

Advanced Mathematics 2 - Linear Algebra

Chapter 3: Determinants and Diagonalization

Department of Mathematics
The FPT university

Chapter 3 Introduction

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Topics:

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3.1 The cofactor expansion

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3.2 Determinants and matrix inverses

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- 3.1 The cofactor expansion
- 3.2 Determinants and matrix inverses
- 3.3 Diagonalization and Eigenvalues

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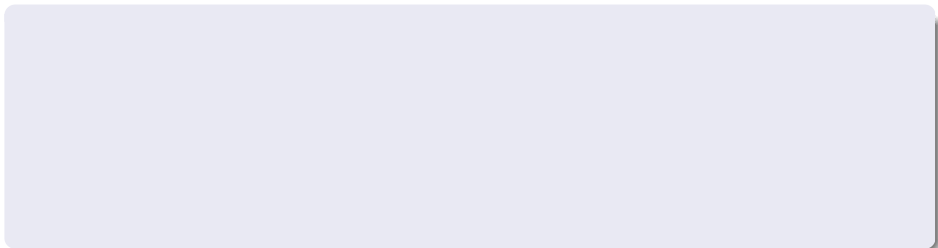
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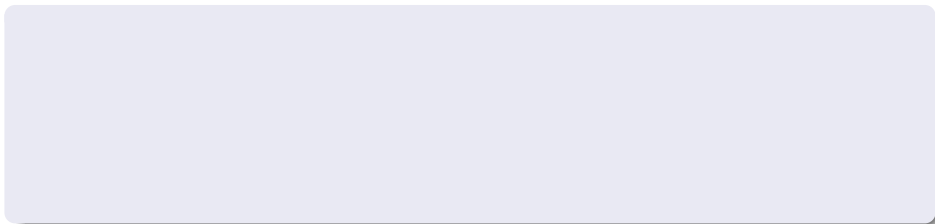
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$$c_{23}(A) = (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = 6$$

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$$= 1(-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$$

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Determinants and row operations

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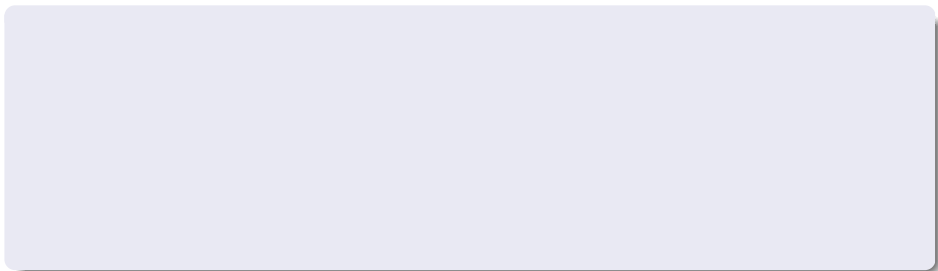
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Example. Find eigenvalues and eigenvectors of the matrices

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Since D^k is easy to compute, A^k is now computable.

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- In this case D is the diagonal matrix with $\lambda_1, \lambda_2, \dots, \lambda_n$ in the diagonal, and P is the matrix whose columns are the basic eigenvectors.

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For $\lambda = 1$, there is only one basic eigenvector $X = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, therefore A is not diagonalizable.

Linear Dynamical System

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A sequence of columns V_0, V_1, \dots , is called a **linear dynamical system** if

$$V_{k+1} = AV_k$$

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Theorem

Let $P = [X_1 \ X_2 \ \cdots \ X_n]$ be an invertible matrix such that $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Let $P^{-1}V_0 = [b_1 \ b_1 \ \cdots \ b_n]^T$. Then

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Therefore $x_k = \frac{1}{5} (3^{k+1} - (-2)^{k+1})$.