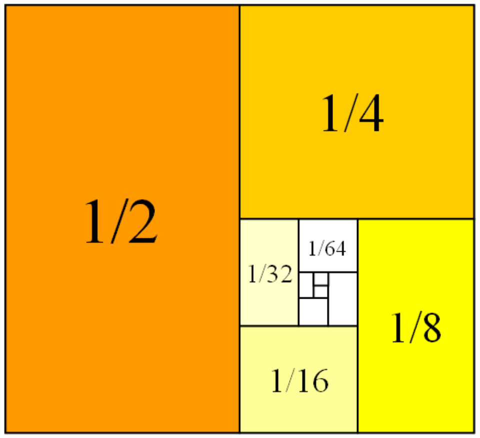


Chapter 1: INTEGRATION

Department of Mathematics, FPT University

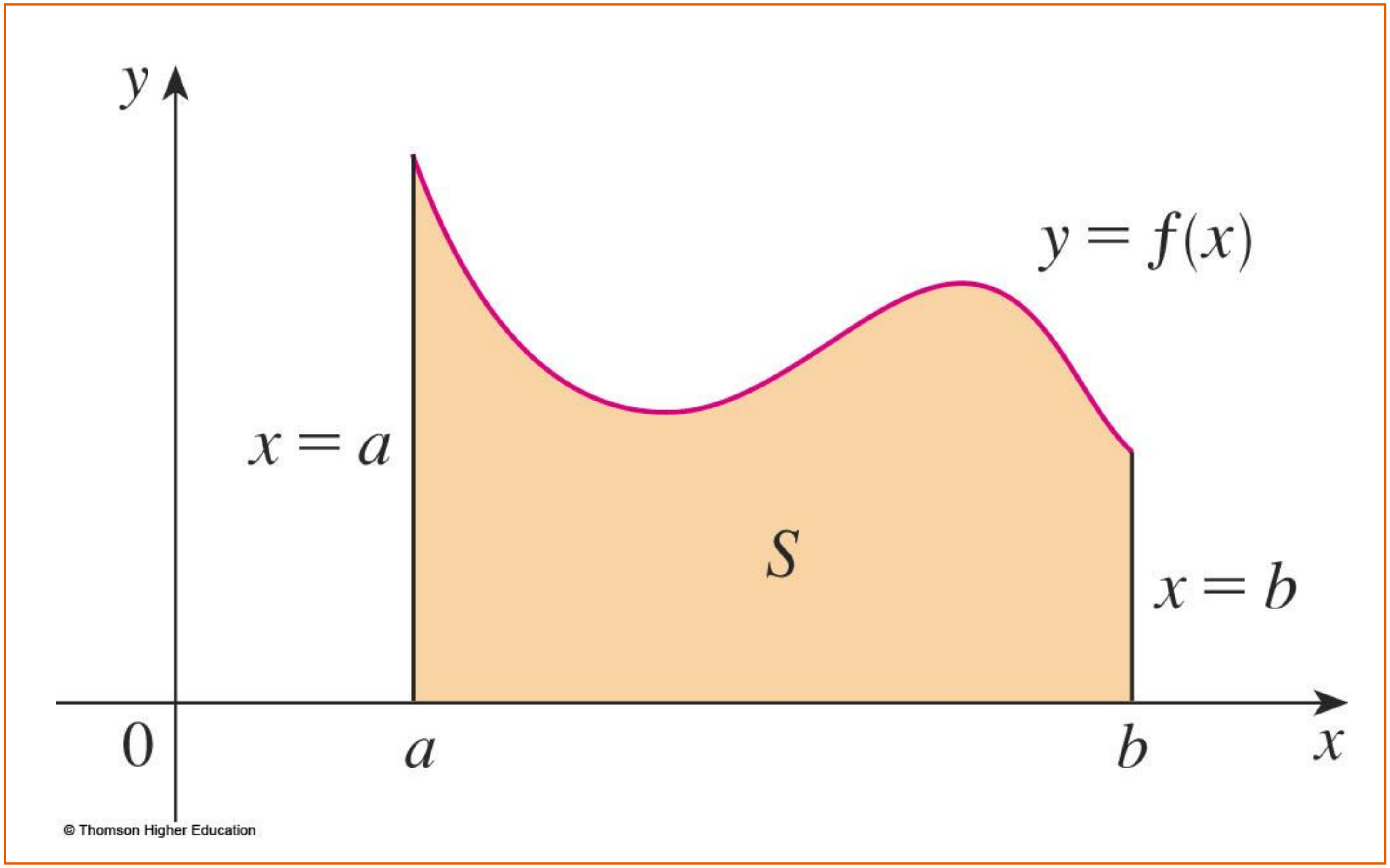
CONTENTS

1.1

Approximating Areas

AREA PROBLEM

Let's consider the area of region *S*

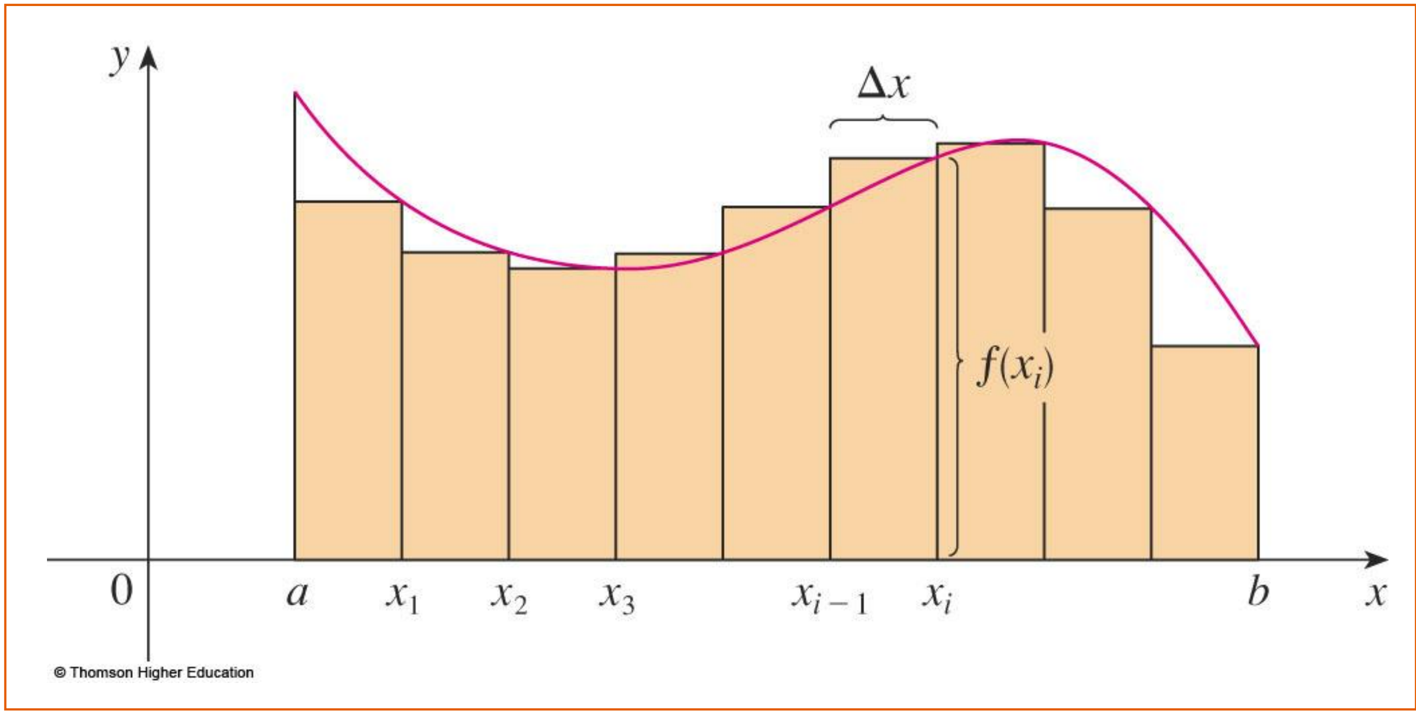


AREA PROBLEM

What we think of intuitively as the area of *S* is approximated by the sum of the areas of these rectangles

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \dots + f(x_n)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$



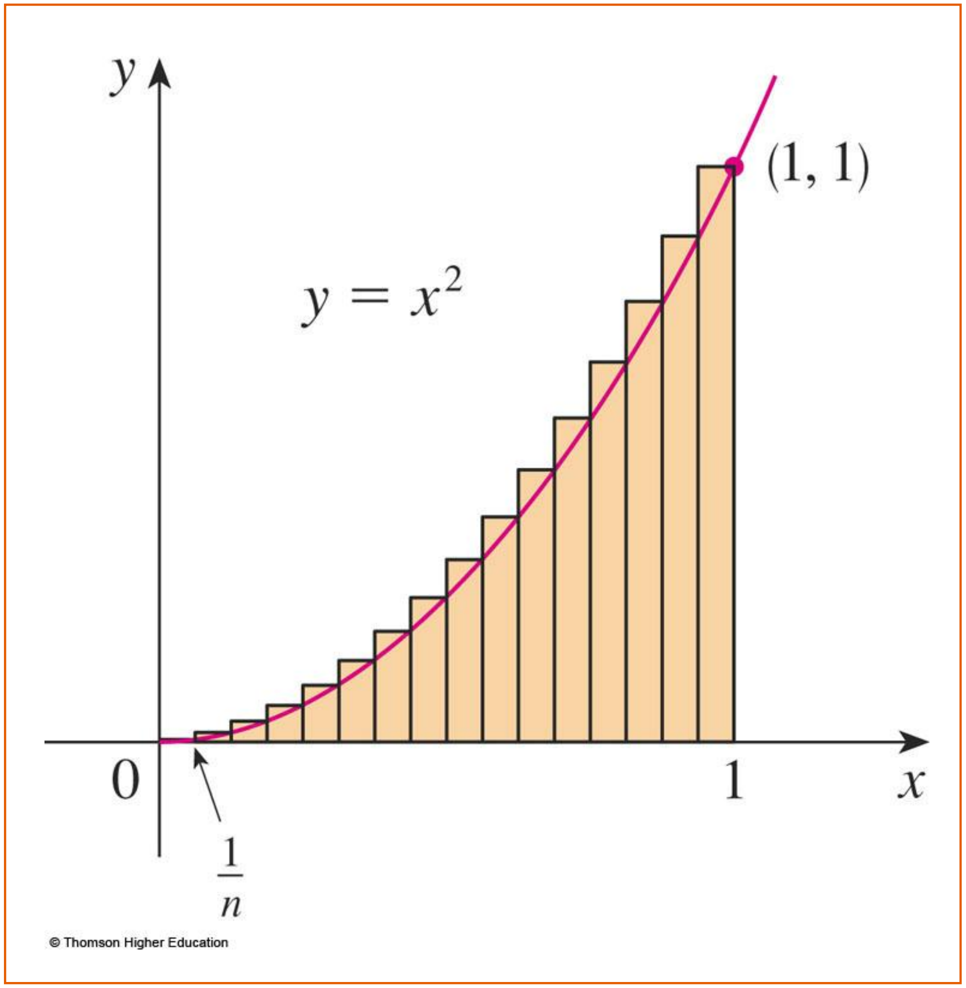
Right endpoints

AREA PROBLEM

R_n is the sum of the areas of the n rectangles.

- Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the **right endpoints** $1/n, 2/n, 3/n, \ldots, n/n$.

- That is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$.



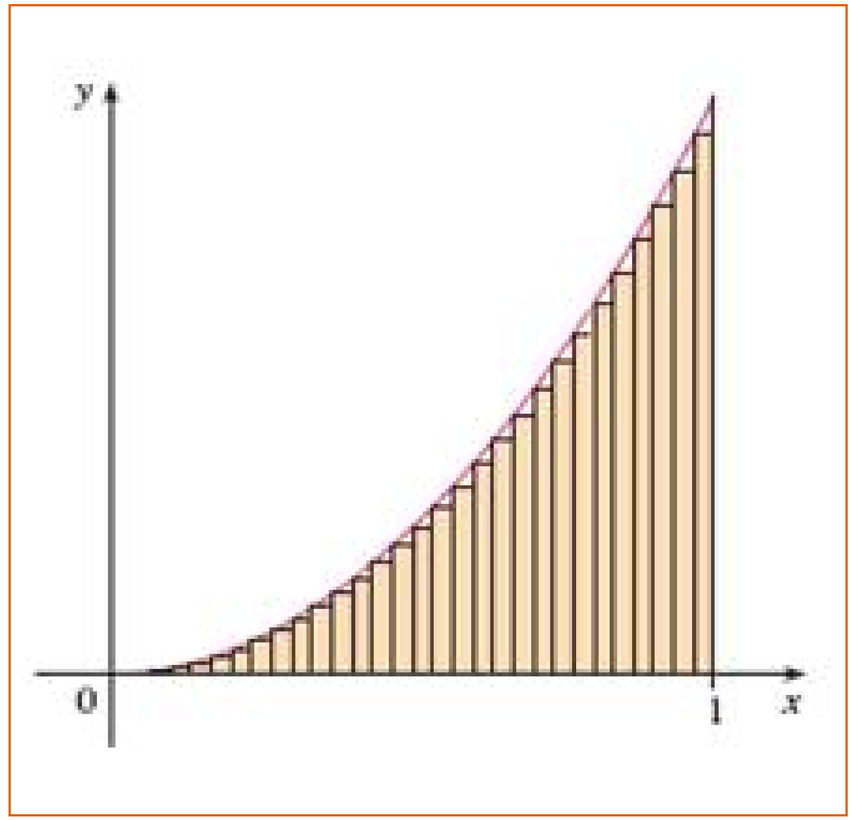
Left endpoints

AREA PROBLEM

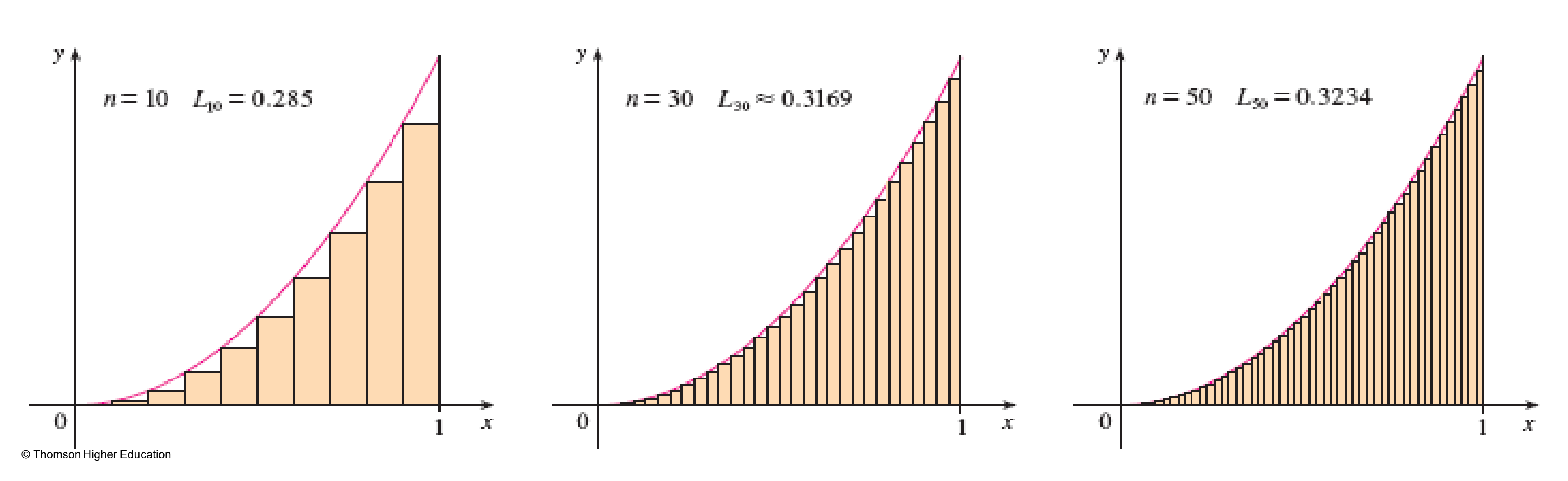
L_n is the sum of the areas of the *n* rectangles.

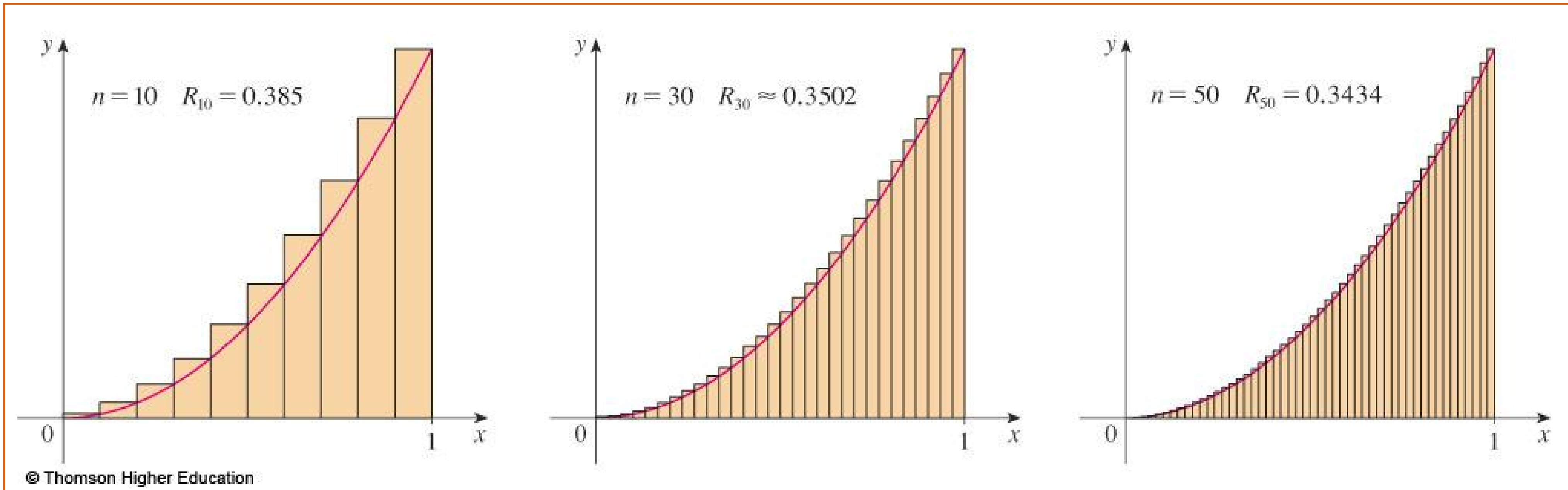
- Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $0, 1/n, 2/n, \ldots, (n-1)/n$.

- That is, the heights are $0, (1/n)^2, (2/n)^2, \ldots, ((n-1)/n)^2$.



AREA PROBLEM





AREA PROBLEM

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$L_n = \frac{1}{n}0^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 \ldots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2$$

$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2)$$

$$L_n = \frac{1}{n^3} . \frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2}$$

AREA PROBLEM

Thus, we define the area A to be the limit of the sums of the areas of the approximating rectangles

$$\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}}$$

AREA PROBLEM

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

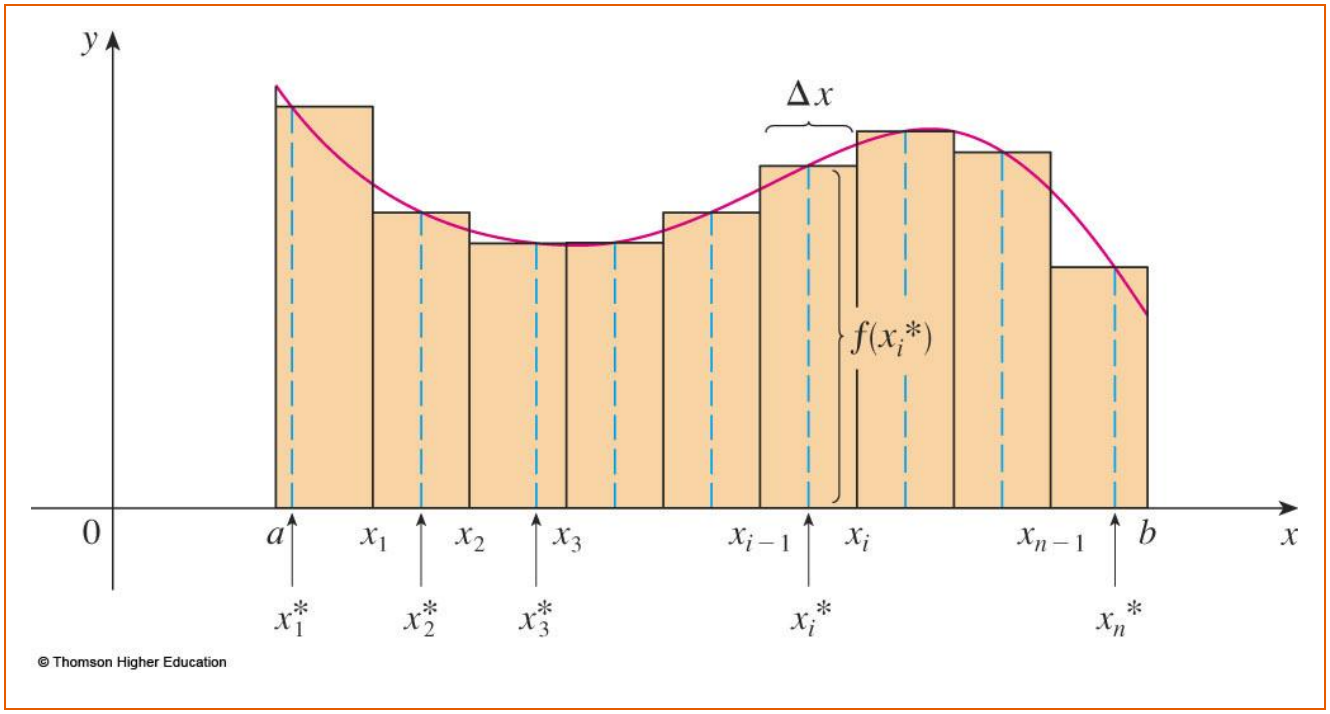
$$
\begin{aligned}
A &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]
\end{aligned}
$$

$$
\begin{aligned}
A &= \lim_{n\to\infty} L_n \\
&= \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x]
\end{aligned}
$$

AREA PROBLEM

Besides that, the sample point can be chosen arbitrarily in each subinterval.

x_i^* : the *sample points* in the i^{th} subinterval $[x_{i-1}, x_i]$



$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

Hence,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum.

1. Evaluating the Riemann sum of the following function with four subintervals. The sample points are the **right endpoints.**

$$f(x) = x - \frac{1}{x} \ (1 \leq x \leq 2)$$

2. Estimate the area of the region under the graph of y = sin x on [0,5] with 3 subintervals [0,1]; [1,2.5] and [2.5,5] by using left-endpoints.

3. Find the lower sum for $f(x) = 10 - x^2$ on [1,2] with 4 subintervals

1.2
The Definite Integral

In this section, we will learn about Integrals with limits that represent a definite quantity.

- If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = (b - a)/n$$

- Let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, i.e. x_i^* lies in the i^{th} subinterval.

Then, the definite integral of *f* from *a* to *b* is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

provided that this limit exists.

☞ If the limit exists, we say *f* is integrable on [*a*, *b*].

☞ The symbol $\int$ is called an integral sign.

DEFINITE INTEGRAL

- The definite integral $\int_a^b f(x)\,dx$ is a number. It does not depend on *x*.

- In fact, we could use any letter in place of *x* without changing the value of the integral

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr}$$

DEFINITE INTEGRAL

Let $f(x)$ be an integrable function defined on $[a, b]$.
Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis.
Let A_2 represent the area between $f(x)$ and the x- axis that lies below the axis.
Then, the net signed area

$$\int_a^b f(x)dx = A_1 - A_2$$

$$\int_{-3}^{3} 2x dx = A_1 - A_2 = 9 - 9 = 0.$$

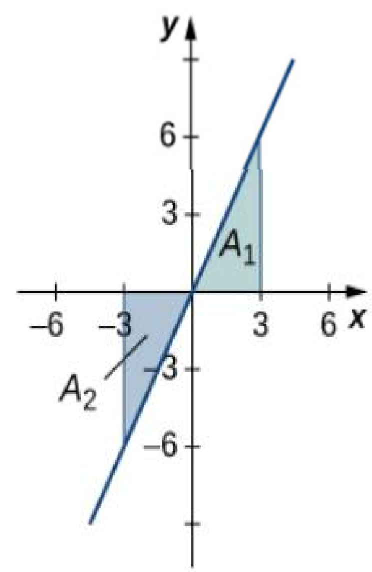


Figure 1.19 The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

DEFINITE INTEGRAL

The total area between $f(x)$ and the x-axis is

$$\int_a^b |f(x)|dx = A_1 + A_2$$

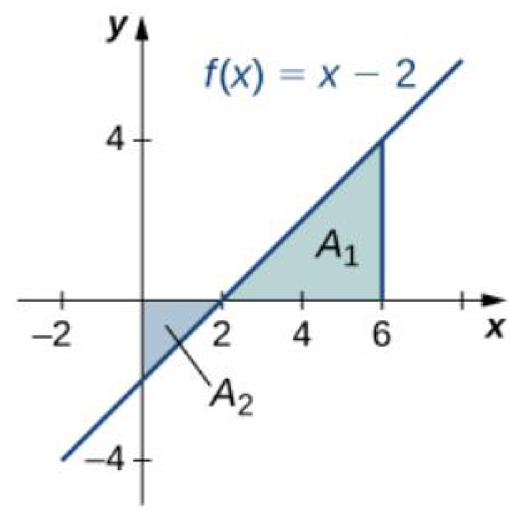


Figure 1.22 The total area between the line and the x-axis over $[0,\ 6]$ is A_2 plus A_1.

$$\int_0^6 |(x-2)|dx = A_2 + A_1.$$

INTEGRABLE FUNCTIONS

Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.

That is, the definite integral $\int_a^b f(x)\,dx$ exists.

INTEGRABLE FUNCTIONS

If f is integrable on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\,\Delta x$$

Example:

$$\lim_{n\to\infty} \sum_{i=1}^{n} 4r^2\,\Delta r = ? \qquad [1; 4]$$

EVALUATING INTEGRALS

The following three equations give formulas for sums of powers of positive integers.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

PROPERTIES OF THE INTEGRAL

Assume *f* and *g* are integrable functions.

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

COMPARISON PROPERTIES OF THE INTEGRAL

These properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leq b$.

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

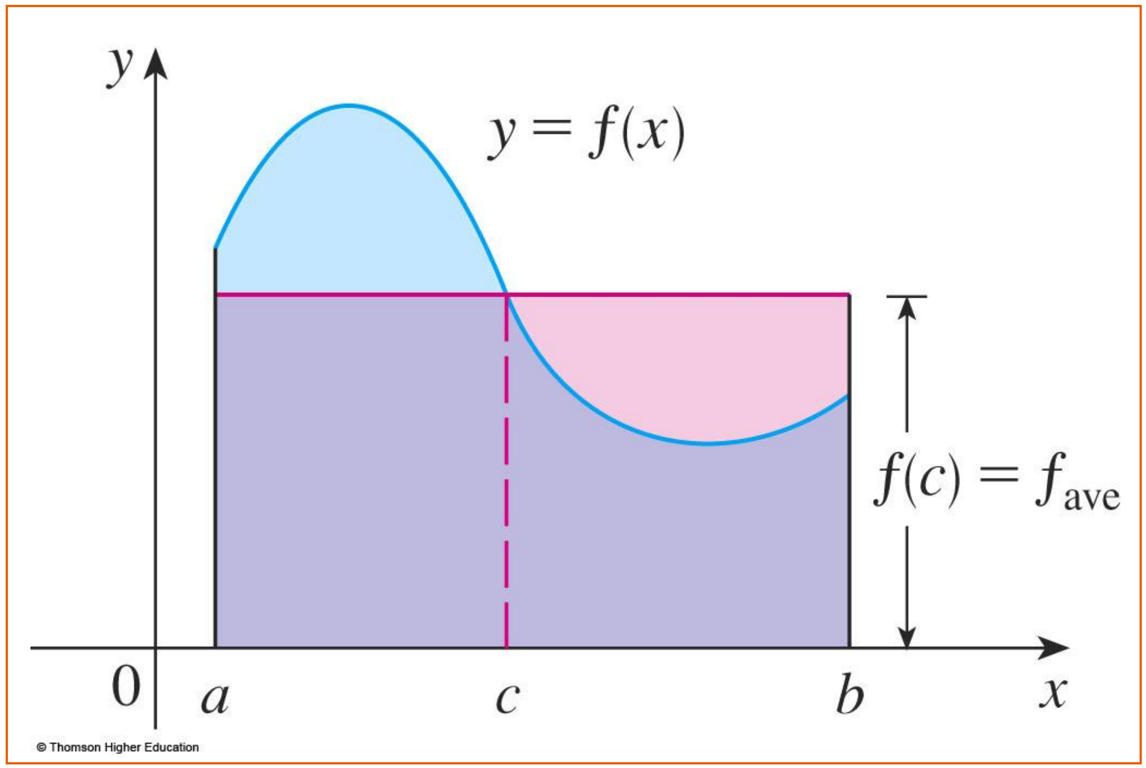
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

The Mean Value Theorem for Integrals

For 'positive' functions *f*, there is a number *c* such that the rectangle with base [*a*, *b*] and height *f*(*c*) has the same area as the region under the graph of *f* from *a* to *b*.



AVERAGE VALUE OF A FUNCTION

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Find the average value of f(x) = 6 − 2x over the interval [0,3] and find c such that f(c) equals that average value.

1.3

The Fundamental Theorem of Calculus

In this section, we will learn about
The Fundamental Theorem of Calculus
and its significance.

FUNDAMENTAL THEOREM OF CALCULUS PART 1: Integrals and Antiderivatives

If $f(x)$ is continuous over $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)$ over $[a, b]$.

Note: Any integrable function has an antiderivative.

Find the derivative of

- $f(x) = \int_1^x \frac{5}{3t} \mathrm{dt}$
- $g(x) = \int_x^{100} (2 - 5t)^6 dt$

Generalization

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f(u(x)) - v'(x)f(v(x))$$

Find the derivative of the following functions

$$\mathrm{H}(x) = \int_{1}^{x^3} \cos t \, dt$$

$$G(x) = \int_{x}^{x^2} \sin t \, dt$$

FUNDAMENTAL THEOREM OF CALCULUS PART 2: Evaluation theorem

If f is continuous on $[a, b]$, and F is any antiderivative of f, i.e. $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Evaluate

$$\int_1^2 x^{-4} dx$$

1.4

Integration Formulas and the Net Change Theorem

NET CHANGE THEOREM

We can reformulate $\int_a^b f(x)\,dx = F(b) - F(a)$ as follows.

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

The new value equals the initial value plus the integral of the rate of change

$$\boxed{F(b) = F(a) + \int_a^b F'(x)dx}$$

NET CHANGE THEOREM

If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

The net change takes into account both births and deaths.

NET CHANGE THEOREM

If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.

The net change of position, or displacement, from t_1 to t_2.

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

The distance is computed by integrating $|v(t)|$, the speed.

$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

Note: $v(t)$ can take positive or negative values.

Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m.

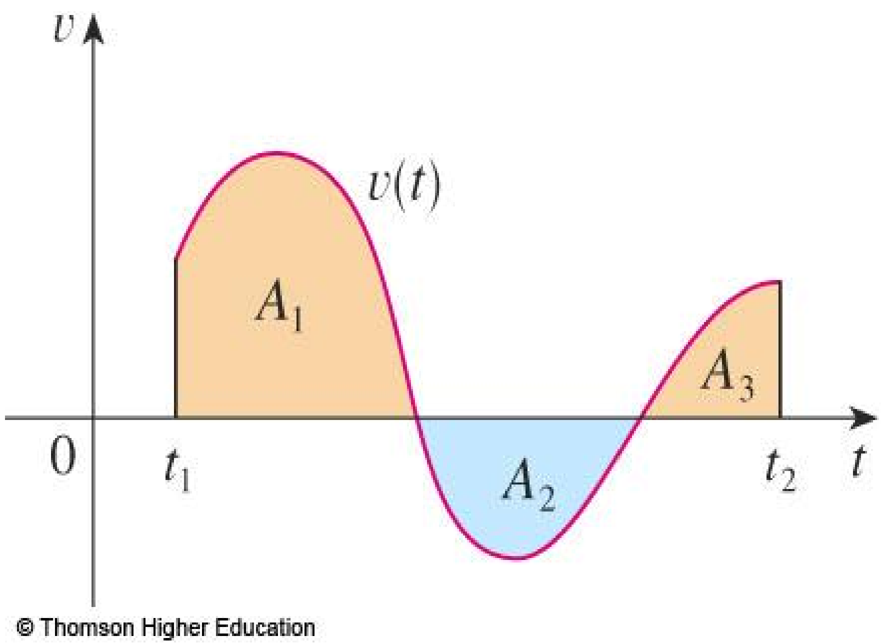
A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (in meters per second)

a) Find the displacement of the particle during the time period $1 \le t \le 4$.

b) Find the distance traveled during this time period.

NET CHANGE THEOREM

The figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

NET CHANGE THEOREM

The acceleration of the object is $a(t) = v'(t)$.

So,

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

SOLVE THE FOLLOWING EXERCISES

1. Suppose that the animal population is increasing at a rate $f(t)=3t-1$ (t measured in years).
How much does the animals increase between the third and the seven years?

2. Suppose the acceleration function and initial velocity are $a(t)=t+3$ (m/s^2), $v(0)=5$ (m/s).
Find the velocity at time t and the distance traveled from the beginning to 20 seconds.

INTEGRATING OF SYMMETRIC FUNCTIONS

Suppose *f* is continuous on [–*a* , *a*].

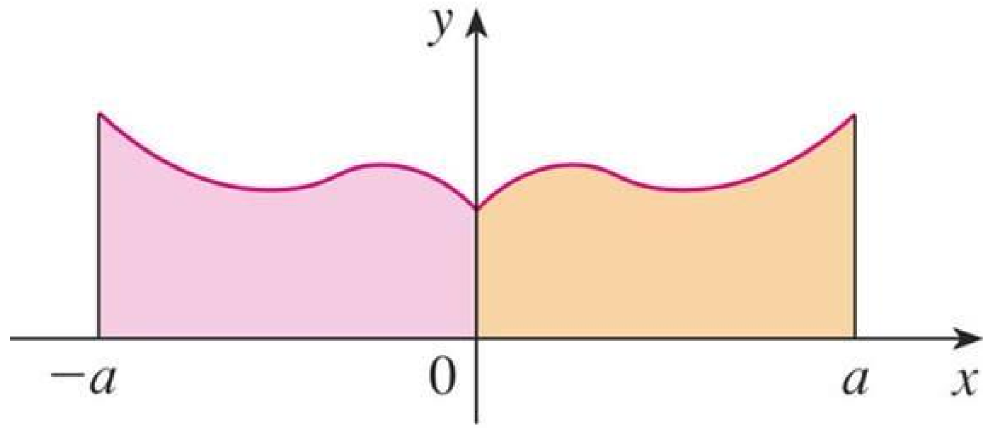
a. If *f* is even, [*f*(–*x*) = *f*(*x*)], then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

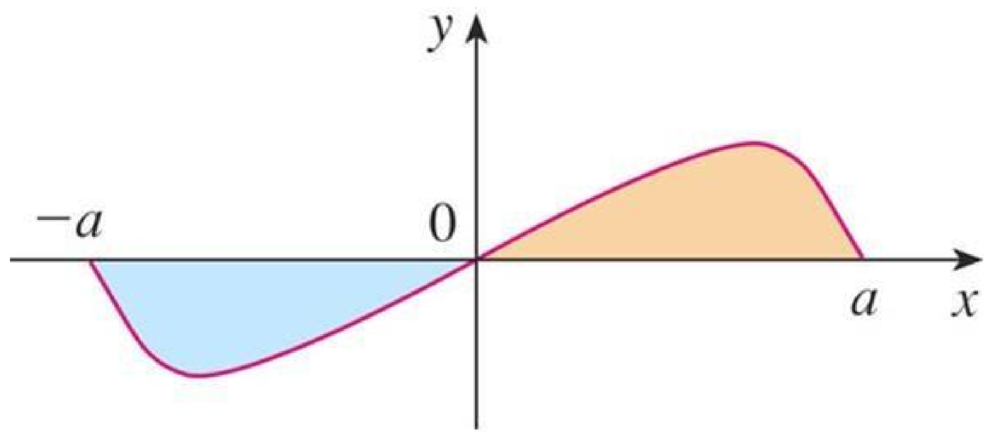
b. If *f* is odd, [*f*(-*x*) = -*f*(*x*)], then

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

This Theorem is illustrated here.



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

1.5
Substitution

In this section, we will learn

To substitute a new variable in place of an existing expression in a function, making integration easier.

INDEFINITE INTEGRAL

The notation $\int f(x)\, dx$ is traditionally used for an antiderivative of f and is called an indefinite integral.

Thus,

$$\boxed{\int f(x)\, dx = F(x) \quad \text{means} \quad F'(x) = f(x)}$$

You should distinguish carefully between definite and indefinite integrals.

- A definite integral $\int_a^b f(x)\,dx$ is a number.
- An indefinite integral $\int f(x)dx$ is a function (or family of functions).

TABLE OF INDEFINITE INTEGRALS

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

SUBSTITUTION RULE

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f\big(g(x)\big)g'(x)dx = \int f(u)du}$$

Exercise: Find $\int 4x^3\sqrt{1+x^4}\,dx$; $\int \cos x\,(\sin x - 1)\,dx$

Evaluate $$\int_1^2 \frac{dx}{(3-5x)^2}$$

- Let $u = 3 - 5x$. Then, $du = -5\ dx$, so $dx = -du/5$
- When $x = 1$, $u = -2$, and when $x = 2$, $u = -7$

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

Let F be an antiderivative of f.

Then, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

So,

$$\boxed{\begin{aligned}\int_a^b f(g(x))g'(x)dx &= F(g(x))\Big]_a^b \\ &= F(g(b)) - F(g(a))\end{aligned}}$$

Exercises (Calculus Volume 2)

- 8,9, 16,17, 42,43 (p.21)
- 61,62, 64,65, 84,85, 88,89, 94,98 (p.42)
- 148, 150, 155 (page 60)
- 207-212, 224, 226, 227 (page 73)
- 267,268, 271,272 (p.90)