FUNCTIONS AND GRAPHS

Department of Mathematics, FPT University

Hanoi
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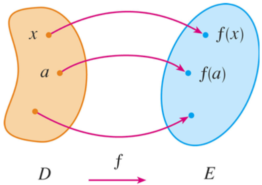
Definition

A ***function*** f is a rule that assigns to each element x in a set **D** *exactly one* element, called $f(x)$, in a set **E**.
The set **D** is called the domain of the function f.



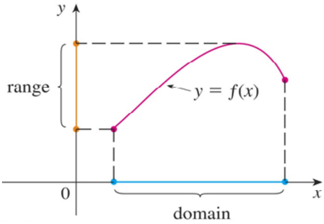
The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Definition

The ***graph*** of f is the set of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of f also allows us to picture:

- The domain of f on the x-axis.
- Its range on the y-axis.



Example

The graph of a function f is shown in Figure 1.

a) Find the values of $f(1)$ and $f(5)$.

b) What is the domain and range of f?

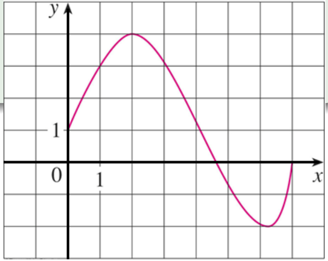


Figure 1:

Solution. a) $f(1) = 3$,
$f(5) = -0.7$.
b) **D** $= [0, 7]$,
Range$(f) = [-2, 4]$.

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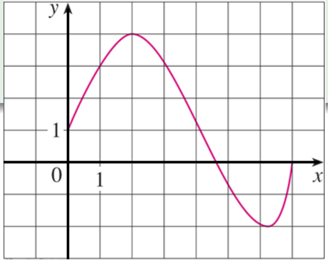

Figure 1:

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Example

Find the domain and region of the functions (if it is a function).

a) $f(n) = \sqrt{n}$ for all natural numbers n.

b) $g(x)$ is any real number such that larger than x.

There are four possible ways to represent a function:

- Algebraically (by an explicit formula)
- Visually (by a graph)
- Numerically (by a table of values)
- Verbally (by a description in words)

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Example

The human population of the world P depends on the time t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

- The table gives estimates of the world population $P(t)$ at time t, for certain years.
- However, for each value of the time t, there is a corresponding value of P, and we say that P is a function of t.

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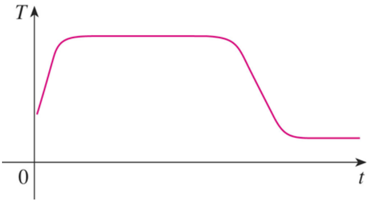
Example

"When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running".
Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.



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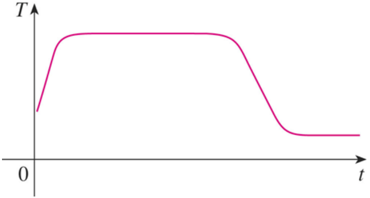
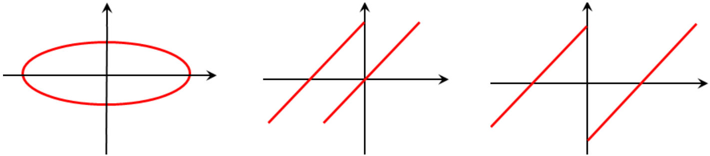


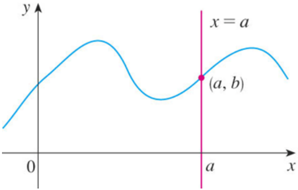
The Vertical Line Test

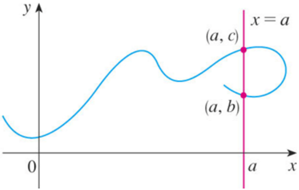
A curve in the xy-plane is the graph of a function of x if and only if ***no vertical line*** intersects the curve ***more than once***.



The Vertical Line Test

The reason for the truth of the Vertical Line Test can be seen in the figure.





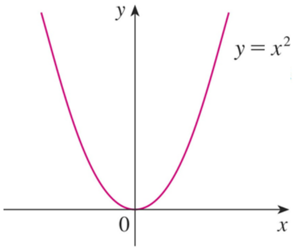
Definition

A function f is called ***increasing*** on an interval I if:

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called ***decreasing*** on I if:

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$



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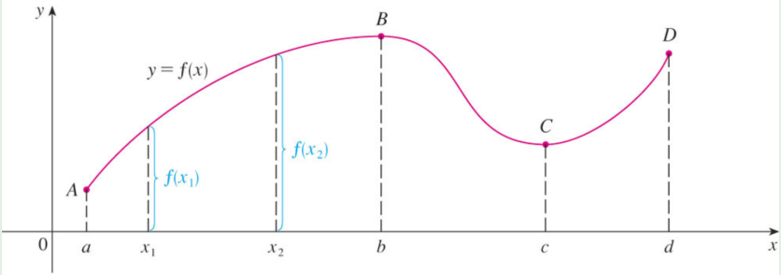

The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.

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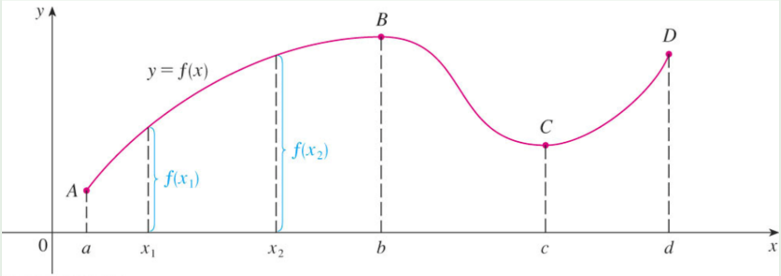

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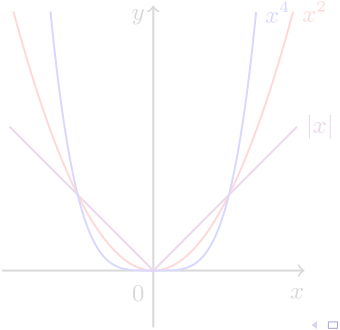
Definition

If a function f satisfies:

$$f(-x) = f(x), \text{ for all } x \text{ in } \mathbf{D}$$

then f is called an ***even function***.

The geometric significance of an even function is that its graph is ***symmetric with respect to the y-axis***.



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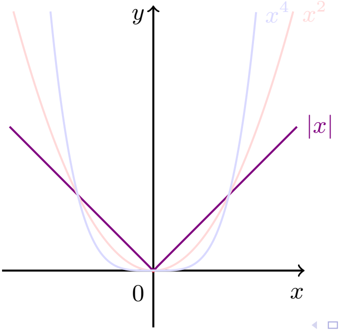

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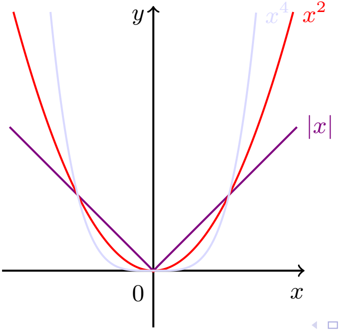

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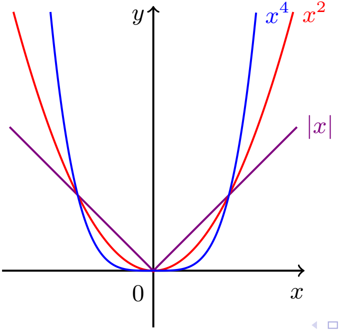

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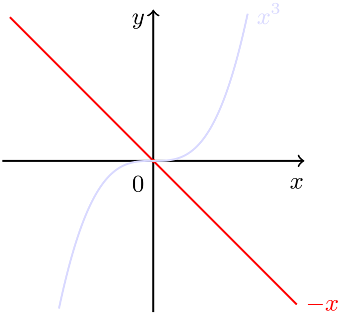

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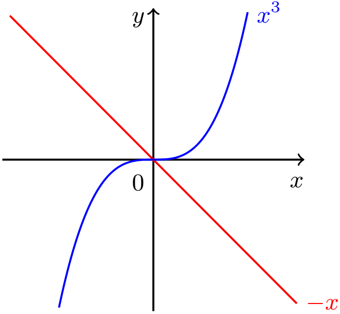

Quiz questions

Choose one correct answer (TRUE or FALSE) for each statement.

1. If f is a function then $f(x+2) = f(x) + f(2)$.
2. If $f(s) = f(t)$ then $s = t$.
3. Let f be a function. We can find s and t such that $s = t$ and $f(s)$ is not equal to $f(t)$.

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Combining Functions

Combinations of Functions

- Combining Functions with Mathematical Operations

  1. $(f+g)(x) = f(x) + g(x)$ Sum
  2. $(f-g)(x) = f(x) - g(x)$ Difference
  3. $(f \cdot g)(x) = f(x) \cdot g(x)$ Product
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Examples

1. If f and g are functions, then $f \circ g = g \circ f$.

A. True

B. False

2. Let f and g are functions described by below table

x	1	2	3	4	5	6
$f(x)$	3	2	1	0	1	2
$g(x)$	6	5	2	3	4	6

$f \circ g(3)$ is

A. 5

B. 1

C. 2

D. None of the others.

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3. Let $h(x) = (f \circ g)(x)$.

a) If $g(x) = x - 1$ and $h(x) = 3x + 2$ then $f(x)$ is:

A. $3x + 3$ B. $3x + 4$ C. $3x + 1$ D. None of them.

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- Algebraic Functions
  1. Linear Functions
  2. Power Functions
  3. Polynomials
  4. Rational Functions
- Transcendental Functions
  1. Trigonometric Functions
  2. Exponential Functions
  3. Logarithmic Functions
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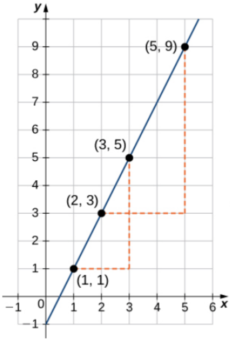
Linear Functions

When we say that y is a ***linear function*** of x, we mean that the graph of the function is a ***line***.
So, we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b,$$

where m is *the slope of the line* and b is *the y-intercept.*



$$\frac{y_2 - y_1}{x_2 - x_2} = \frac{9-3}{5-2} = 2$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{3-1} = 2$$

For any linear function, the slope $\dfrac{y_2 - y_1}{x_2 - x_1}$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

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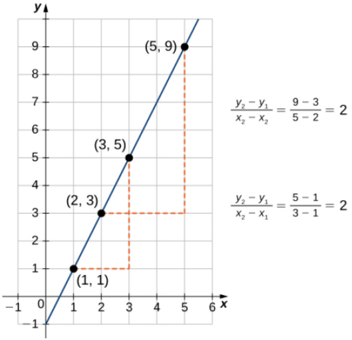

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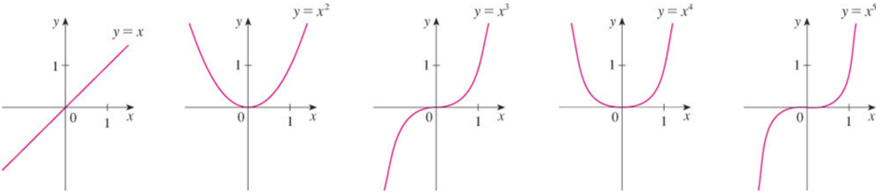
Power Functions

A function of the form $f(x) = x^a$, where a is constant, is called a ***power function***.

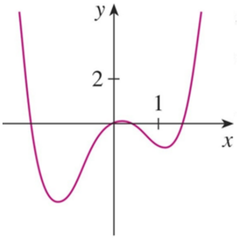


Polynomial Functions

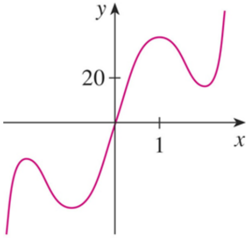
A funtion P is called a ***polynomial*** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer and the numbers $a_0,\ a_1,\ a_2, \cdots,\ a_n$ are constants called the coefficients of the polynomial.



$y = x^4 - 3x^2 + x$



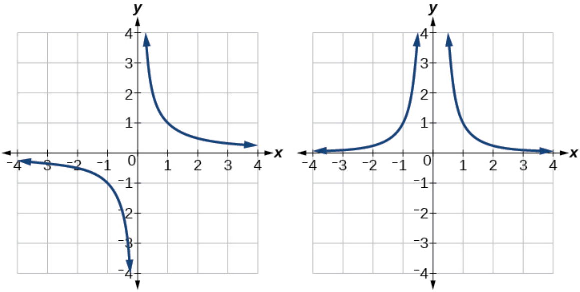
$y = 3x^5 - 25x^3 + 60x$

Rational Functions

A rational function f is a ratio of two polynomials

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials.
The domain consists of all value of x such that $Q(x) \neq 0$.

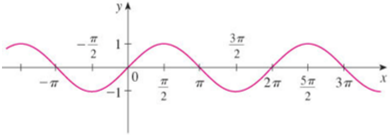


$f(x) = \frac{1}{x}$ $\quad$ $f(x) = \frac{1}{x^2}$

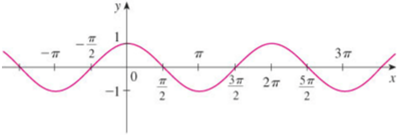
Trigonometric Functions

$$f(x) = \sin x, \qquad \textbf{D} = (-\infty, \infty)$$
$$g(x) = \cos x \qquad \textbf{R} = [-1, 1]$$

$$\sin(x + k2\pi) = \sin x, \qquad \cos(x + k2\pi) = \cos x, \qquad k \in \mathbb{Z}$$



(a) $f(x) = \sin x$

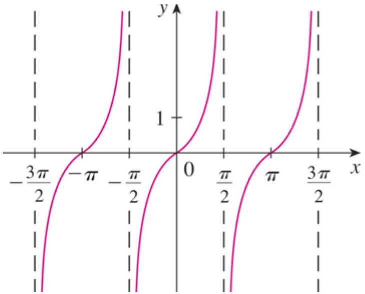


(b) $g(x) = \cos x$

Trigonometric Functions

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad x \neq \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \cdots$ $\qquad \mathbf{R} = (-\infty, +\infty)$

$$\tan(x + k\pi) = \tan x, \qquad k \in \mathbb{Z}$$



Trigonometric Functions

The reciprocals of the sine, cosine, and tangent functions are

$$\text{cosec}x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

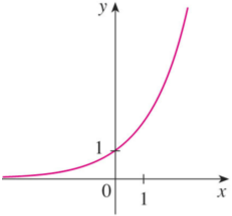
Exponential Functions

The exponential functions are the functions of the form

$$f(x) = a^x,$$

where the base a is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown. In both cases, the domain is $\mathbf{D} = \mathbb{R}$ and the range is $\mathbf{R} = (0, +\infty)$.



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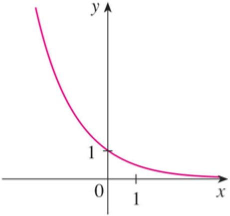


(b) $y = (0.5)^x$

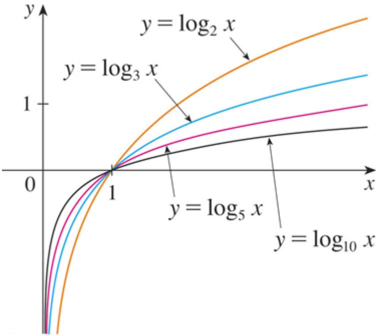
Exponential Functions

The logarithmic functions

$$f(x) = \log_a x,$$

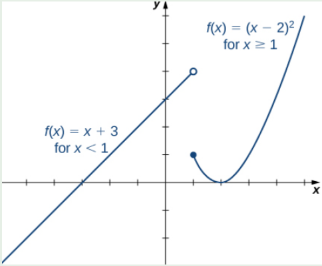
where the base a is a positive constant, are the inverse functions of the exponential functions.

The figure shows the graphs of four logarithmic functions with various bases.



Example

$$f(x) = \begin{cases} x+3 & x < 1 \\ (x-2)^2 & x \geq 1 \end{cases}$$ is a piecewise-defined function



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Transformations of f $(c > 0)$	Effect on the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $c > 1$; Horizontal compression if $0 < c < 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Example

Suppose $c > 0$.
To obtain
the graph of $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward.

To obtain the graph of $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward.

To obtain the graph of $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left.

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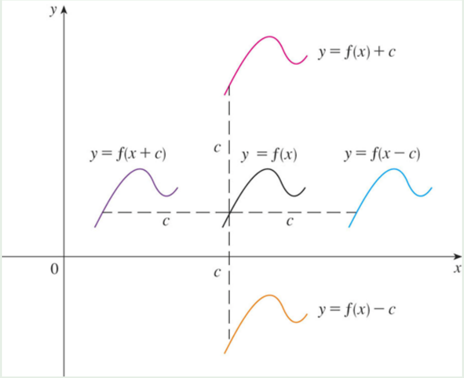

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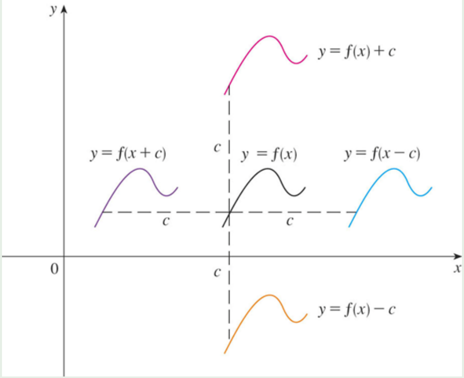

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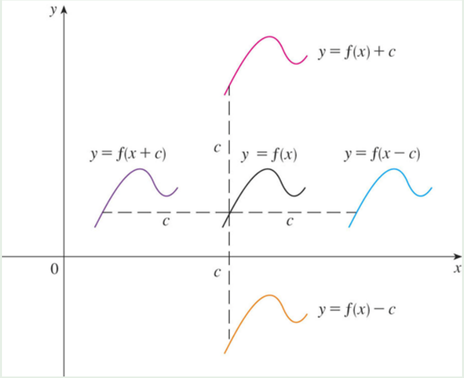

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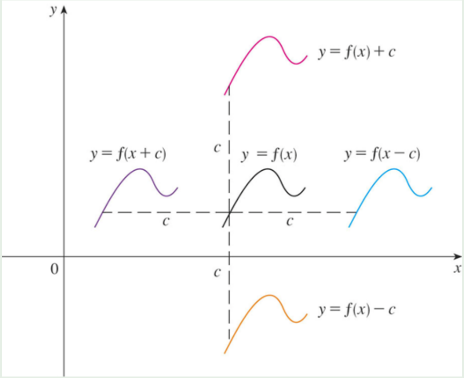

Example

Suppose $c > 1$.
To obtain
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To obtain the graph of $y = \frac{1}{c} f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c.

To obtain the graph of $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c.

To obtain the graph of $y = f\left(\frac{1}{c} x\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of c.

To obtain the graph of $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

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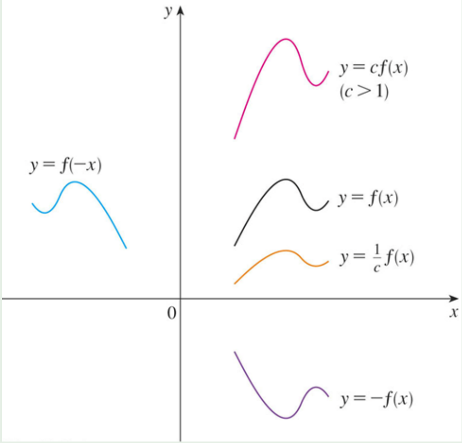

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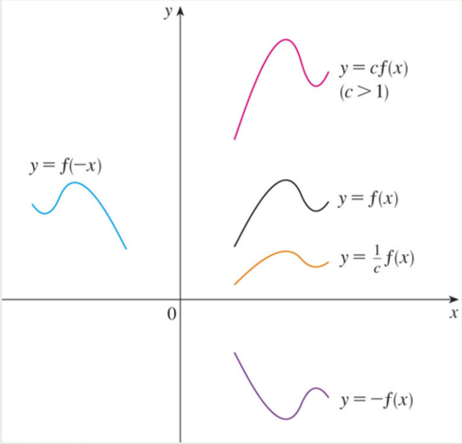

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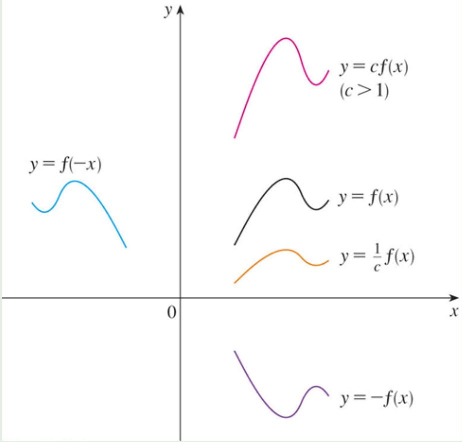

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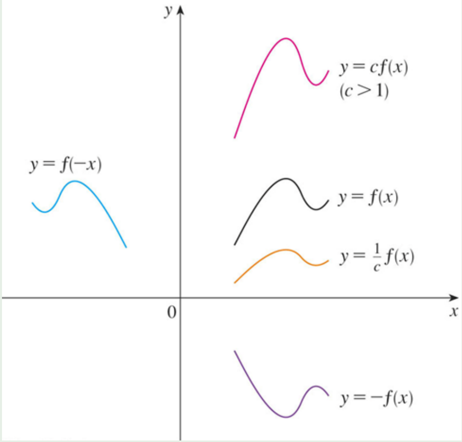

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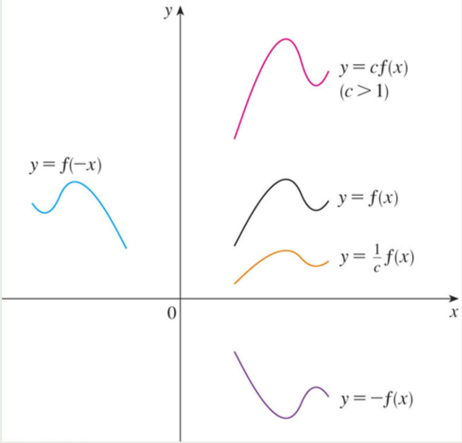

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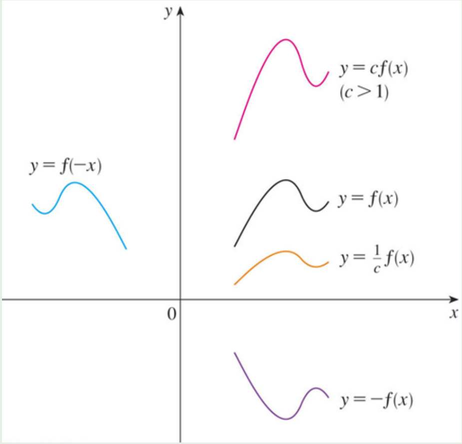

Example

Suppose that the graph of f is given.
Describe how the graph of the function $f(x-2)+2$ can be obtained from the graph of f.
Select the correct answer.

A. Shift the graph 2 units to the left and 2 units down.
B. Shift the graph 2 units to the right and 2 units down.
C. Shift the graph 2 units to the left and 2 units up.
D. Shift the graph 2 units to the right and 2 units up.
E. None of these

THANK YOU!