# Chapter 5. The Vector Space $\mathbb{R}^n$

# 5.1 Subspace and Spanning

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_i \in \mathbb{R} \right\}$$

We consider $\mathbb{R}^n$ as a set of all matrices of size $n \times 1$ that has two operations: addition and scalar multiplication. Denote

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the standard basis of $\mathbb{R}^n$. For any vector $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ can be expressed via the standard basis: $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n$ .

## Definition

A set $U$ of vectors in $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it satisfies the following properties:

(i) If $\boldsymbol{x}, \boldsymbol{y} \in U$ then $\boldsymbol{x} + \boldsymbol{y} \in U$.

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## Example

Which of the following are subspaces of $\mathbb{R}^3$?

(i) $U = \{(x, y, z) | z = 2x + 3y + 2\}$

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(ii) $V$ is not closed under addition (although $V$ is closed under scalar multiplication as well as $\mathbf{0} \in V$). Indeed, we can see that

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}^T, \boldsymbol{v}_2 = \begin{bmatrix} 2 & 2 & -2 \end{bmatrix}^T \in V$$

but $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{bmatrix} 4 & 4 & 0 \end{bmatrix}^T \notin V$. Hence $V$ is not a subspace. ($V$ is a cone in the space $\mathbb{R}^3$)

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## Example

Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 2 \end{bmatrix}$. We have

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$$U = \{\begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 | \quad 2x + y - z = 0 \text{ and } 3x - y + 5z = 0\}$$

is a subspace of $\mathbb{R}^3$. In fact, $U = \text{null}(A)$, for $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 5 \end{bmatrix}$.

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$$V = \left\{ \begin{bmatrix} 2x - y + z + t & y - 3z - t \end{bmatrix}^T \middle| x, y, z, t \text{ are real numbers} \right\}$$

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*For any matrix $A$ of size $m \times n$,*

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# Spanning sets

Given vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ in $\mathbb{R}^n$.

## Definition

- A vector of the form

$$t_1\boldsymbol{x}_1 + t_2\boldsymbol{x}_2 + \cdots + t_k\boldsymbol{x}_k$$

where the $t_i \in \mathbb{R}$ are scalars is called a linear combination of the $\boldsymbol{x}_i$, and $t_i$ is called the coefficient of $\boldsymbol{x}_i$ in the linear combination.

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- The set of all such linear combinations is called the span of the $\boldsymbol{x}_i$ and is denoted

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If $V = \operatorname{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$, we say that $V$ is spanned by the vectors $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$.

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Find all values of $a$ so that the vector $[5 \;\; 3 \;\; a]^T$ is in $\text{span}\{[3 \;\; 2 \;\; 0]^T, [1 \;\; 0 \;\; 3]^T\}$.

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$$= \operatorname{im}\left( \begin{bmatrix} 1 & 1 & -3 \\ 2 & -3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \right)$$

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## Example

Let $\boldsymbol{x} = (2, -1, 1, 1)$ and $\boldsymbol{y} = (1, 1, 3, 4)$ in $\mathbb{R}^4$. Determine whether $\boldsymbol{p} = (0, -2, 4, 5)$ or $\boldsymbol{q} = (7, -5, 1, 0)$ are in $U = \text{span}\{\boldsymbol{x}, \boldsymbol{y}\}$.

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## Theorem

*Let* $U = span\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ *in* $\mathbb{R}^n$. *Then:*

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Moreover, we have

$$\begin{aligned} \boldsymbol{x} &= \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{y}) \\ \boldsymbol{y} &= \frac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{y}). \end{aligned}$$

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### Example

It is easy to see that $\mathbb{R}^n = \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$. For example $\mathbb{R}^2 = \text{span}\left\{\boldsymbol{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Indeed, for any vector $\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ can be written as $\boldsymbol{u} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$.

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**Answer:** Any vector $\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$ can be written as

$$\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix} = s\begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}-1\\2\end{bmatrix} = \frac{2x+y}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{-x+y}{3}\begin{bmatrix}-1\\2\end{bmatrix},$$

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**Answer:** Any vector $\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$ can be written as

$$\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix} = s\begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}-1\\2\end{bmatrix} = \frac{2x+y}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{-x+y}{3}\begin{bmatrix}-1\\2\end{bmatrix},$$

this shows that $\boldsymbol{u} \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$.

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It is easy to see that $\mathbb{R}^n = \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$. For example $\mathbb{R}^2 = \text{span}\left\{\boldsymbol{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Indeed, for any vector $\boldsymbol{u} = \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ can be written as $\boldsymbol{u} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$.

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If $U$ is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ of vectors in $U$ is called a basis of $U$ if it satisfies the following two conditions:

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*Suppose that* $U = span\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$. *Then*

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## Theorem

*Any set consist of n linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.*

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Which of the following sets are bases of $\mathbb{R}^3$?

(i) $\{(1, 2, 1), (-1, 0, 3)\}$

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Given $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ are two $n$-tuples (vectors) in $\mathbb{R}^n$.

**Definition**

- The dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

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4. $(a\boldsymbol{u}) \cdot \boldsymbol{v} = a(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \cdot (a\boldsymbol{v})$ *for any scalar $a$.*
5. $\boldsymbol{u} \cdot (\boldsymbol{v} \pm \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} \pm \boldsymbol{u} \cdot \boldsymbol{w}$.
6. $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ *(Cauchy inequality)*
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Find the number $c$ such that the set

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## Theorem

*Let* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ *be an orthogonal basis of* $U$. *Then any* $\mathbf{u} \in U$:

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Let $\boldsymbol{x} = [a, b, c]$. Let $U$ be the subspace spanned by the orthogonal basis $\{\boldsymbol{u} = [1, 1, 1],\ \boldsymbol{v} = [1, -1, 0],\ \boldsymbol{w} = [1, 1, -2]\}$.
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Express $\boldsymbol{X} = \begin{bmatrix} -3 & 2 & 2 & 7 \end{bmatrix}^T$ as a linear combination of the orthogonal basis of the subspace
$U = \text{span}\left\{ \boldsymbol{u} = \begin{bmatrix} -1 & 2 & 0 & 3 \end{bmatrix}^T, \boldsymbol{v} = \begin{bmatrix} 1 & 2 & -2 & -1 \end{bmatrix}^T \right\}.$

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# 5.4 Rank of a Matrix

Let $A$ be $m \times n$ matrix.

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$$= \left\{ x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x + 2y + 3z \\ -2x + 5z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

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## Lemma

*Let $A$ and $B$ denote $m \times n$ matrices.*

1. *If $A \to B$ by elementary row operations, then $row(A) = row(B)$.*
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## Example

Let $A$ be a $3 \times 6$ matrix and let $\dim(\text{null}(A)) = 2$. Which of the following statements are true?

(i) All bases of the $\text{col}(A)$ have four vectors.

(ii) $\dim(\text{row}(A)) = 1$.

## Example

Let $A$ be a $3 \times 6$ matrix and let $\dim(\text{null}(A)) = 2$. Which of the following statements are true?

(i) All bases of the $\text{col}(A)$ have four vectors.

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**Answer:**

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**Answer:** We have $\dim(\text{null}(A)) = 6 - \text{rank}(A) = 2$. Hence $\text{rank}(A) = 4$.

(i) True. Since $\dim(\text{col}(A)) = \text{rank}(A) = 4$. Hence all bases of the $\text{col}(A)$ have four vectors.

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## Example

Let $A$ is a $100x350$ matrix. Which of the following statements are true?

(i) $\dim(\text{Null}(A))$ must be at least 250.

(ii) $\dim(\text{row}(A)) + \dim(\text{col}(A)) = 450$.

**Answer:**

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**Answer:**

(i) True. Since $\dim(\text{Null}(A)) = 350 - \text{rank}(A) = 4$. Moreover, $\text{rank}(A) \leq \min\{100, 350\} = 100$. Thus,

$$\dim(\text{Null}(A)) = 350 - \text{rank}(A) \geq 250 \ .$$

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$$\dim(\mathrm{Null}(A)) = 350 - \mathrm{rank}(A) \geq 250 \ .$$

(ii) False. Since $\dim(\mathrm{row}(A)) + \dim(\mathrm{col}(A)) \leq 200$.

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$$\dim(\text{Null}(A)) = 350 - \text{rank}(A) \geq 250 \ .$$

(ii) False. Since $\dim(\text{row}(A)) + \dim(\text{col}(A)) \leq 200$.

# Exercises

Section 5.1: 1, 2, 3, 4 (page 267-268)

Section 5.2: 1, 2, 3, 4, 6, 7abcfg (page 278)

Section 5.3: 1, 2, 3, 4,5, 6, 7, 12 (page 286-287)

Section 5.4: 1, 2, 3, 7 (page 294-295)