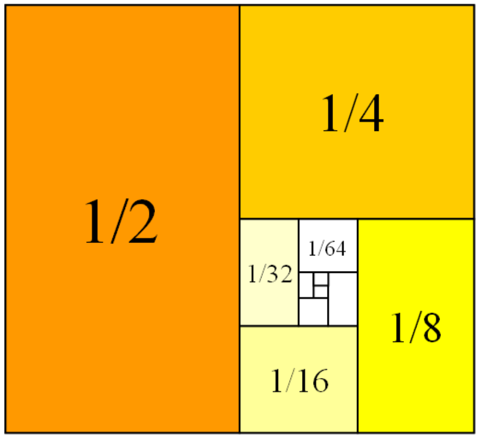


Chapter 3: TECHNIQUES OF INTEGRATION

Department of Mathematics, FPT University

Content

3.1 Integration by Parts

3.6 Numerical Integration

3.7 Improper Integrals

3.1
Integration by Parts

In this section, we will learn:

How to integrate complex functions by parts.

INTEGRATION BY PARTS

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

- Let $u = f(x)$ and $v = g(x)$.
- Then, the differentials are:

$$du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)\,dx$$

Thus, by the Substitution Rule, the formula for integration by parts becomes:

$$\int u\,dv = uv - \int v\,du$$

Evaluating both sides of Formula 1 between *a* and *b*, assuming *f ʹ* and *g ʹ* are continuous, and using the FTC, we obtain:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

Example 1: Find $\int x \sin x \, dx$

Let $u = x$ $\quad dv = \sin x \, dx$

Then, $du = dx$ $\quad v = -\cos x$

Using Formula 2, we have:

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Example 2: Evaluate $\int e^x \sin x \, dx$

$u = e^x$ and $dv = \sin x \, dx$

Then, $du = e^x \, dx$, $v = -\cos x$

Integration by parts gives:

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

This time, we use

$u = e^x$ and $dv = \cos x \, dx$

Then, $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

We get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

Hence,

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x\,dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

Suspect f(x) is continuous and differentiable, f(1)=4 and $\int_0^1 f(x)dx = 5$ Find $\int_0^1 xf'(x)dx$	
a	4/5
b	5/4
c	1
d	None of the others
e	-1

Example 4

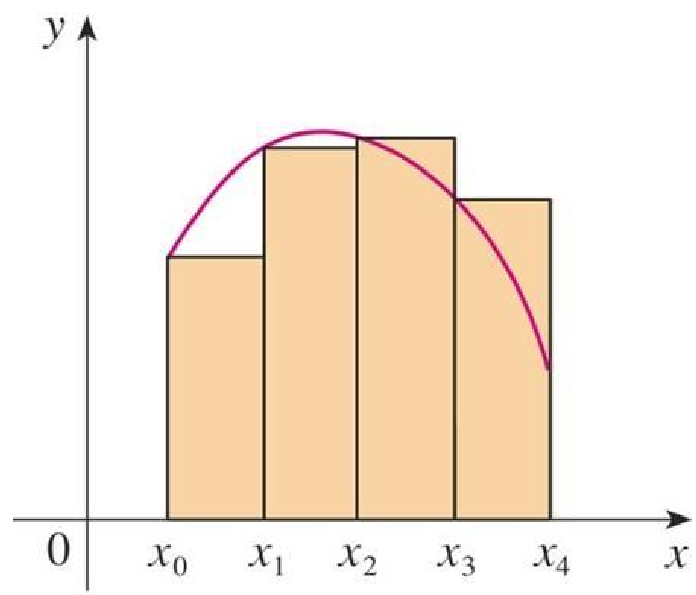
Suppose f(x) is continuous and differentiable, f(1)=3, f(3)=1 and $\int_1^3 xf'(x)dx = 13$

What is the average value of f on the interval [1,3]?

3.6

Numerical Integration

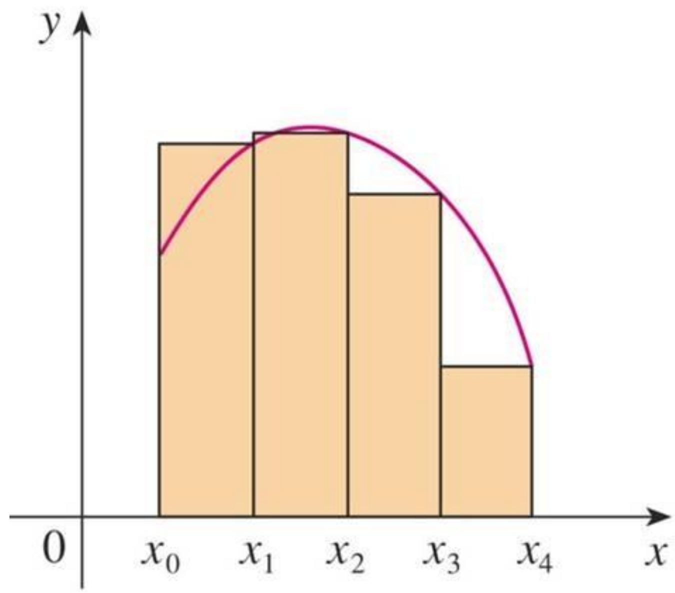
Left endpoint Method



(a) Left endpoint approximation

$$\int_a^b f(x)dx \approx \Delta x[f(x_0)+f(x_1)+\ldots+f(x_{n-1})]$$

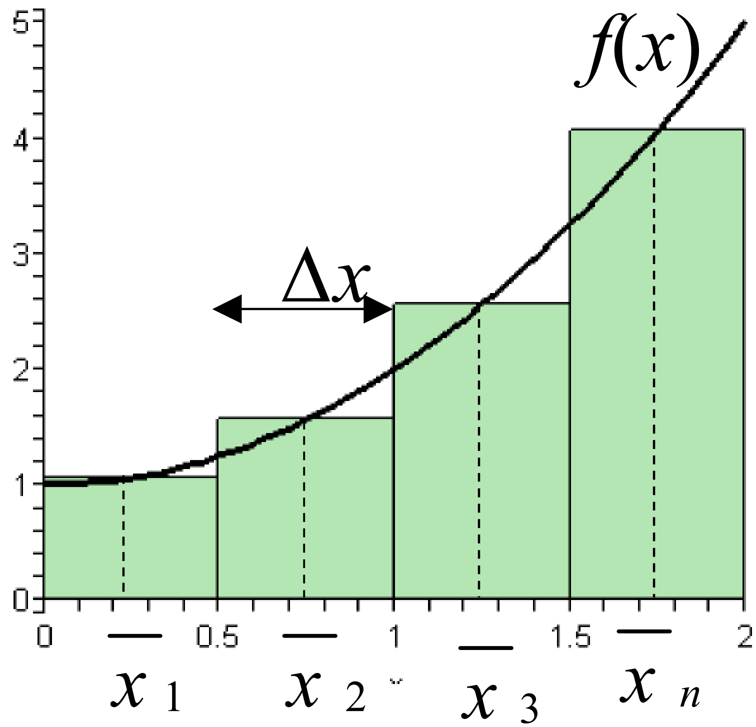
Right endpoint Method



(b) Right endpoint approximation

$$\int_a^b f(x)dx \approx \Delta x[f(x_1)+f(x_2)+\ldots+f(x_n)]$$

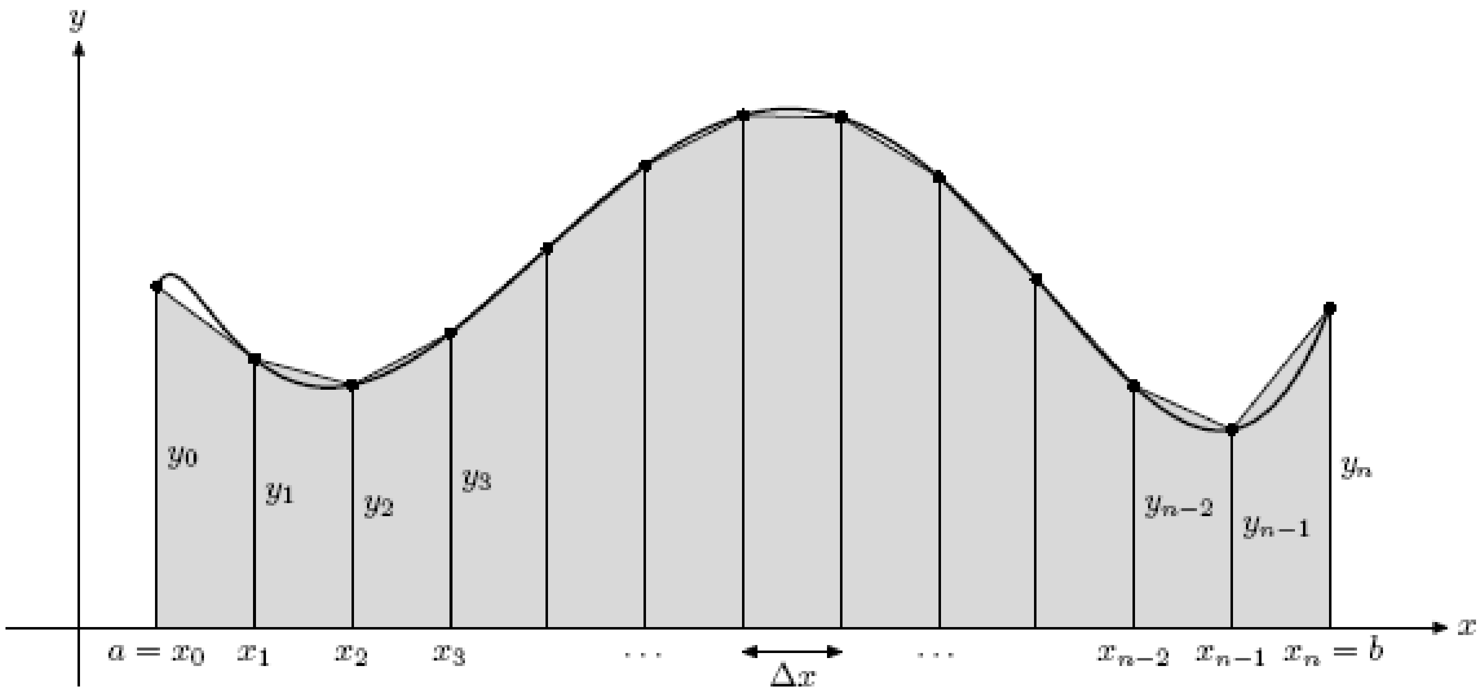
Midpoint Method



$$\int_a^b f(x)dx \approx \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\ldots+f(\bar{x}_n)]$$

Numerical Integration

Trapezoidal Method

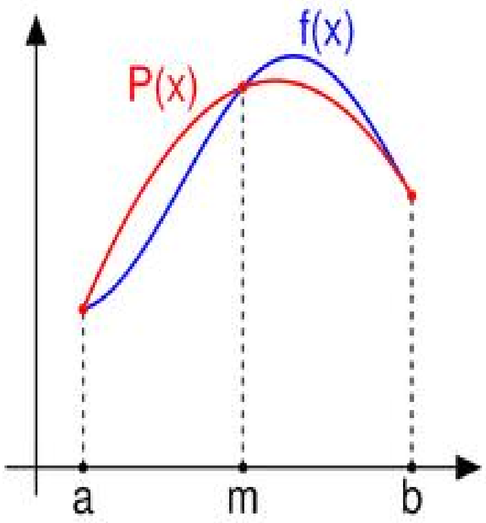


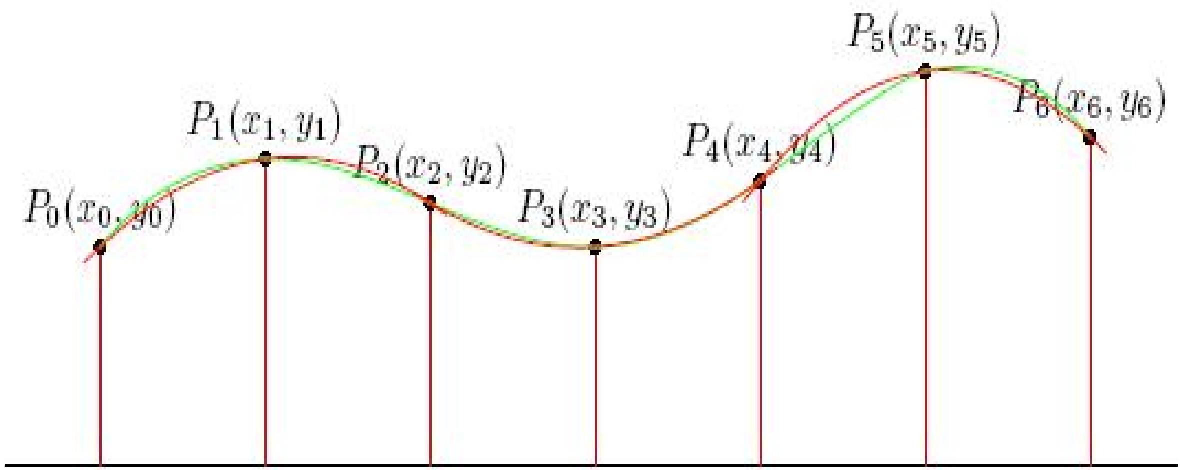
$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0)+f(x_1)]+\ldots+\frac{\Delta x}{2}[f(x_{n-1})+f(x_n)]$$

$$\approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n)]$$

Numerical Integration

Simpson's Method





$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+f(x_2)]+\ldots+\frac{\Delta x}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

$$\approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_{n-1})+f(x_n)]$$

Example

Approximate the integral with n = 8, using:

$$\int_1^2 (1/x)\,dx$$

a. Left/Right endpoints

b. Midpoints

c. Trapezoidal method

d. Simpson's method

Estimate error for Midpoint and Trapezoidal method

- Suppose $|f''(x)| \le K$ for $a \le x \le b$.
- If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

If B is our estimate of some quantity having an actual value of A, the absolute error is given by |A − B|.

The relative error is the error as a percentage $\left|\frac{A-B}{A}\right|$

Estimate error for Simpson's method

- Suppose $| f^{(4)}(x) | \leq K$ for $a \leq x \leq b$.
- If E_S is the error in the Simpson's method, then

$$|E_s| \leq \frac{K(b-a)^5}{180n^4}$$

Example

How large should we take *n* in order to guarantee that the Trapezoidal, Midpoint Rule, Simpson's rule approximations for

$$\int_1^2 (1/x)\,dx$$

are accurate to within 0.0001?

$| f''(x) | \leq 2$ for $1 \leq x \leq 2$

Accuracy to within 0.0001 means that error < 0.0001

Trapezoidal: Choose smallest *n* so that $\frac{2(1)^3}{12n^2} < 0.0001$

$\rightarrow n = 41$

Midpoint: $\frac{2(1)^3}{24n^2} < 0.0001$ $\rightarrow n = 30$

Simpson: $\left|f^{(4)}(x)\right| = \left|\frac{24}{x^5}\right| \leq 24$

$\rightarrow \frac{24(1)^5}{180n^4} < 0.0001$ $\rightarrow n = 8$

3.7
Improper Integrals

In this section, we will learn

How to solve definite integrals

where the interval is infinite and

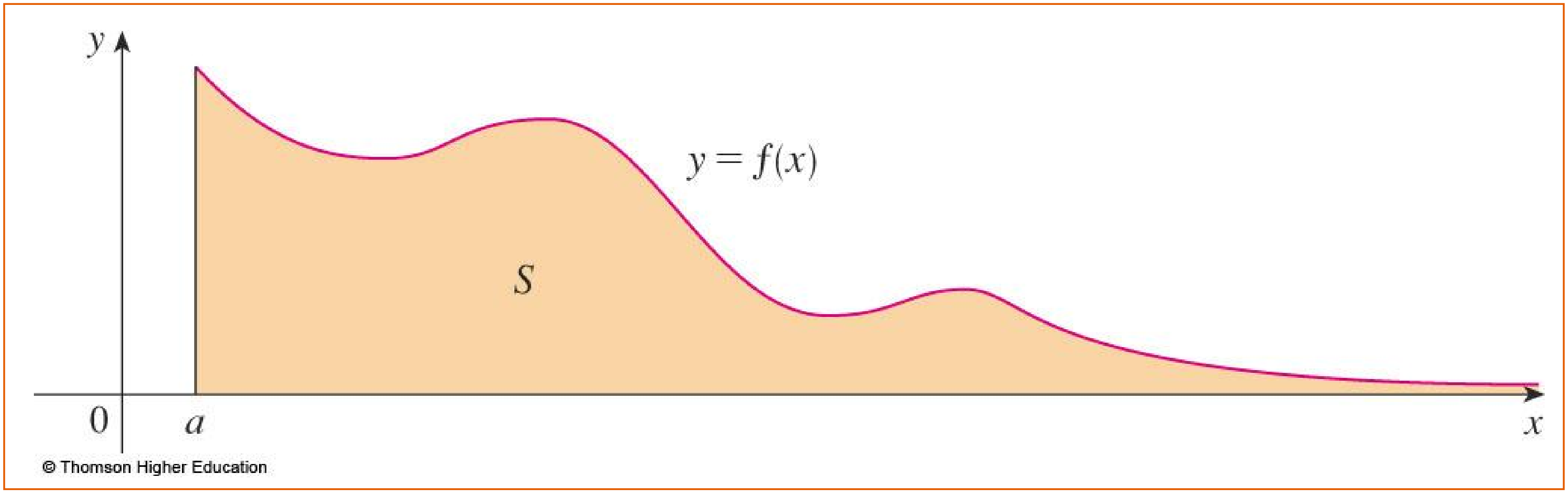
where the function has an infinite discontinuity.

IMPROPER INTEGRAL OF TYPE 1

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx}$$

provided this limit exists (as a finite number).



If $\int_t^b f(x)\,dx$ exists for every number $t \le a$, then

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx}$$

provided this limit exists (as a **finite number**).

IMPROPER INTEGRAL OF TYPE 1

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called

– Convergent if the corresponding limit exists.

– Divergent if the limit does not exist.

If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx}$$

- Here, any real number *a* can be used.

Example 1

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

- Convergent if $p > 1$
- Divergent if $p \leq 1$

Example 2

Investigate the convergence of the improper integrals

(a)

$$\int_{-\infty}^{0} x^2 e^{x^3}\,dx$$

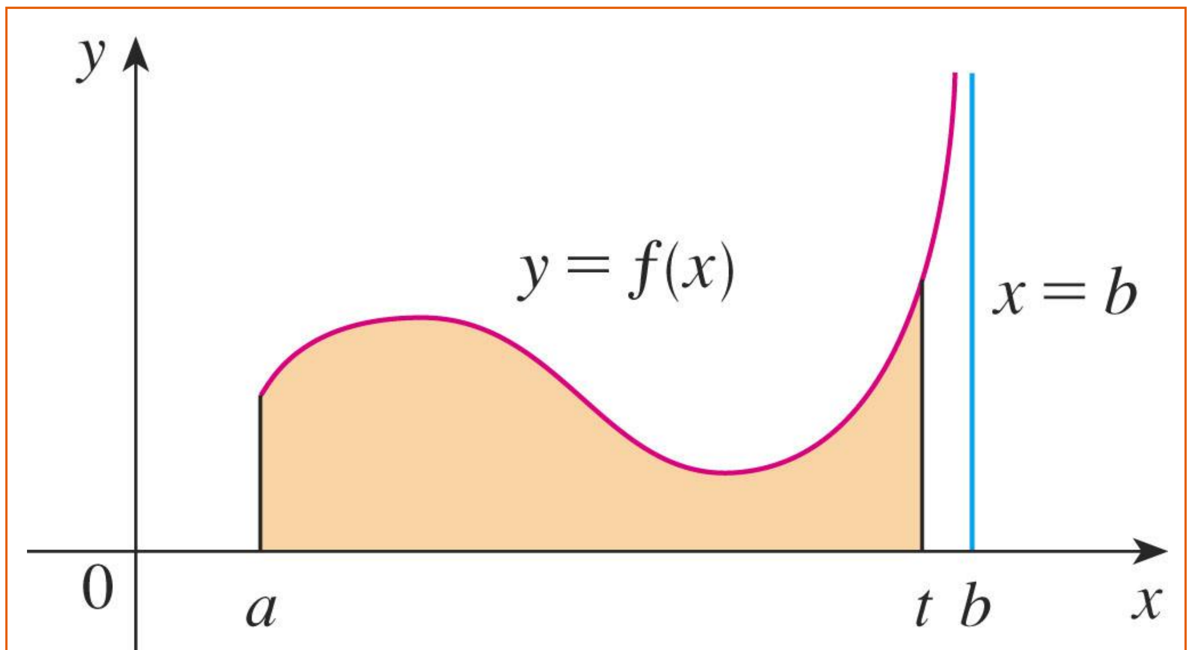
(b)

$$\int_{-\infty}^{\infty} x^2 e^{x^3}\,dx$$

If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx}$$

if this limit exists (as a finite number).

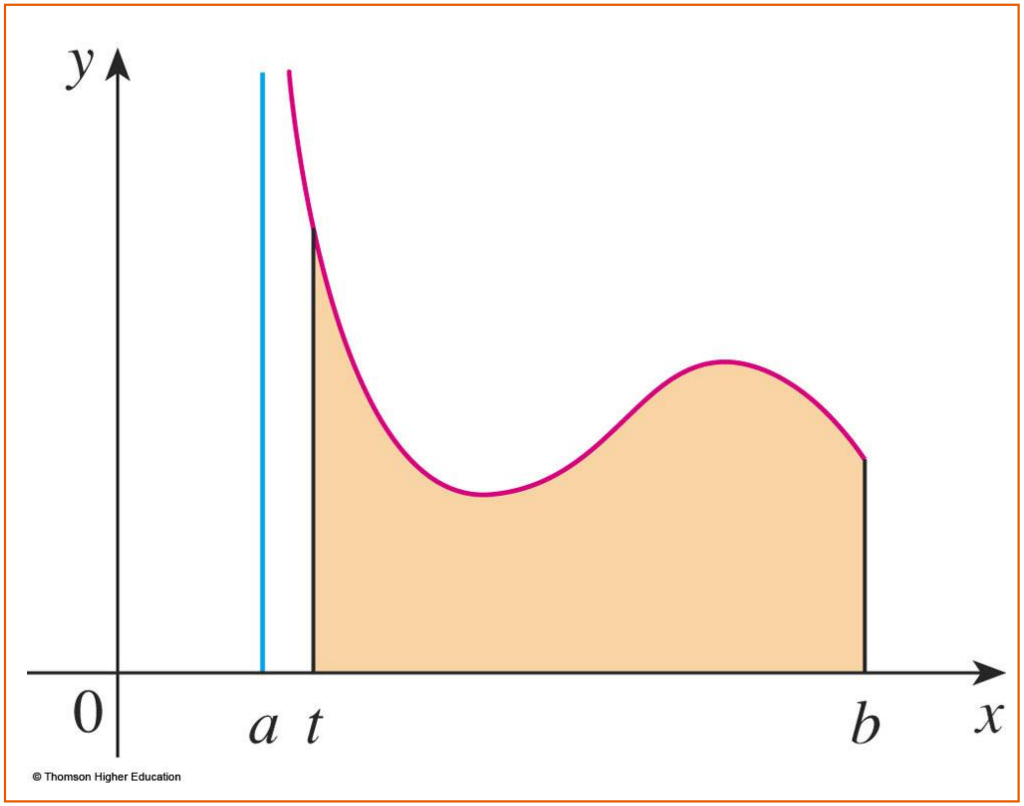


IMPROPER INTEGRAL OF TYPE 2

If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx}$$

if this limit exists (as a finite number).

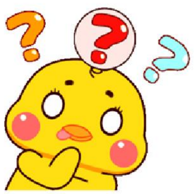


IMPROPER INTEGRAL OF TYPE 2

The improper integral $\int_a^b f(x)\,dx$ is called

– Convergent if the corresponding limit exists.

– Divergent if the limit does not exist.



Investigate the convergence of the following improper integrals

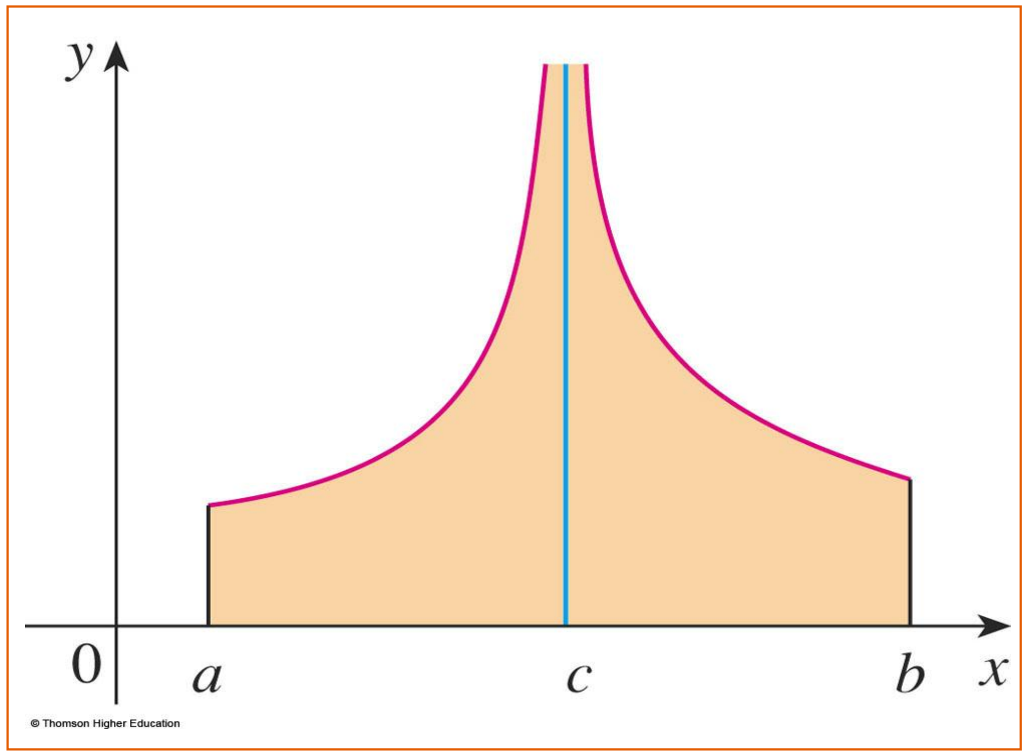
a, $\int_0^3 \frac{dx}{x-1}$ Convergent

b, $\int_a^b \frac{dx}{(x-a)^p} \quad (b > a)$ Divergent if $p \geq 1$, convergent if $p < 1$

IMPROPER INTEGRAL OF TYPE 2

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

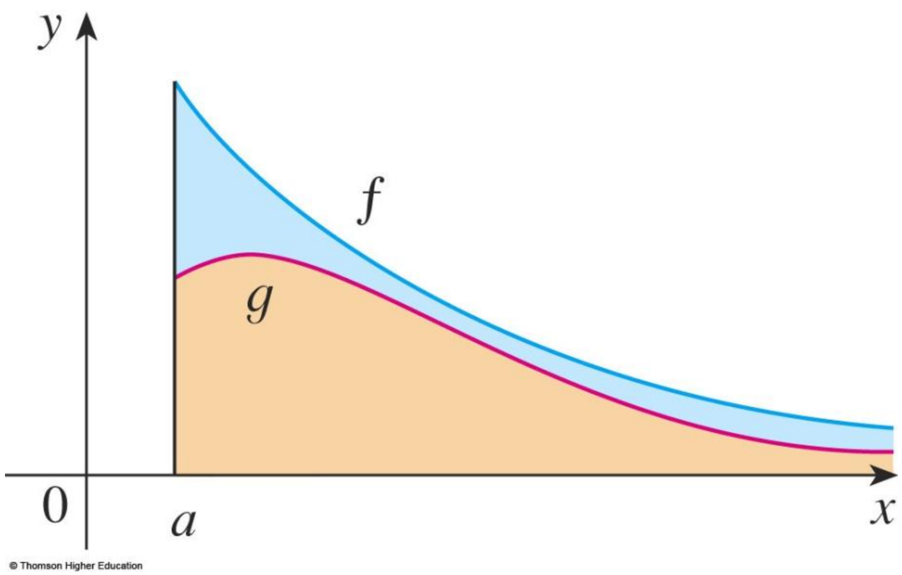


COMPARISON THEOREM

Suppose *f* and *g* are continuous functions with ***f*(*x*) ≥ *g*(*x*) ≥ 0** for $x \geq a$.

a. If $\int_a^{\infty} f(x)\,dx$ is convergent, then $\int_a^{\infty} g(x)\,dx$ is convergent.

b. If $\int_a^{\infty} g(x)\,dx$ is divergent, then $\int_a^{\infty} f(x)\,dx$ is divergent.



Example

Does $I = \int_1^\infty \frac{|\cos(x)|\,dx}{x^2}$ converge?

We have

$$0 \le \frac{|\cos(x)|}{x^2} \le \frac{1}{x^2}$$

and $\int_1^\infty \frac{dx}{x^2}$ converges

Hence, I converges.

Example

Investigate the convergence of the improper integrals

(a) $$\int_1^{\infty} \frac{\sqrt{x+2}}{x}\,dx$$

(b) $$\int_3^{\infty} \frac{\sqrt{x-2}}{x^3}\,dx$$

(c) $$\int_0^{\infty} e^{-x^2}\,dx$$

Exercises

- 1,2,3,5, 20-23 (p.270)
- 62, 64, 66 (p.271)
- 304, 305, 311 (p.327)
- 355-371 (p.343)

347. divergent
349. $\frac{\pi}{2}$
351. $\frac{2}{e}$
353. Converges
355. Converges to 1/2.
357. −4
359. π
361. diverges
363. diverges
365. 1.5
367. diverges
369. diverges
371. diverges
373. Both integrals diverge.
375. diverges
377. diverges
379. π
381. 0.0
383. 0.0
385. 6.0
387. $\frac{\pi}{2}$