Chapter 1. Systems of Linear Equations

1 Solutions and Elementary Operations

2 Gaussian Elimination

3 Homogeneous Systems

1.1 Solutions and Elementary Operations

Definition

A system of m linear equations and n variables (unknowns):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \quad (\star) \text{ where } a_{ij}, b_i \in \mathbb{R}$$

We say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the coefficient matrix

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- A solution to one equation is a sequence of real numbers $s_1, s_2, \ldots, s_n$ such that $a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b$.
- $s_1, s_2, \ldots, s_n$ is called a solution to a system of linear equations if $s_1, s_2, \ldots, s_n$ is a solution to every equation of the system.
- Two systems are said to be equivalent if they have the same set of solutions.

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Find the solution of $\begin{cases} x + y - z = 1 \\ 2x - y + z = 5 \\ -x + 2y - 2z = -4 \end{cases}$.

The system is equivalent to (by using operations $R_2' = -2R_1 + R_2$, $R_3' = R_1 + R_3$) the following system

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$$\begin{cases} x + y - z = 1 \\ 0x - 3y + 3z = 3 \\ 0x + 3y - 3z = -3 \end{cases} \Leftrightarrow \begin{cases} x + y - z = 1 \\ \quad\;\; y - z = -1 \end{cases}$$

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The augmented matrix

$$\bar{A} = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 3 \end{array}\right]$$

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Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

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Rank of a matrix

- The reduced row-echelon form of a matrix A is uniquely determined by A, but the row-echelon form of A is not unique.
- The number r of leading 1's is the same in each of the different row-echelon matrices.
- As r depends only on A and not on the row-echelon forms, it is called the rank of the matrix A, denoted by $rank(A)$.

Rank of a matrix

Definition

The rank of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.

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Find the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}$.

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Hence $\text{rank}(A) = 2$.

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Find all values of m so that the system

$$\begin{cases} x + y + 3z = 1 \\ -x + 3y + 2z = 3 \\ 3x - y + 4z = m \end{cases}$$

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The augmented matrix is given for a system of equations. If the system is consistent, find the general solution. Otherwise state that there is no solution.

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A homogeneous system consists of m linear equations and n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases} \quad (\star) \text{ where } a_{ij} \in \mathbb{R}.$$

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1. A homogeneous system always has a trivial solution $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$.
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The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 4 & -2 & 3 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Example

Solve the following system
$$\begin{array}{cccccc} x_1 & +2x_2 & -x_3 & +2x_4 & +x_5 & =0 \\ x_1 & +2x_2 & +2x_3 & & +x_5 & =0 \\ 2x_1 & +4x_2 & -2x_3 & +3x_4 & +x_5 & =0 \end{array}.$$

The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 4 & -2 & 3 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$\text{Rank}(A) = 3.$

Example

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$$\begin{array}{cccccc} x_1 & +2x_2 & -x_3 & +2x_4 & +x_5 & =0 \\ x_1 & +2x_2 & +2x_3 & & +x_5 & =0 \\ 2x_1 & +4x_2 & -2x_3 & +3x_4 & +x_5 & =0 \end{array}.$$

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$$\rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Rank$(A) = 3$.

The system has infinite solution:

$$\begin{cases} x_1 &= -2x_2 + x_3 - 2x_4 - x_5 \\ &= -2t + 1/3s \\ x_2 &= t \\ x_3 &= 2/3x_4 = -2/3s \\ x_4 &= -x_5 = -s \\ x_5 &= s \end{cases}$$

Example

Solve the following system
$$\begin{array}{cccccc} x_1 & +2x_2 & -x_3 & +2x_4 & +x_5 & =0 \\ x_1 & +2x_2 & +2x_3 & & +x_5 & =0 \\ 2x_1 & +4x_2 & -2x_3 & +3x_4 & +x_5 & =0 \end{array}.$$

The coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 4 & -2 & 3 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Rank$(A) = 3$.
The system has infinite solution:

$$\begin{cases} x_1 & = -2x_2 + x_3 - 2x_4 - x_5 \\ & = -2t + 1/3s \\ x_2 & = t \\ x_3 & = 2/3x_4 = -2/3s \\ x_4 & = -x_5 = -s \\ x_5 & = s \end{cases}$$

Example

Find all values of a such that the homogeneous system

$$\begin{cases} x - 2y + z = 0 \\ x - y + 3z = 0 \\ 2x + ay + 4z = 0 \end{cases}$$

has only the trivial solution.

Answer: The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & 3 \\ 2 & a & 4 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & a+4 & 2 \end{bmatrix}$$

$$\xrightarrow{-(a+4)R_2+R_3} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2-2(a+4) \end{bmatrix}.$$

We need $\text{rank}(A) = 3$. This implies that $2 - 2(a+4) \neq 0$, i.e., $a \neq -3$.

Exercises

1.1 : 10, 14, 18, 19 (page 8, 9).

1.2 : 1, 2a, 3, 4d, 5ace, 7ab, 8ad, 9a, 11acd, 12 (page 17-19).

1.3 : 1, 2ac, 3bc, 5ab (page 25, 26).