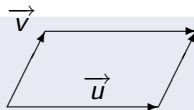
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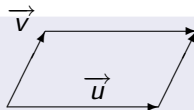
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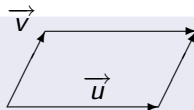
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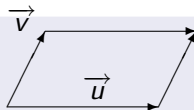
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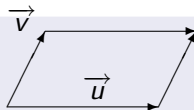
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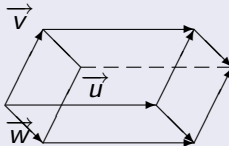
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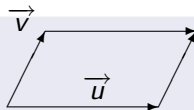


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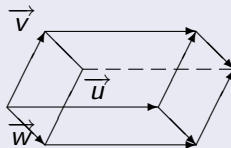


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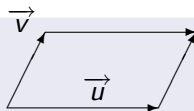


**Note.**

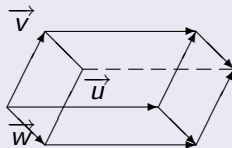


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**Note.** If  $\vec{u} = [x_0 \ y_0 \ z_0]^T$ ,  $\vec{v} = [x_1 \ y_1 \ z_1]^T$ ,  $\vec{w} = [x_2 \ y_2 \ z_2]^T$  then the volume of the parallelepiped determined by  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  is the *absolute*

*value* of the determinant of the matrix 
$$\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$$

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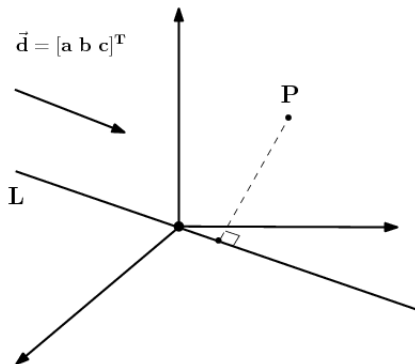
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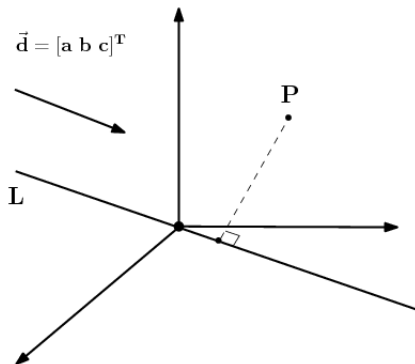


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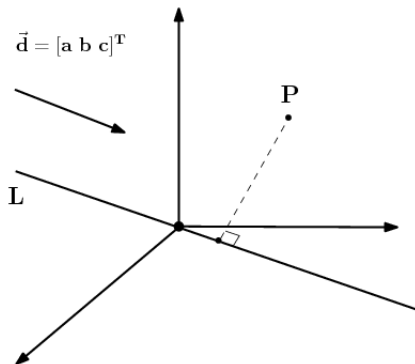
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- The matrix of projection on  $L$  is

$$\frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

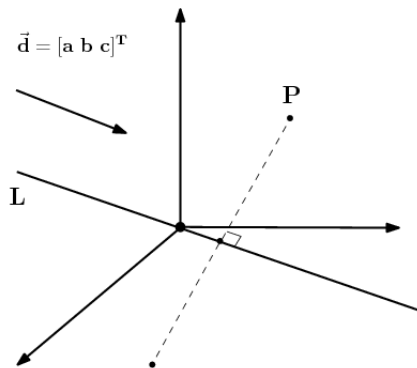
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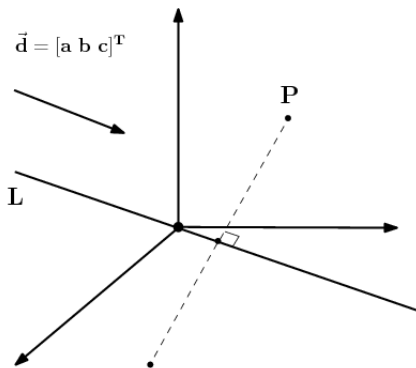


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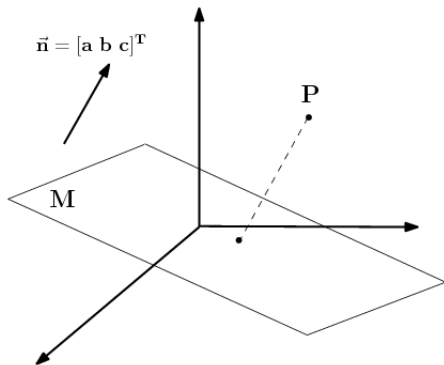


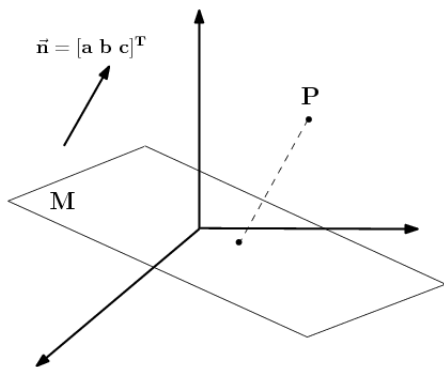
- The matrix of reflection in  $L$  is

$$\frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} a^2 - b^2 - c^2 & 2ab & 2ac \\ 2ab & b^2 - a^2 - c^2 & 2bc \\ 2ac & 2bc & c^2 - a^2 - b^2 \end{bmatrix}$$





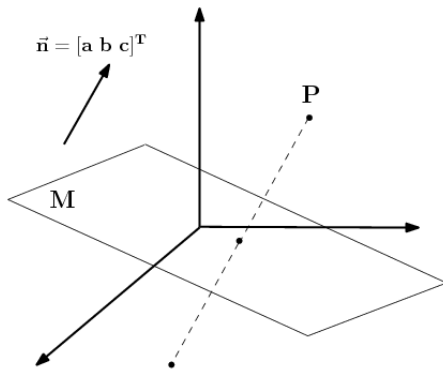


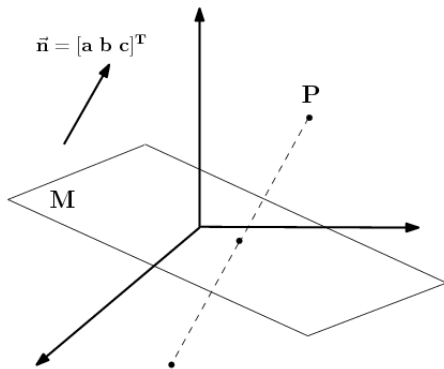


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$$\frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix}$$







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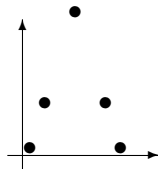
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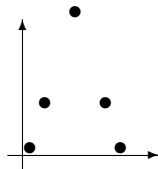
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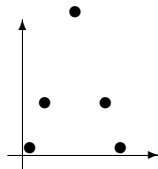


To rotate the letter  $A$  about the origin through  $\pi/6$ , we can multiply the data matrix  $D$  by the matrix  $R_{\pi/6} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$ . This gives

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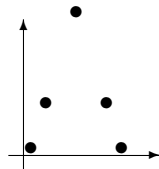
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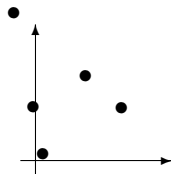
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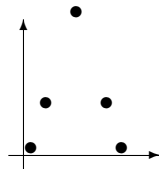
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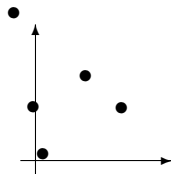
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To make this letter narrower, we can multiply the matrix above with the matrix  $\begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}$  to shrink the  $X$ -coordinates by 0.8. The resulting data is

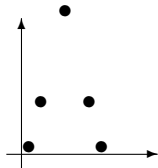
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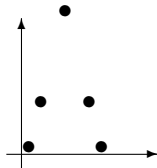
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## Practice problems

- Slant the letter to the right
- Turn the letter upside down
- Rotate the letter through  $\pi/6$  about the point  $(1, 2)$ .