Chapter 3. Determinants and Diagonalization

3.1 The Cofactor Expansion

Let $A = [a_{ij}]_{n \times n}$ be a square matrix.

Definition

- A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.
- (i, j)-cofactor of A

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$$= a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
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Sarrus rule

Example

We have

$$\det\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & 2\\ -2 & 1 & -1\end{bmatrix} = \left|\begin{array}{ccc|cc}1 & -1 & 2 & 1 & -1\\ 3 & 0 & 2 & 3 & 0\\ -2 & 1 & -1 & -2 & 1\end{array}\right.$$

$$=1.0.(-1)+(-1).2.(-2)+2.3.1-2.0.(-2)-1.2.1-(-1).3.(-1)=5.$$

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Find the determinant of $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & 1 & 0 & 5 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & -2 & 0 \end{bmatrix}$.

Answer:

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Answer: Expanse along the fourth column of A:

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Answer: Expanse along the fourth column of A:

$$\begin{aligned} |A| &= 5.(-1)^{2+4} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 3 & 0 & -2 \end{vmatrix} \\ &\overset{\text{expanse along the second column}}{=} 5.3.(-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \\ &= -15.[2.(-2) - 3.1] = 105 \ . \end{aligned}$$

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Let $A = \begin{bmatrix} 4 & -5 & 7 \\ 3 & 6 & -2 \\ 1 & 8 & -9 \end{bmatrix}$. Find $(2,3)$-cofactor of A.

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$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 5 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} - \begin{vmatrix} 3 & 4 & 5 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} \overset{1/2R_2}{=} -2 \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} \overset{-R_2+R_3}{=} -2 \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

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$$\begin{vmatrix} p+x & q+y & r+z \\ 3a-2x & 3b-2y & 3c-2z \\ 3x & 3y & 3z \end{vmatrix} \overset{1/3R_3}{=} 3 \begin{vmatrix} p+x & q+y & r+z \\ 3a-2x & 3b-2y & 3c-2z \\ x & y & z \end{vmatrix}$$

$$\overset{-R_3+R_1, 2R_3+R_2}{=} 3 \begin{vmatrix} p & q & r \\ 3a & 3b & 3c \\ x & y & z \end{vmatrix} \overset{1/3R_2}{=} 9 \begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{=} -9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -9 * 5 = -45.$$

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Let $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 10$. Find $\begin{vmatrix} x & 0 & y & z \\ p & 0 & q & r \\ a & 0 & b & c \\ 2 & 3 & 4 & 5 \end{vmatrix}$.

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Answer: Expanse the determinant along the second column,

$$\begin{vmatrix} x & 0 & y & z \\ p & 0 & q & r \\ a & 0 & b & c \\ 2 & 3 & 4 & 5 \end{vmatrix} = 3 * (-1)^{4+2} \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} -3 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -30$$

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Properties of Determinant

- $\det(AB) = \det(A)\det(B)$. In particular $det(A^m) = [det(A)]^m$, for $m \in \mathbb{N}^\star$.
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$$\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = 9 \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad \begin{vmatrix} 2 & 4 & -8 & 9 \\ 0 & -1 & 5 & 6 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -5 \end{vmatrix} = 2 * (-1) * 4 * (-5) = 40$$

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B is the matrix obtained from A by interchanging R_3 and R_2.
C is the matrix obtained from B by adding 12 times R_3 to R_2.
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Find $det(BCD)$.

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Definition

The adjugate of A, denoted $\mathrm{adj}(A)$:

$$\mathrm{adj}(A) = \left[c_{ij}\right]^T = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \dots & \dots & \ddots & \dots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

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$$\text{adj}(A) = \left[c_{ij}\right]^T = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \ldots & \ldots & \ddots & \ldots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

where c_{ij} is (i, j)-cofactor of A.

Let A be a square $n \times n$ matrix.

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Matrix A is invertible if and only if $\det(A) \neq 0$, *and*

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if $det(A) = ad - bc \neq 0$ and

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Find the inverse matrix of $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$.

Answer: We have

$$\det(A) = 2 * (-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} + 1 * (-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 1 & 5 \end{vmatrix} = 2 * 1 + (-8) = -6.$$

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Crammer's rule

Theorem

If A is an invertible $n \times n$ matrix, the solution to the system

$$Ax = b$$

of n equations in the variables $x_1, x_2, \ldots, x_n$ is given by

$$x_1 = \frac{det\ A_1}{det\ A}$$
$$x_2 = \frac{det\ A_2}{det\ A}$$
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where A_k is the matrix obtained from A by replacing column k by b.

Example

Find y satisfying the following system:

$$\begin{array}{rcrcrcr} 4x & - & y & + & 3z & = & 1 \\ 6x & + & 2y & - & z & = & 0 \\ 3x & + & 3y & + & 2z & = & -1 \end{array}$$

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A diagonal matrix is $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$

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Example

Find all eigenvectors of $A = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$.

Answer: As above example, A has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$.

- -1-eigenvectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is nonzero solution of the system

$$\begin{bmatrix} -2 & -8 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 + 4x_2 = 0 \Rightarrow x = \begin{bmatrix} 4t \\ -t \end{bmatrix} = t \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ t \neq 0.$$

- 5-eigenvectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is nonzero solution of the system

$$\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 - 2x_2 = 0 \Rightarrow X = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ t \neq 0.$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has eigenvectors $x_1, x_2, ..., x_n$ such that the matrix $P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ is invertible.

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Let A be a 2×2 matrix with eigenvalues 2 and 3, and with corresponding eigenvectors $[2\ 1]^T$ and $[5\ 3]^T$. Find the $(2,1)$-entry of A.

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$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 15 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Hence $(2,1)$-entry of A is $2 * 3 + 9 * (-1) = -3$.

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Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$. Given that $\lambda = -1$ is an eigenvalue of A. Which of the following are basic eigenvectors corresponding to λ?

Answer:

$$(\lambda I_3 - A)x = 0 \Leftrightarrow \begin{bmatrix} -2 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; (\star)$$

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$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 4 & -2 \end{bmatrix}.$$

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$$= (x-2)(x^2+x-6) = (x-2)^2(x+3).$$

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Hence, A has two eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

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Exercises

Section 3.1 : $1bci, 5bd, 7, 9, 10, 16a$ (page $155 - 157$)

Section 3.2 : $1ac, 2bdf, 3, 5, 6, 7, 9, 10, 20, 33$ (page $169 - 171$)

Section 3.3 : $1ace, 8a, 12, 13, 14$ (page $189 - 190$)