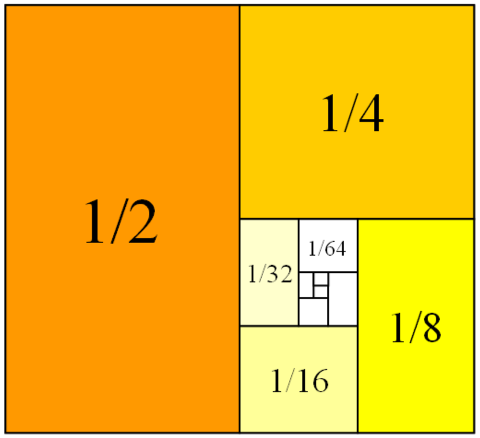


Chapter 4: APPLICATIONS OF DERIVATIVES

Department of Mathematics, FPT University

Contents

4.1
Related Rates

In this section, we will learn

How to compute the rate of change of one quantity in terms of that of another quantity.

Example 1

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/sec.

How fast is the radius of the balloon increasing when the diameter is 50 cm?

The key thing to remember is that rates of change are derivatives.

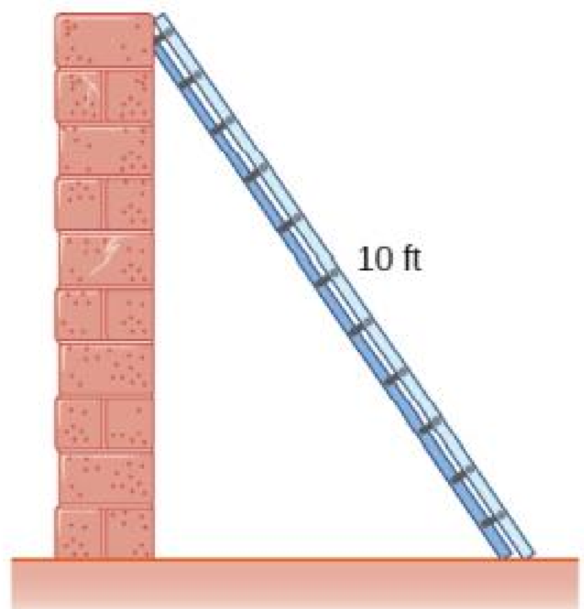
$$V = \frac{4}{3}\pi r^3$$

Chain rule: $$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

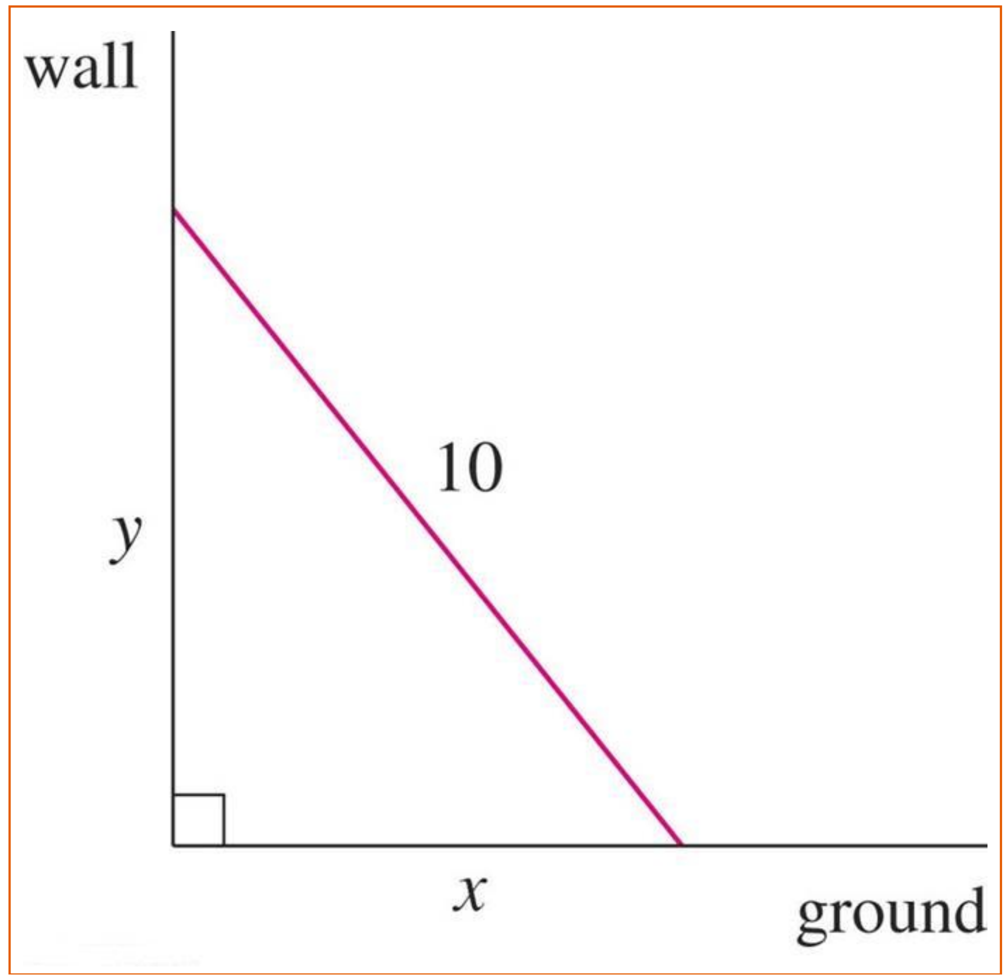
Example 2

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



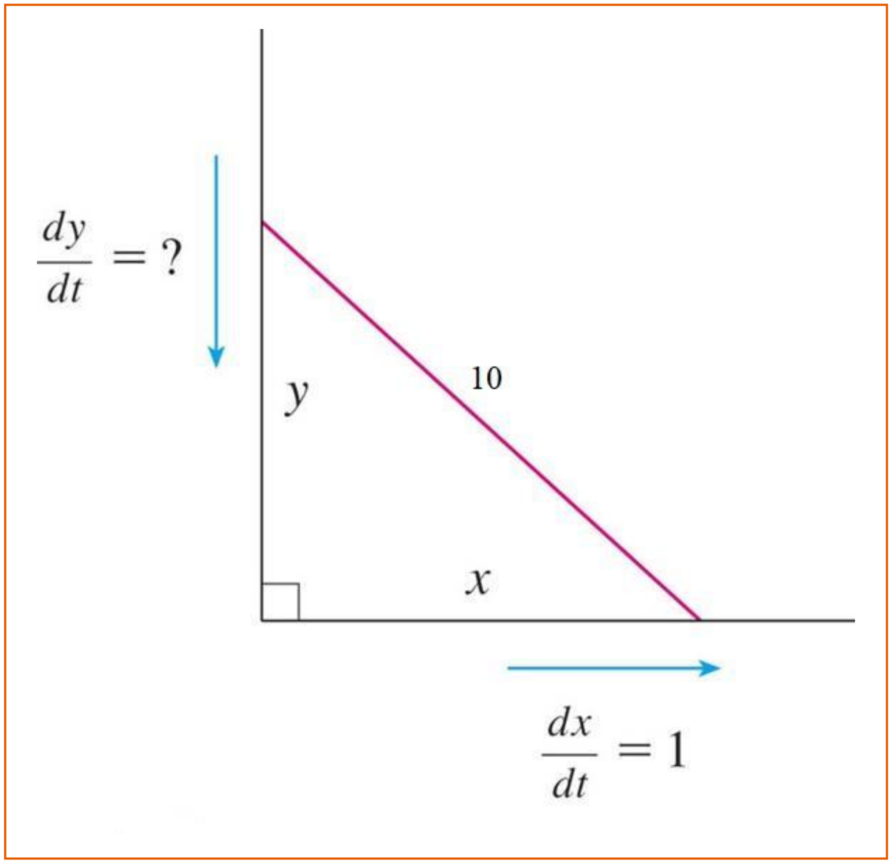
Let x denote the distance from the bottom of the ladder to the wall and y denote the distance from the top of the ladder to the ground.

Note that x and y are both functions of t (time)



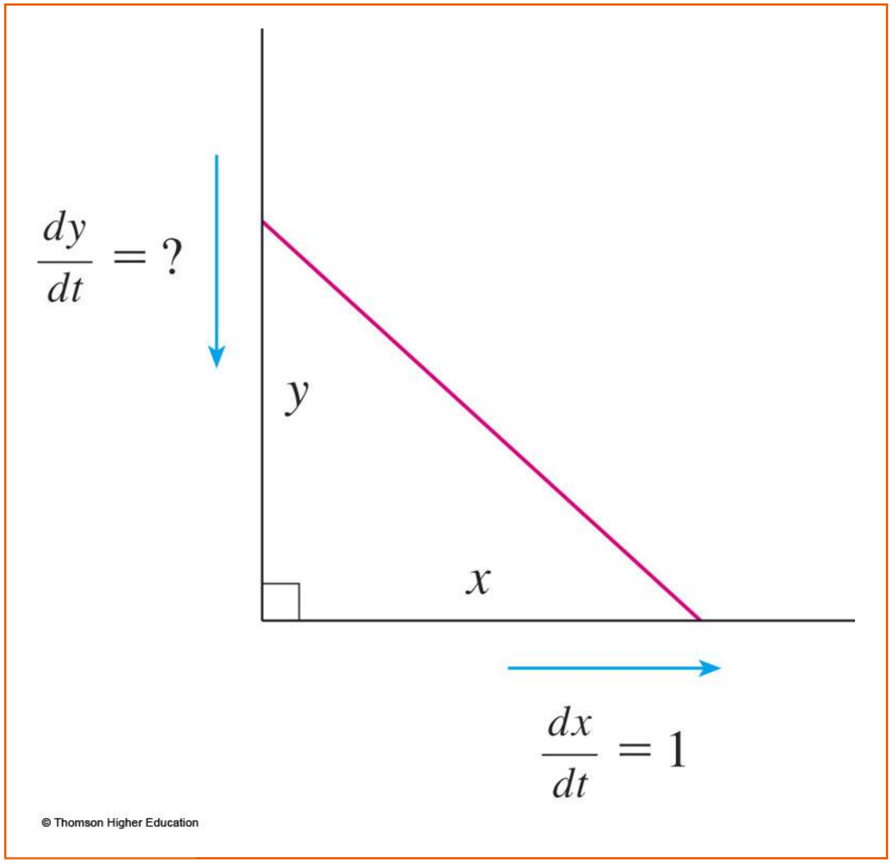
Given that $\frac{dx}{dt} = 1$ ft/sec

and we need to find $\frac{dy}{dt}$ when $x = 6$ ft.



Pythagorean Theorem: $x^2 + y^2 = 100$

Chain rule:



$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{6}{8}.1 = -\frac{3}{4} \text{ ft/sec}$$

4.2

Linear Approximations and Differentials

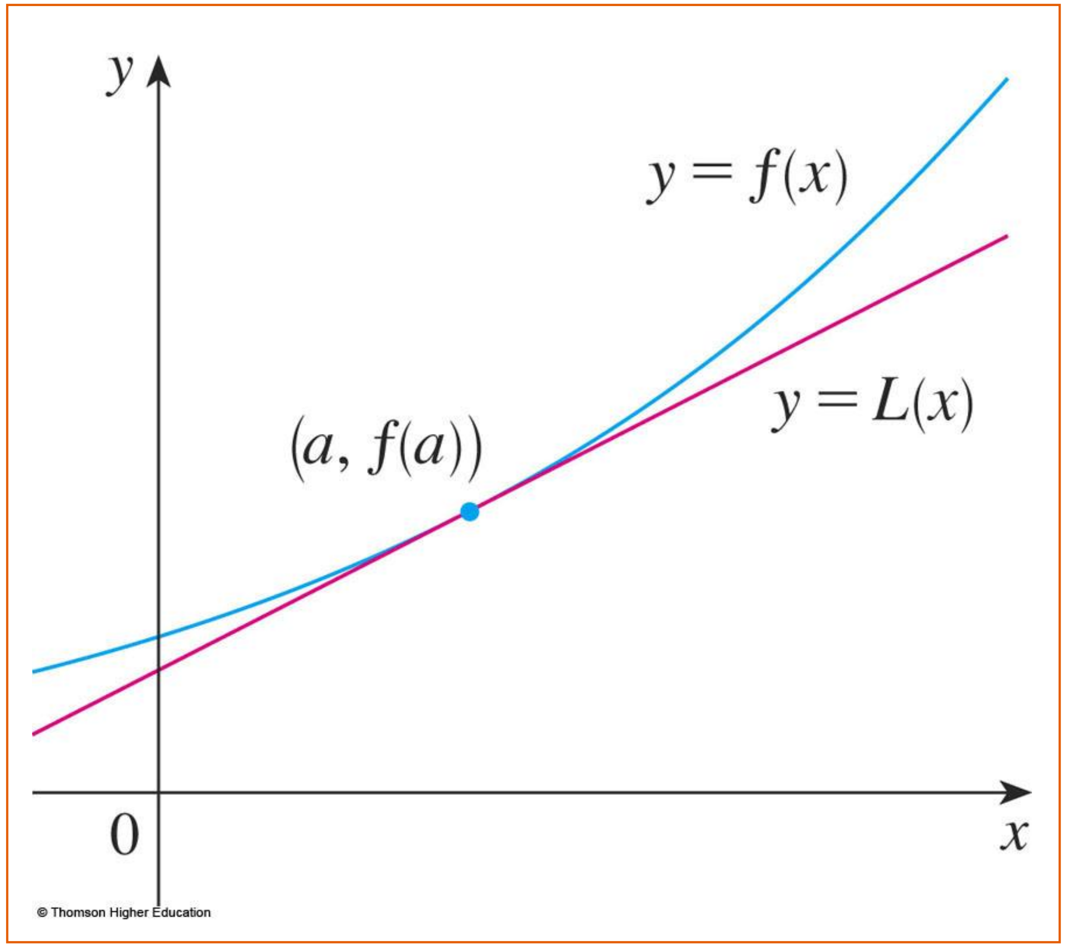
In this section, we will learn about linear approximations and differentials and their applications.

LINEAR APPROXIMATIONS

Tangent line: $L(x) = y = f(a) + f'(a)(x - a)$

When x is near a:

$$f(x) \approx L(x)$$
$$\approx f(a) + f'(a)(x - a)$$



Example 1

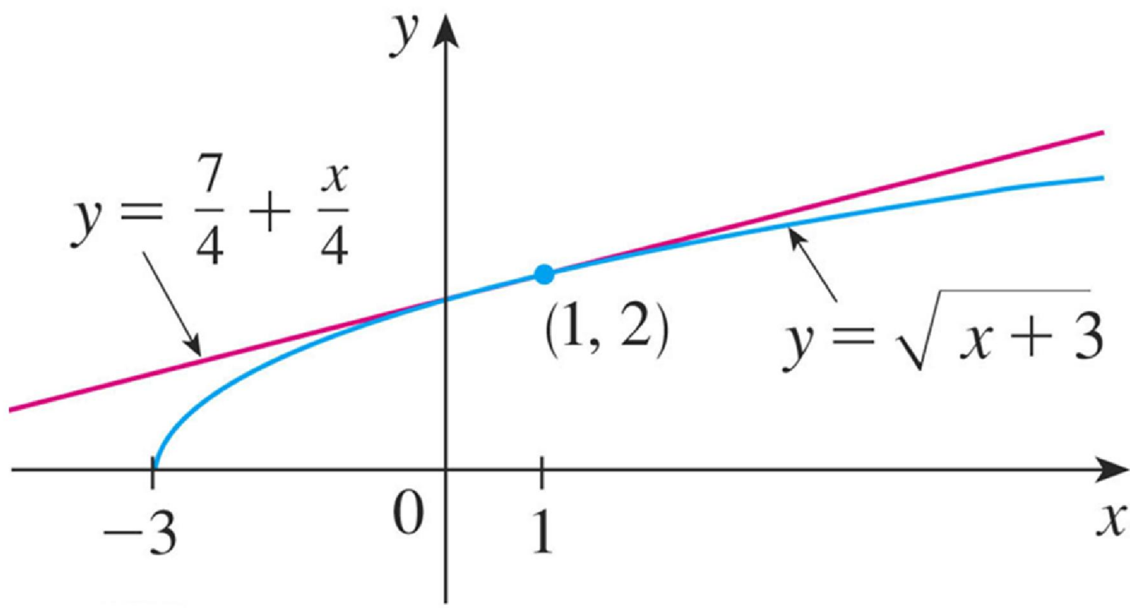
Find the linearization of the function $f(x)=\sqrt{x+3}$

at $x = 1$ and use it to approximate the numbers

$\sqrt{3.98}$ and $\sqrt{4.05}$

The linear approximation is the tangent line of $f(x)$ at $x = 1$.

$$\begin{aligned} L(x) &= f(1)+f'(1)(x-1) \\ &= 2+\tfrac{1}{4}(x-1) \\ &= \frac{7}{4}+\frac{x}{4} \end{aligned}$$



Example 1

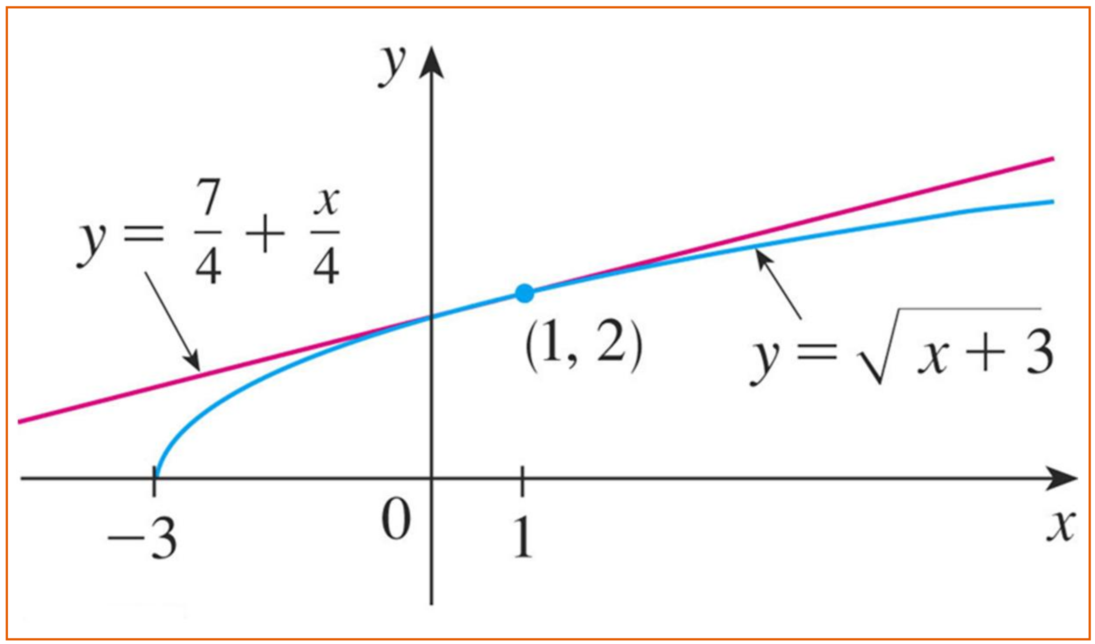
The corresponding linear approximation is

$$\boxed{\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

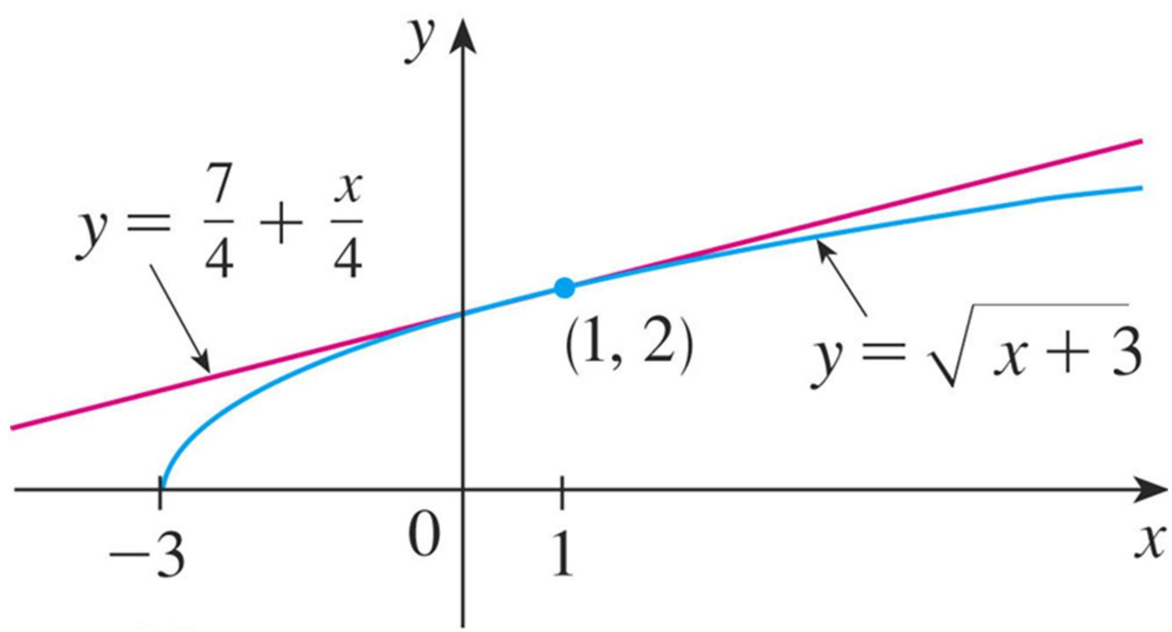


LINEAR APPROXIMATIONS

- The tangent line approximation gives good estimates

  if *x* is close to 1.

- However, the accuracy decreases when *x* is further away from 1.



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Use the linear approximation to of the function $f(x)=\sqrt{x+1}$ at $x = 3$ to approximate the numbers. Select the correct ones.

a) $\sqrt{3.95} \approx 1.9875$

b) $\sqrt{4.05} \approx 2.0125$

c) $\sqrt{3.95} \approx 1.98746$

d) $\sqrt{4.05} \approx 2.01246$

$$f(x) \approx f(a) + f'(a)(x - a)$$

Use the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ to approximate the numbers. Select the most correct answer.

A. $\sin 62° \approx 0.88$

$B.\ \sin 62° \approx 0.967$

$C.\ \cos 0° = 1$

$D.\ \sin 81° \approx 0.981$

DIFFERENTIALS

Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$\boxed{dy = f'(x)dx}$$

Differential form

Note: dy is a functions of both x and dx.

dy and dx are called differentials.

a.For $y = x^2 + 2x$, find dy.
b.Find dy and evaluate when $x = 5$ and $dx = 0.1$
Compare to Δy.

<u>Solution</u>

a. $dy = f'(x)dx$

Hence, $dy = (2x + 2).dx$

b. $dy = (2.5 + 2)(0.1) = 1.2$

$\Delta y = y(5.1) - y(5) = (5.1^2 + 2.(5.1)) - (5^2 + 2.5)$

$\Delta y = 1.21$

- If x changes from a to $a + dx$, then the change of x is $\Delta x = dx$.
- The change of y is $\Delta y = f(a + dx) - f(a)$.
- Linear approximation:

$$f(a + dx) \approx f(a) + f'(a)dx$$
$$f(a + dx) - f(a) \approx f'(a)dx$$
$$\Delta y \approx dy$$

Example: Given $y = x^2 + 2x$, when $x = 5$ and $dx = 0.1$

- dy =1.2
- $\Delta y = 1.21$

Calculating the Amount of Error

Example

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm.

What is the maximum error in using this value of the radius to compute the volume of the sphere?

If the radius of the sphere is r, then its volume is

$$V = 4/3\pi r^3$$

The error in the measured value of r is $\Delta r = dr$.

The error in the calculated value of V is $\Delta V \approx dV$.

$$dV = 4\pi r^2 dr$$

$$dV = 4\pi(21)^2\, 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3

RELATIVE ERROR

Relative error in the radius: $\frac{\Delta r}{r} = \frac{0.05}{21} \approx 0.0024$

Relative error in the volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r} = 3\frac{\Delta r}{r}$$

The relative error in the volume is about three times the relative error in the radius.

Find the limit if $g(x) = \ln x$

$$\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2}$$

a. 1 b. $\frac{1}{2}$ c. $\frac{1}{3}$ d. 2

Find $\dfrac{dy}{dt}$ when $x = 2$ if $\dfrac{dx}{dt} = 1,\ y = x^3 - 3x^2$

a. 1 b. 2 c. 3 d. 4 e. 0

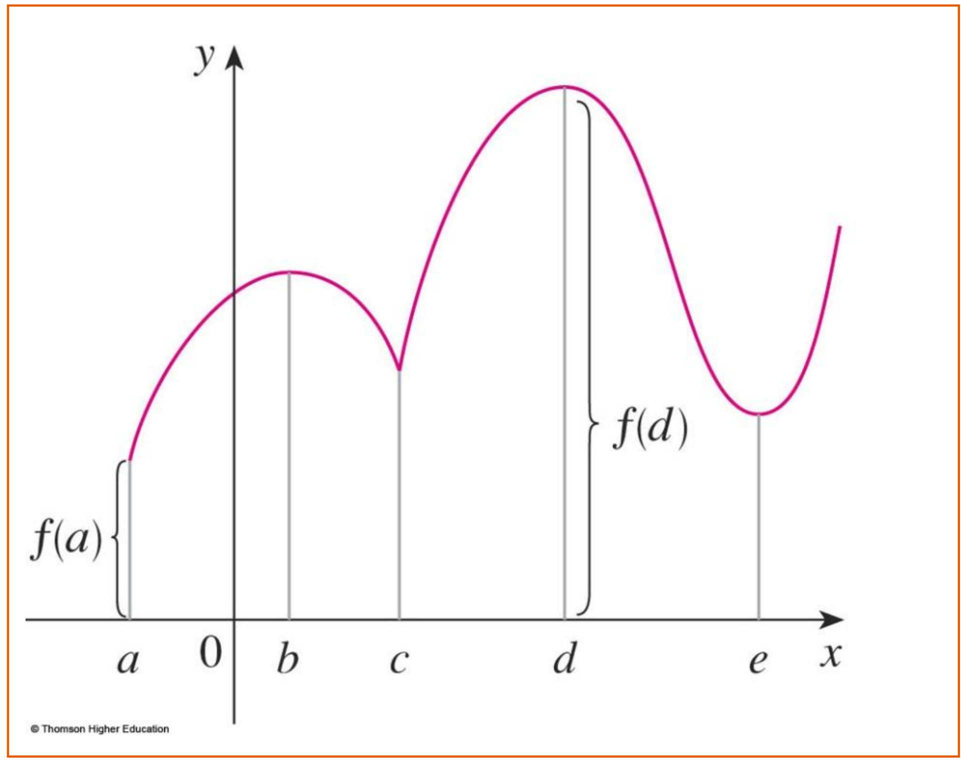
4.3
Maxima and Minima

In this section, we will learn
How to find the maximum
and minimum values of a function.

MAXIMUM & MINIMUM VALUES

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f.

The number $f(c)$ is called the maximum value of f on D.



MAXIMUM & MINIMUM VALUES

f has an absolute minimum at *c* if $f(c) \le f(x)$ for all *x* in *D* and the number $f(c)$ is called the minimum value of *f* on *D*.

The maximum and minimum values of *f* are extreme values of *f*.

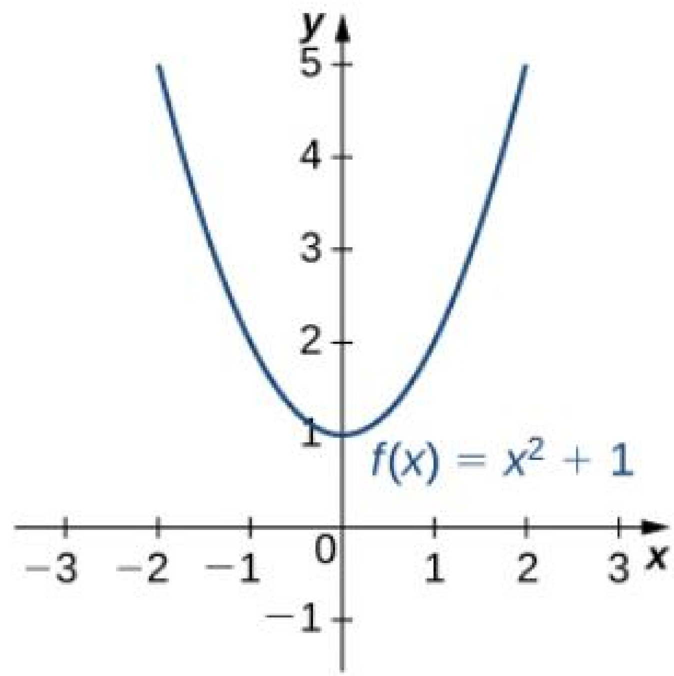
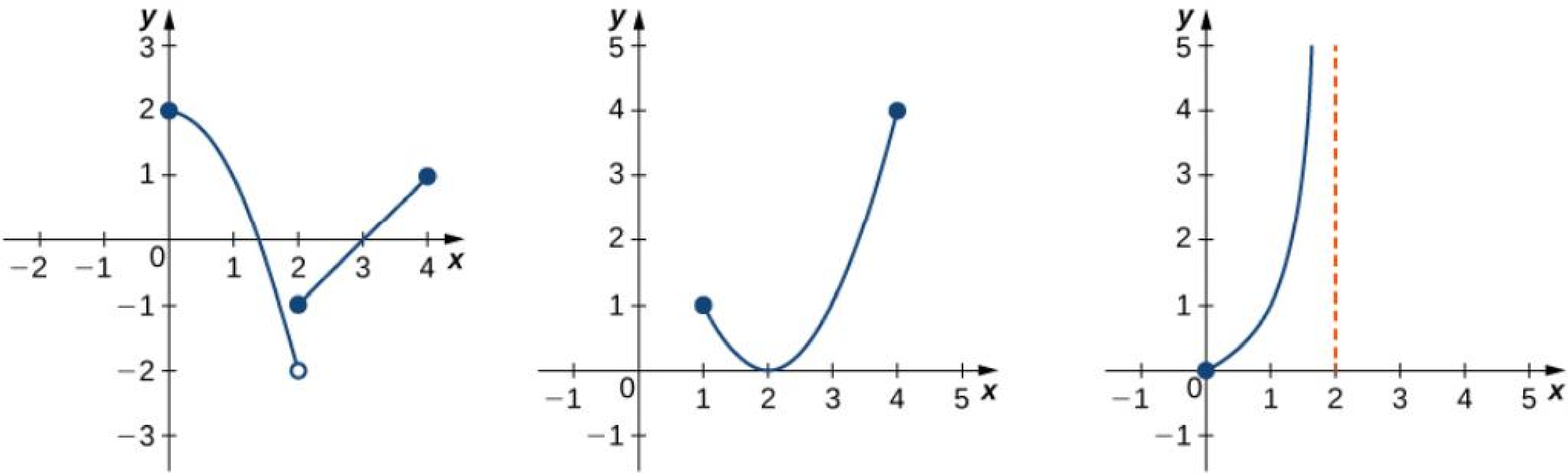


Figure 4.12 The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.



$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$

Absolute maximum of 2 at $x = 0$

No absolute minimum

(d)

$f(x) = (x - 2)^2$ on $[1, 4]$

Absolute maximum of 4 at $x = 4$

Absolute minimum of 0 at $x = 2$

(e)

$f(x) = \frac{x}{2 - x}$ on $[0, 2)$

No absolute maximum

Absolute minimum of 0 at $x = 0$

(f)

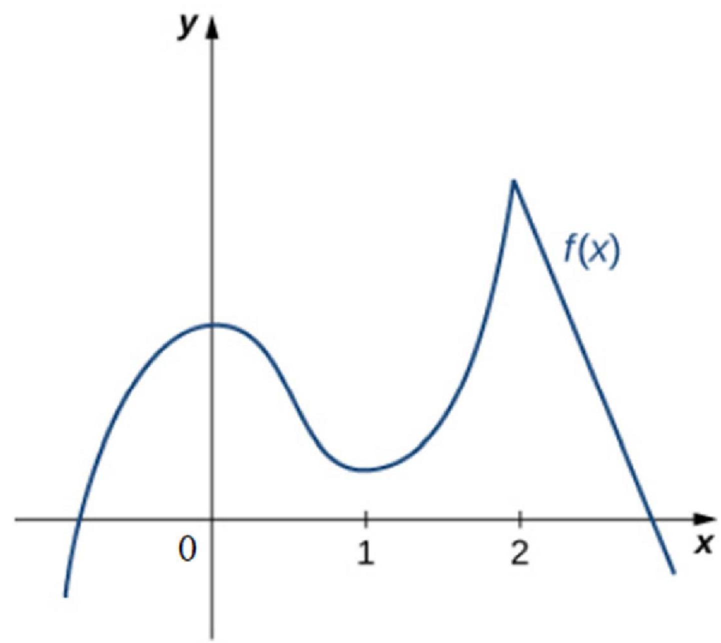
MAXIMUM & MINIMUM VALUES

- A function f has a **local maximum** (or relative maximum) at c

  if $f(c) \geq f(x)$ when ***x* is near *c***.

- Similarly, f has a **local minimum**

at c if $f(c) \leq f(x)$ when ***x* is near *c***.



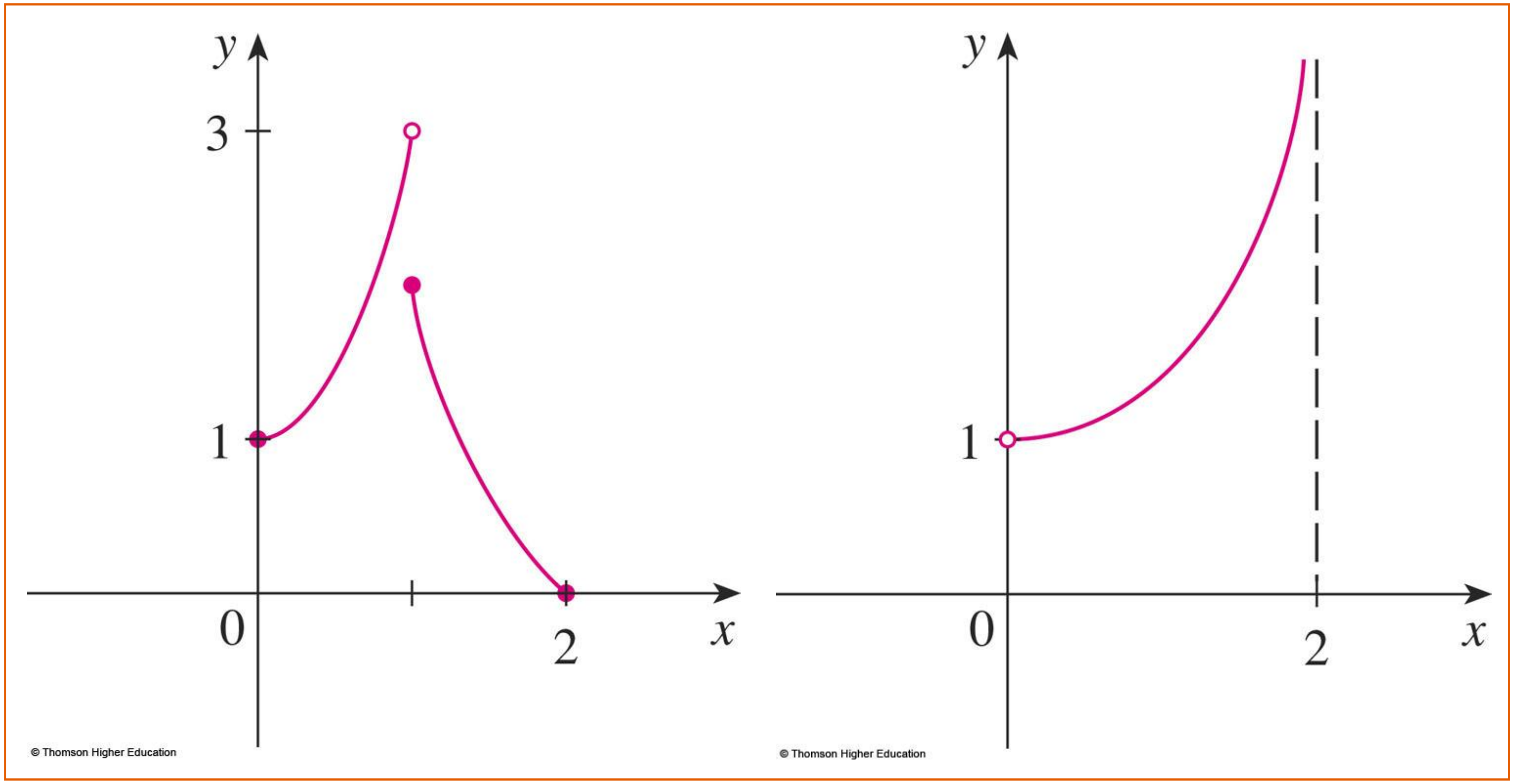
$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 4.14 This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

EXTREME VALUE THEOREM

In the first figure, why isn't 3 the absolute maximum value?

In the second, does it have the absolute maximum and minimum value?



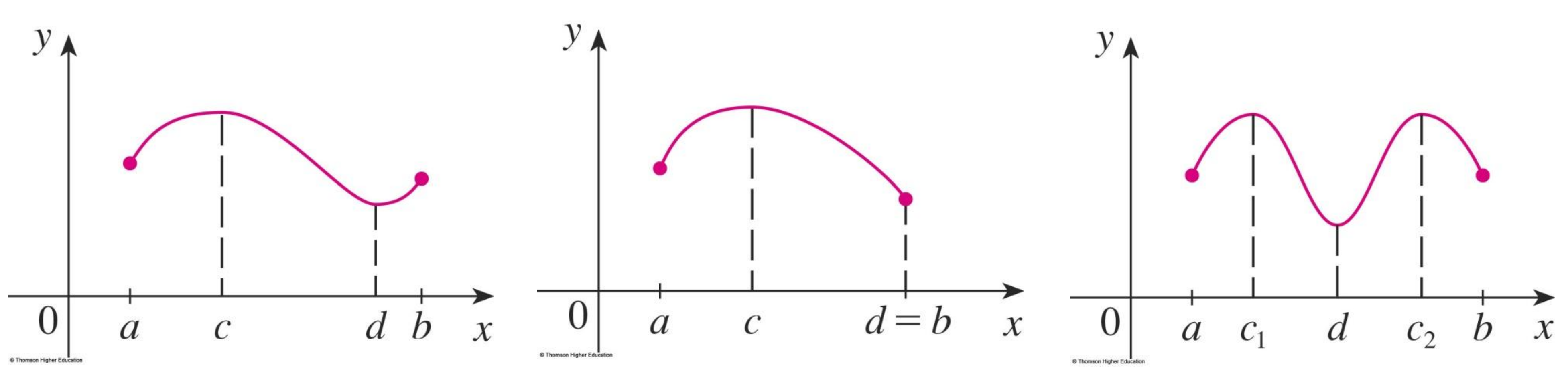
EXTREME VALUE THEOREM

If *f* is continuous on a closed interval [*a*, *b*],

then *f* attains an absolute maximum value *f*(*c*)

and an absolute minimum value *f*(*d*)

at some numbers *c* and *d* in [*a*, *b*].

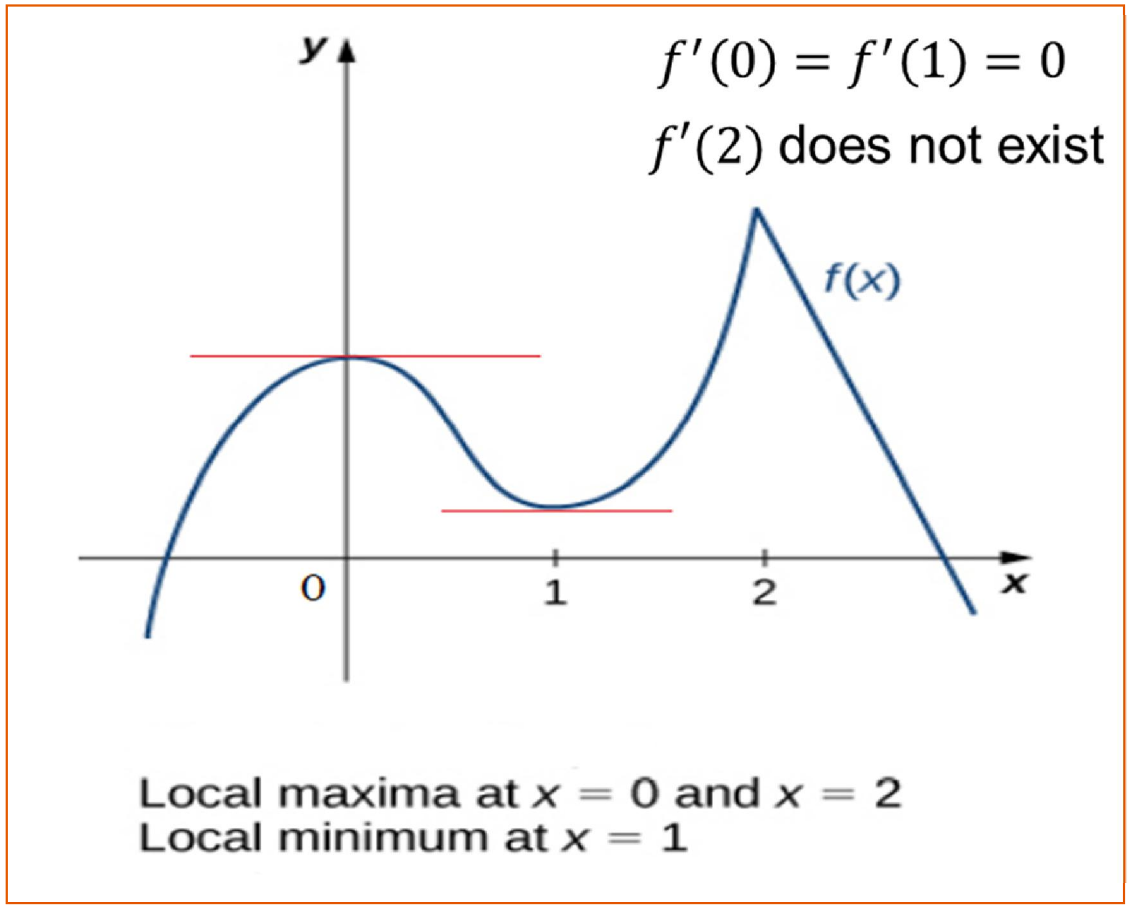


EXTREME VALUE THEOREM

The theorem does not tell us how to find these extreme values.

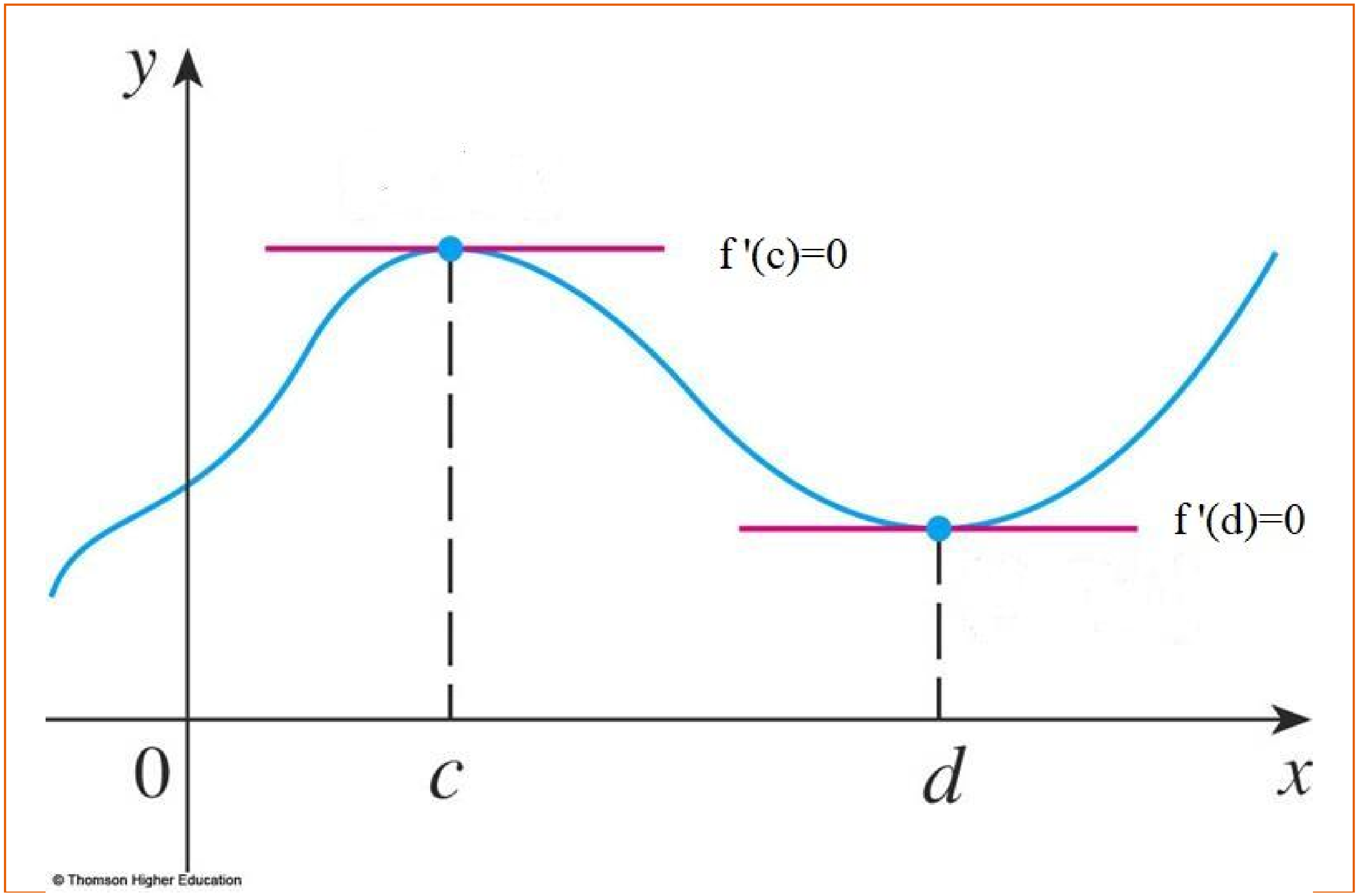
We start by looking for local extreme values.

$c \in Domain$ is called a critical number of f if f ′(c) = 0 or f ′(c) does not exist.



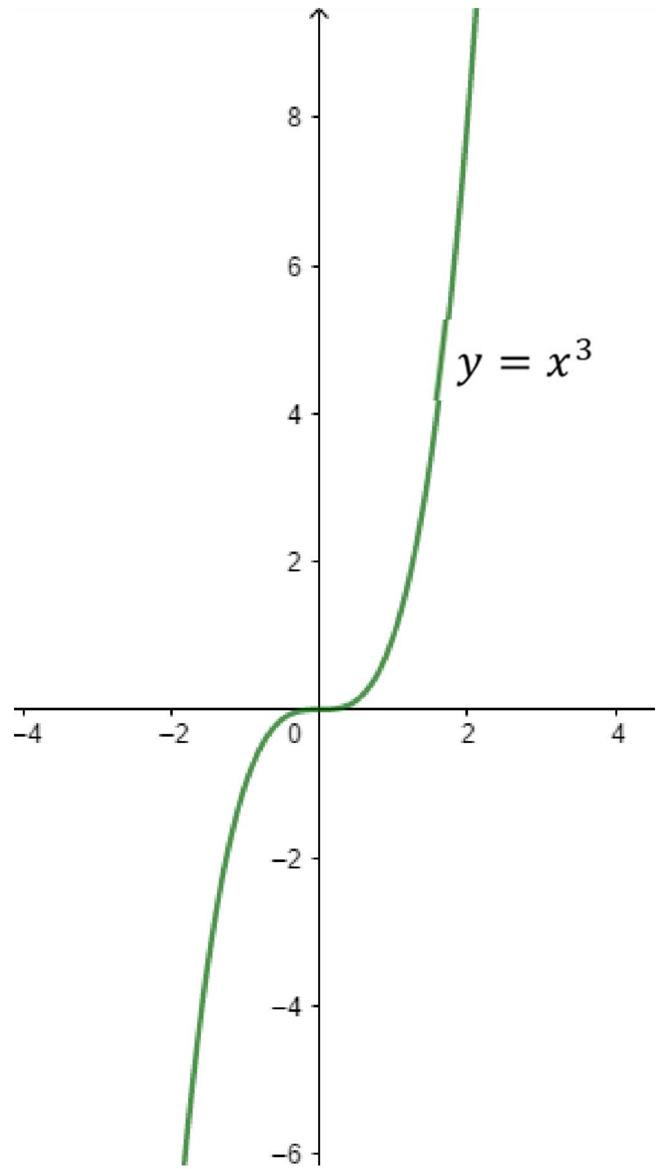
FERMAT'S THEOREM

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



Given $f'(c) = 0$.
Can we conclude that f has local extreme value at c?

No



Theorem

If f has a local maximum or minimum at c, then c is a critical number of f

Find all critical points for $f(x) = x^3 - x$.
Sketch the graph to determine whether the function has a local extremum at each of the critical points.

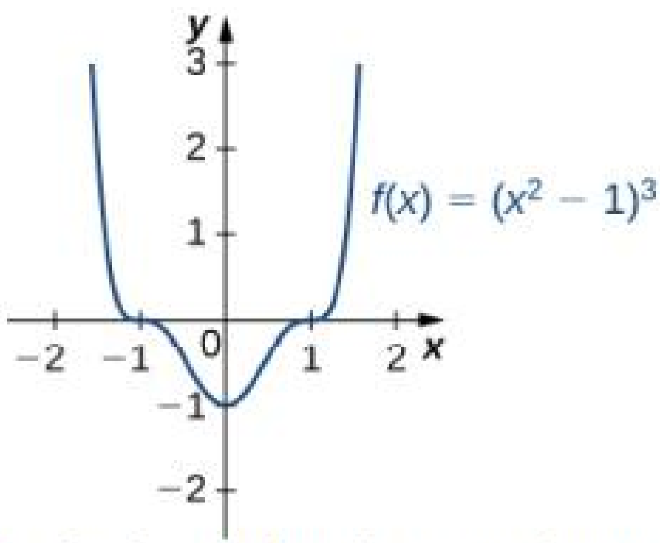
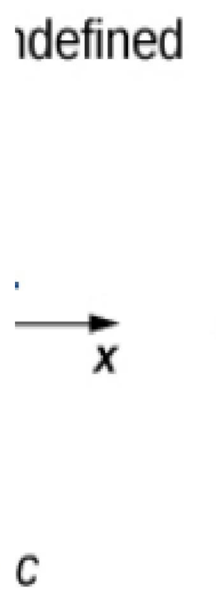
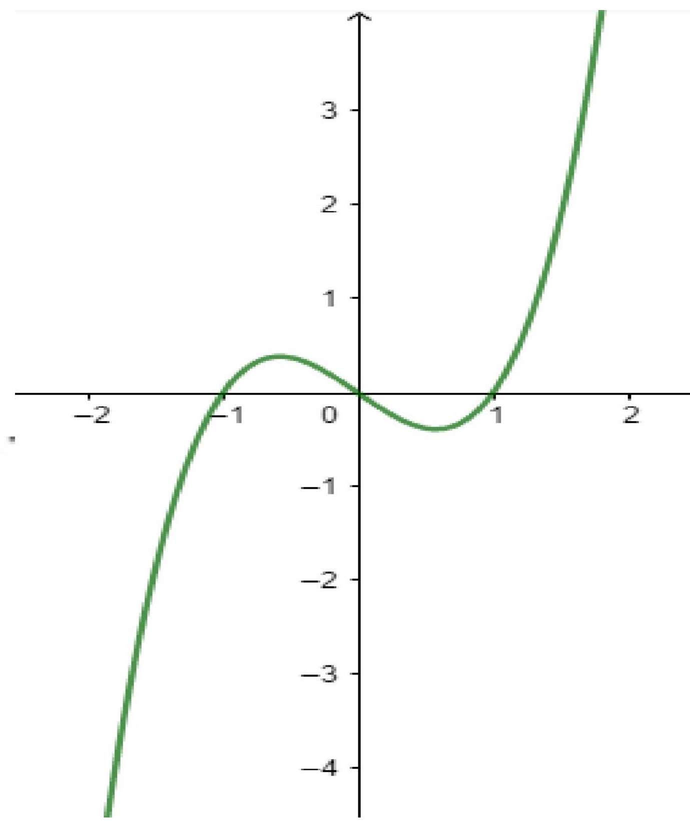


Figure 4.17 This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.





To find the **absolute** maximum and minimum values of a continuous function *f* on a closed interval [*a*, *b*]:

1. Find the values of *f* at the critical numbers of *f* in (*a*, *b*).

2. Find the values of *f* at the endpoints of the interval.

3. The largest value from step 1 and step 2 is the absolute maximum value. The smallest is the absolute minimum value.

Exercise: Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over [1,4]

Select the correct ones.

a. If $f'(c)=0$ then f has the local maximum or minimum at c.

b. If f has the absolute minimum value at c then $f'(c)=0$.

c. If f is continuous on (a,b) then f attains an absolute maximum value f(c) and an absolute minimum value f(d) for some c and d in (a,b).

d. All of the above.

e. None of the above.

4.4
The Mean Value Theorem

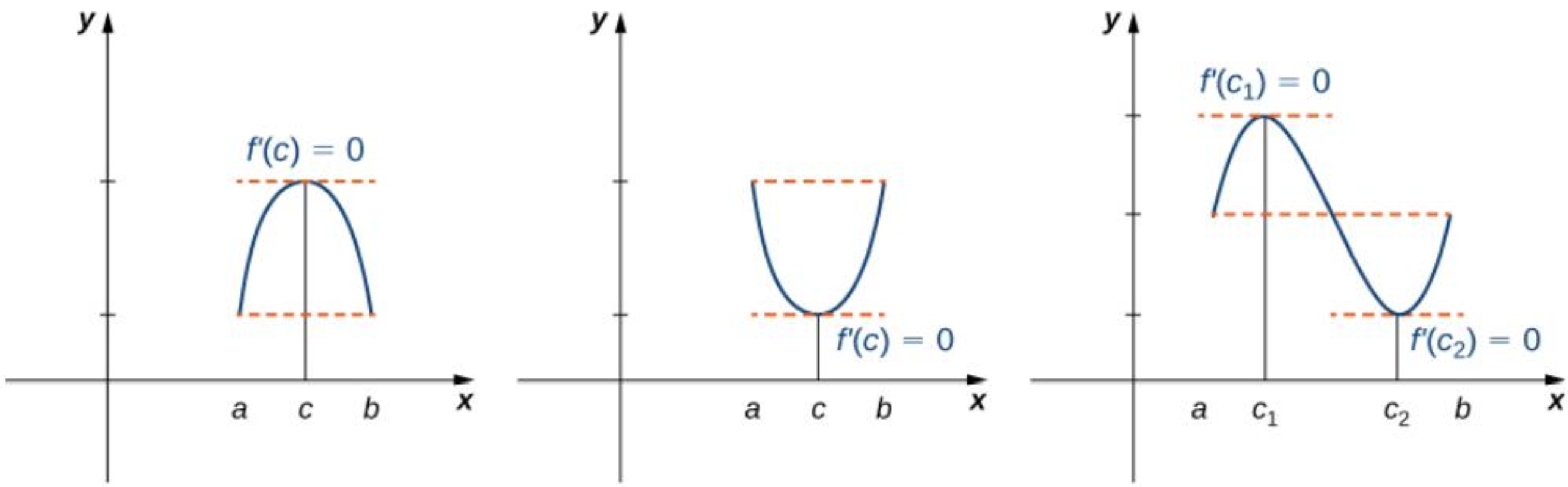
In this section, we will learn about the significance of the Mean Value Theorem.

ROLLE'S THEOREM

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then, there is a number c in (a, b) such that $f'(c) = 0$.



(a) (b) (c)


Let $f(x) = 2x^2 - 8x + 6$ defined over the interval [1,3]. Verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where f ′(c) = 0.

Solution

- $f(x)$ is a polynomial

=> it is continuous and differentiable everywhere.

- $f(1) = f(3)$
- $f'(x) = 4x - 8$

Hence, $f'(x) = 0 \Leftrightarrow x = 2$

MEAN VALUE THEOREM

Let f be a function that fulfills two hypotheses:

1. f is continuous on the closed interval [a, b].

2. f is differentiable on the open interval (a, b).

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

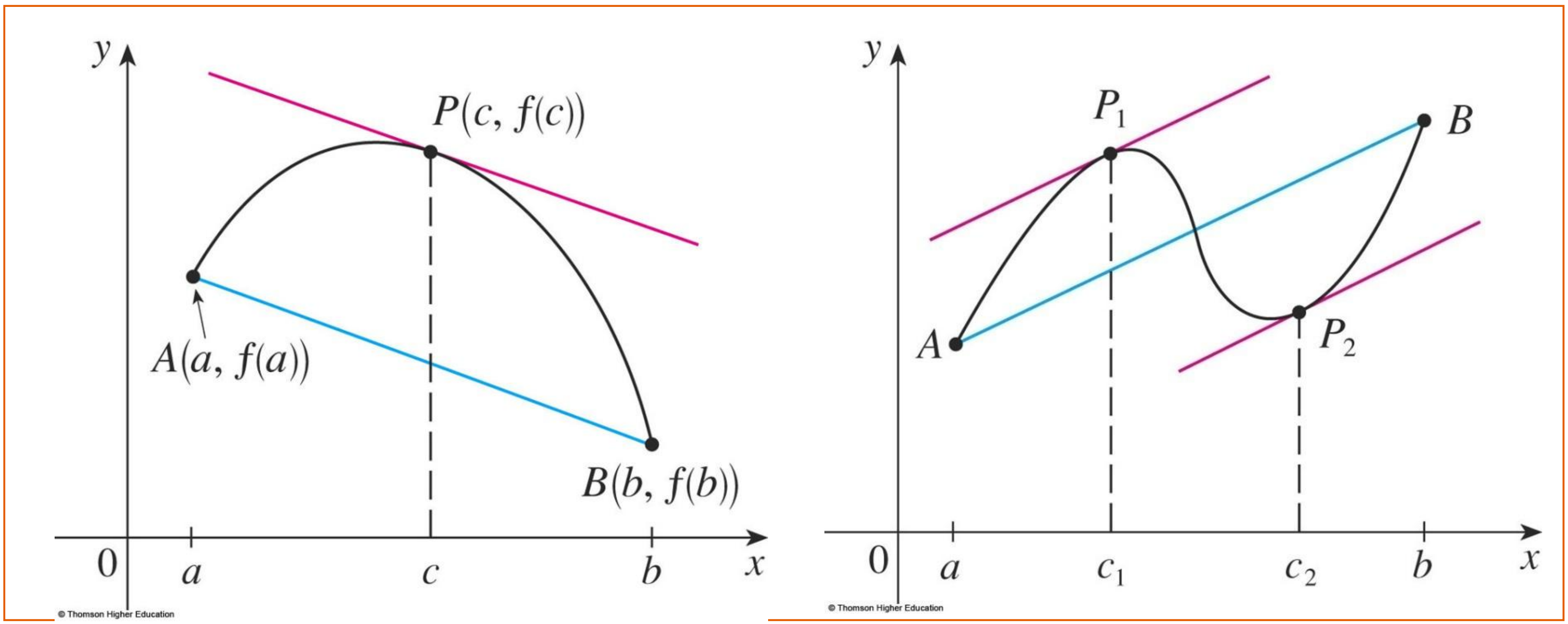
or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

MEAN VALUE THEOREM

Assume f is continuous over [a, b] and differentiable over (a, b).

There is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.



Example 1

Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can $f(2)$ possibly be?

Since f is differentiable ($f'(x) \le 5 \ \forall x$), f is continuous everywhere.

In particular, we can apply the Mean Value Theorem on [0, 2].

There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

$$f(2) = -3 + 2f'(c)$$

$$\Rightarrow \qquad f(2) \le -3 + 10 = 7$$

Hence, the largest possible value for $f(2)$ is 7.

Example 2

Find number $c \in [1,3]$ that satisfy the Mean Value Thoerem for

$f(x) = \frac{1}{x}$?

Example 3

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function s(t) = $-16t^2+100$.

a) Determine how long it takes before the rock hits the ground.

b) Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.

c) Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

MEAN VALUE THEOREM

Corollary 1

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary 2

If $f'(x) = g'(x)$ for all x in an interval (a, b),

then $f - g$ is constant on (a, b).

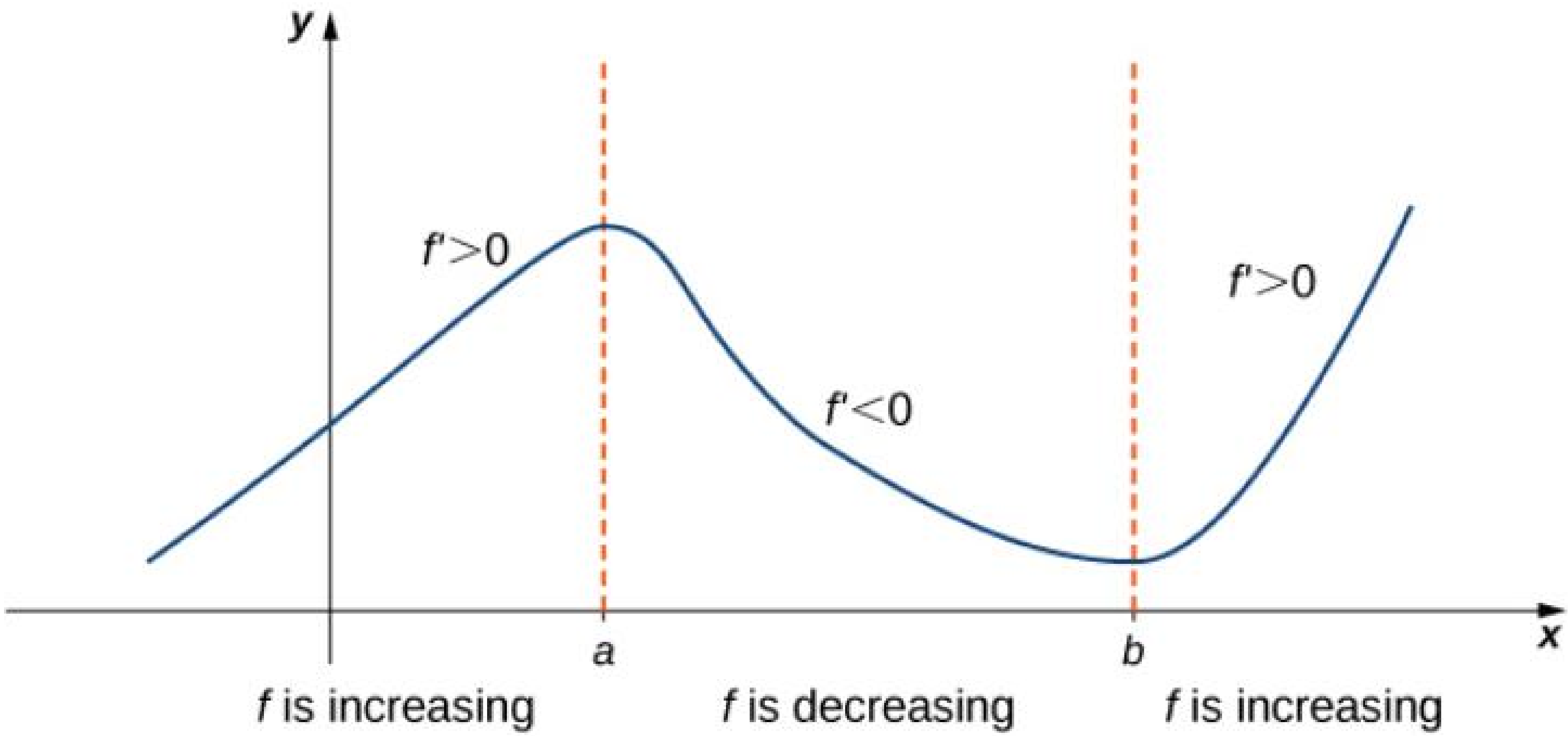
That is, $f(x) = g(x) + c$ where c is a constant.

Corollary 3

Let f be continuous over the closed interval [a, b] and differentiable over the open interval (a, b).

a. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over [a, b].

b. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over [a, b].



4.5
Derivatives and the Shapes of Graphs

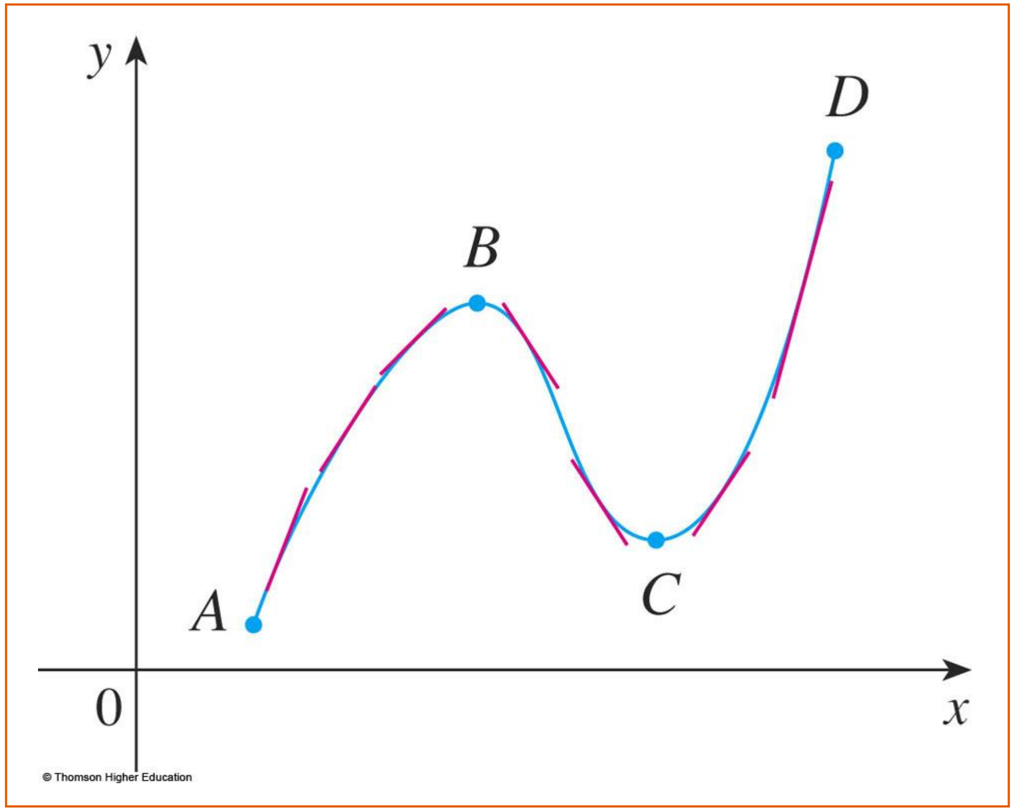
In this section, we will learn

How the derivative of a function gives us the direction in which the curve proceeds at each point.

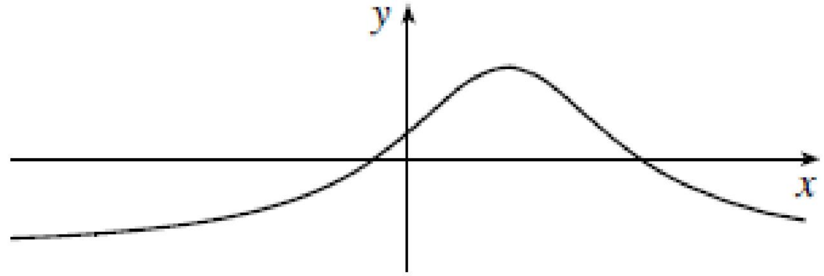
The increasing/decreasing test

a. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

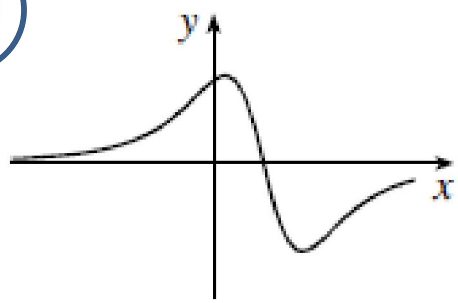
b. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



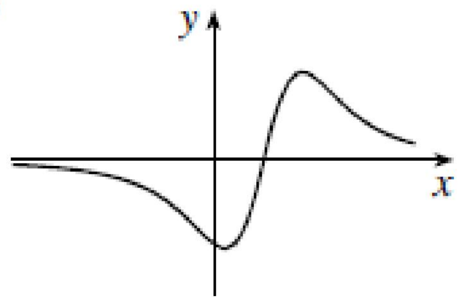
• **Drill Question:** The graph of f is shown below. Which of the following could be the graph of f'?



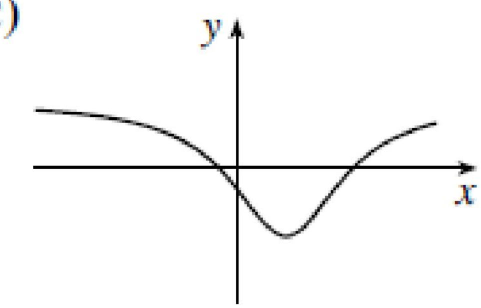
(A)



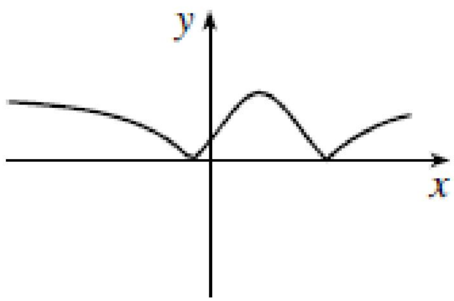
(B)



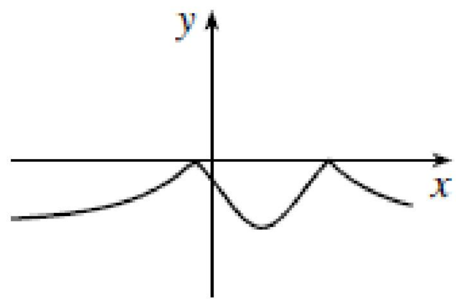
(C)



(D)

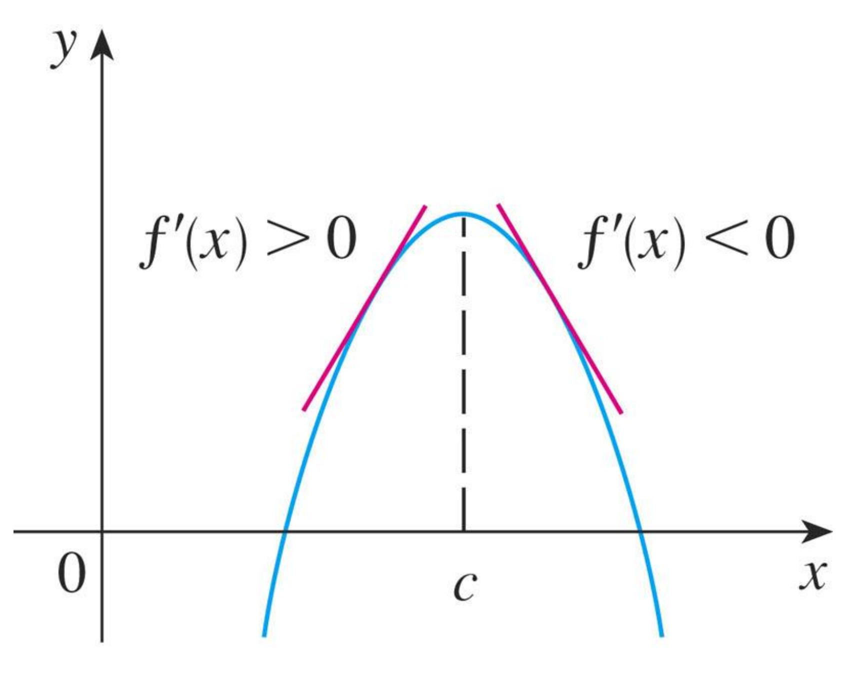


(E)



FIRST DERIVATIVE TEST

Suppose that *c* is a critical number of a continuous function *f*. If f' changes from positive to negative at *c*, then *f* has a local maximum at *c*.

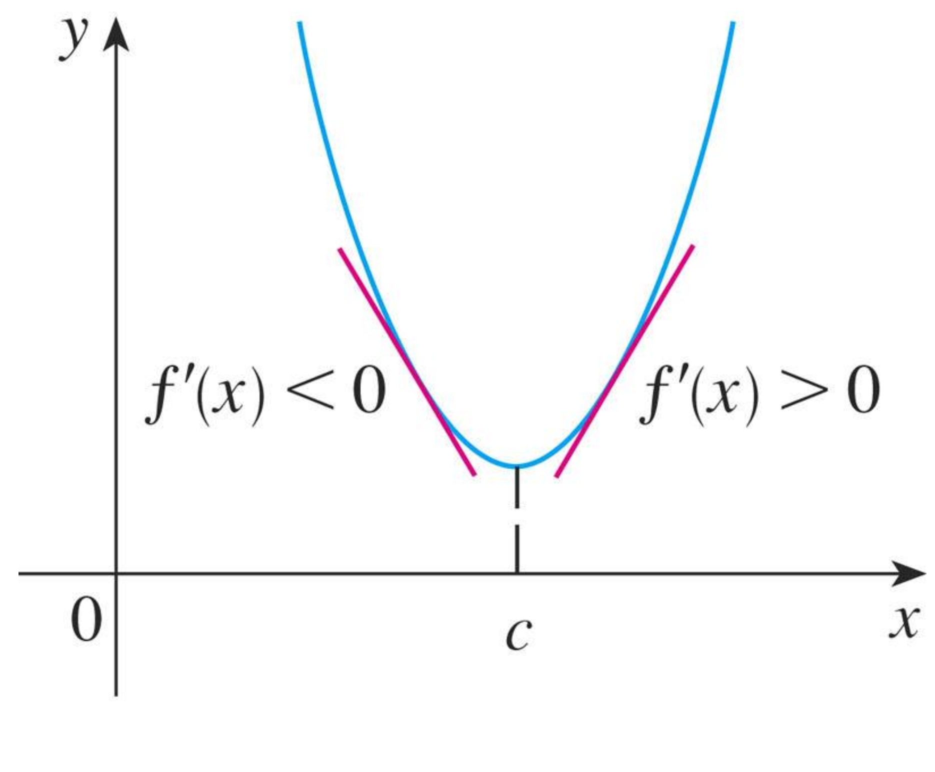


(a) Local maximum

FIRST DERIVATIVE TEST

Suppose that *c* is a critical number of a continuous function *f*.

If *f* ' changes from negative to positive at *c*, then *f* has a local minimum at *c*.

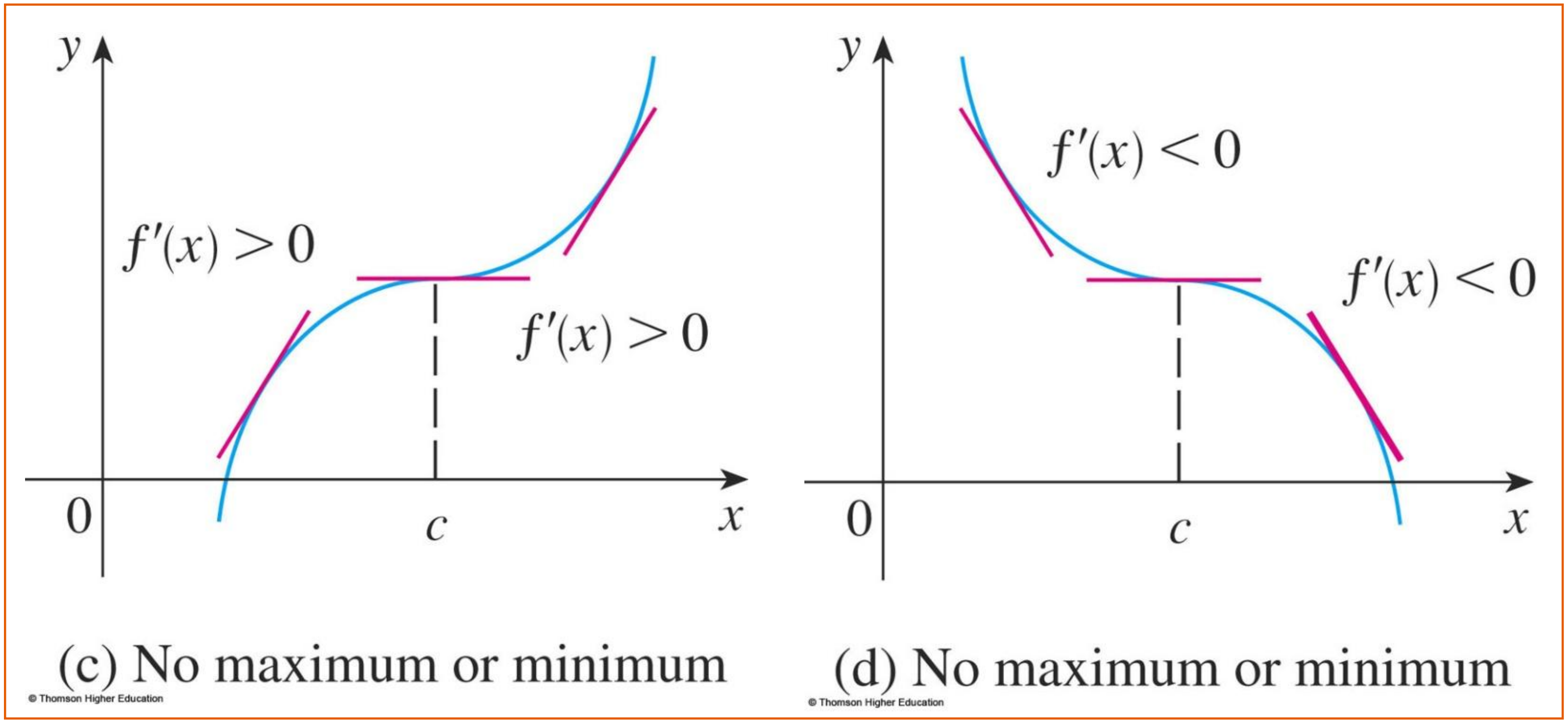


(b) Local minimum

FIRST DERIVATIVE TEST

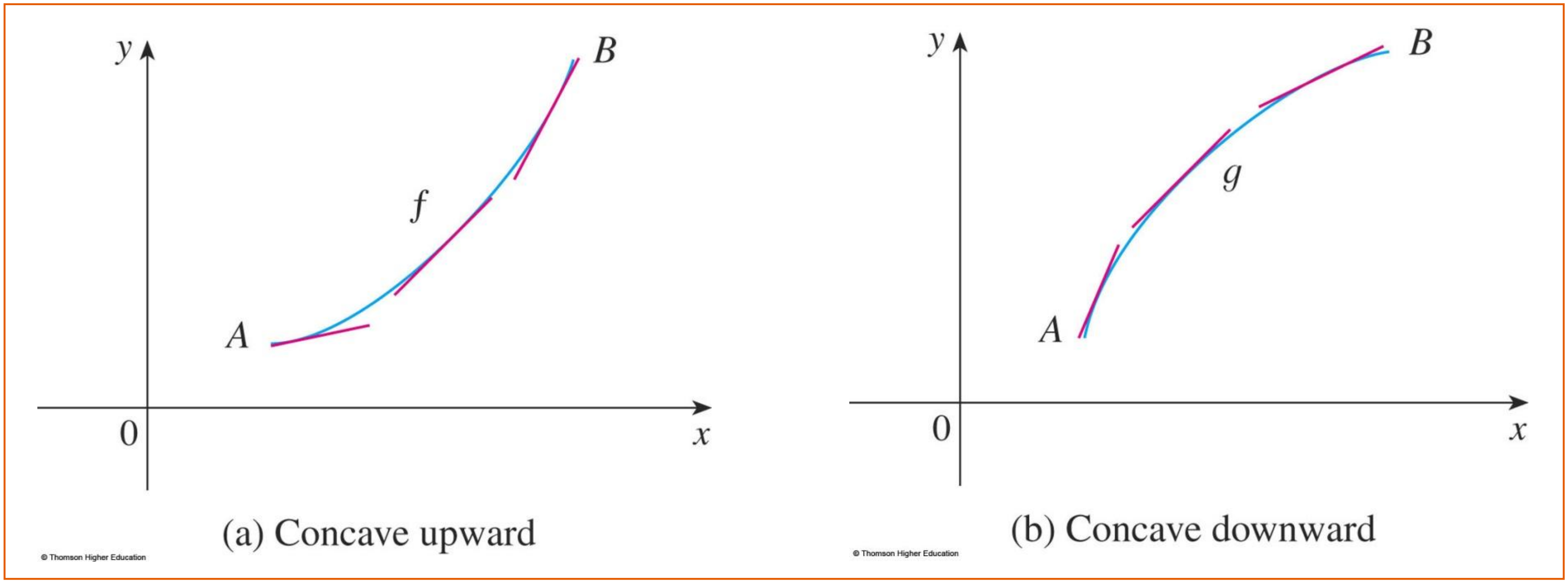
Suppose that *c* is a critical number of a continuous function *f*.

If *f'* does not change sign at *c* then *f* has no local maximum or minimum at *c*.



CONCAVE UPWARD/DOWNWARD

- The curve lies above the tangents and f is called concave upward on (a, b).
- The curve lies below the tangents and g is called concave downward on (a, b).

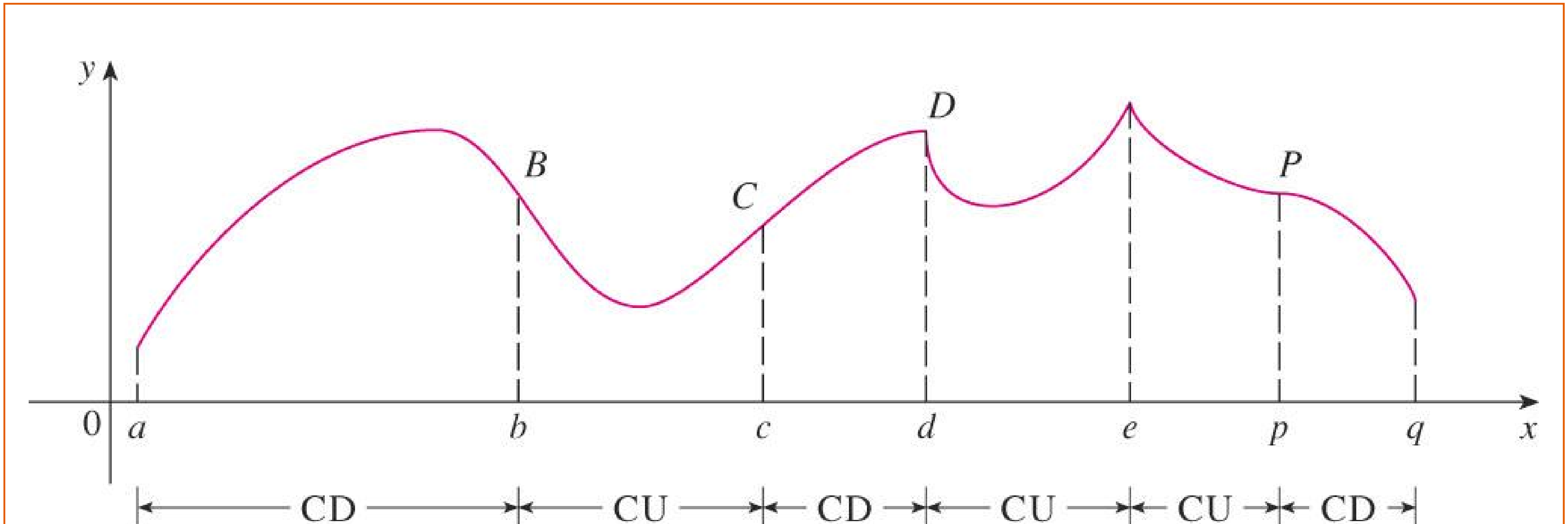


(a) Concave upward (b) Concave downward

CONCAVITY TEST

a. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I

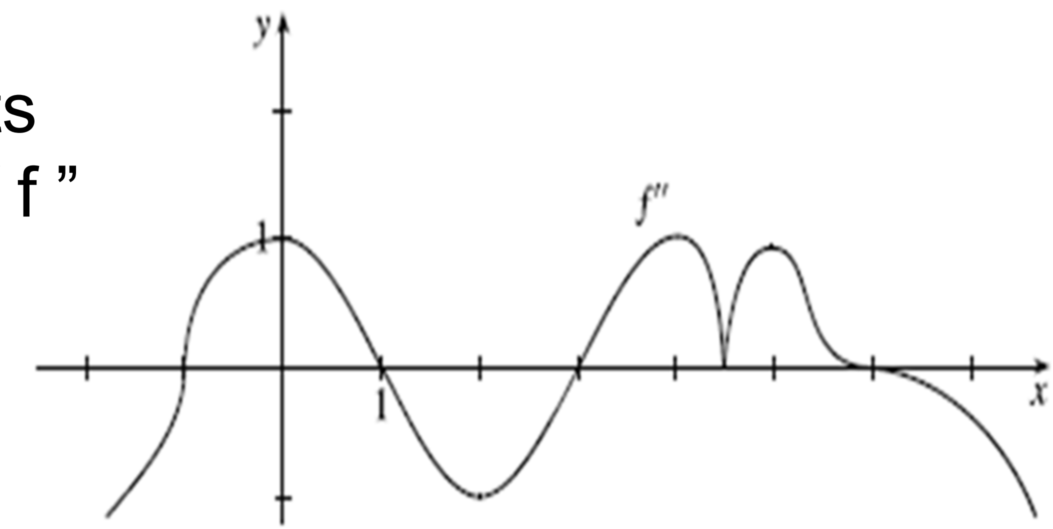
b. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I



INFLECTION POINT

A point P on the graph of function $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or from concave downward to concave upward) at P.

Indicate the inflection points of f if we have the graph of f ” as the figure



Answer: (-1, f(-1)); (1; f(1));
(3; f(3)); (6;f(6))

SECOND DERIVATIVE TEST

Suppose f'' is continuous near c.

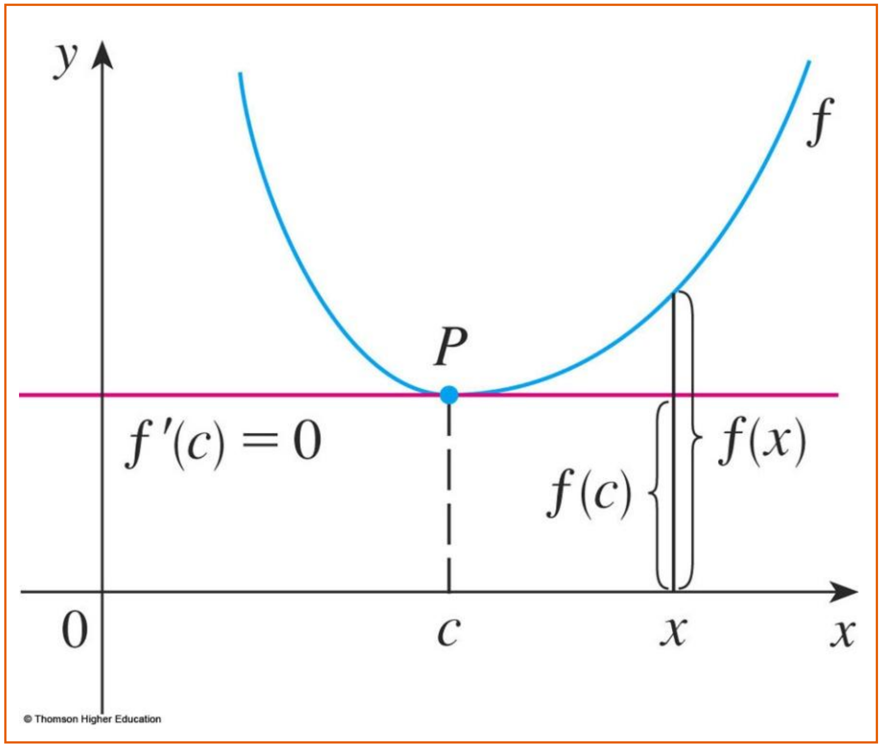
a. If $f'(c) = 0$ and $f''(c) > 0$,
then f has a local minimum at c.

b. If $f'(c) = 0$ and $f''(c) < 0$,
then f has a local maximum at c.

Find all local extrema for

$$f(x) = x^4 - 5x^2$$

$$g(x) = x^3 - \frac{3}{2}x^2 - 18x$$



Choose the correct one.	
A	If f has local extreme value at c then f'(c)=0.
B	If f'(c)=0 then f has local extreme value at c.
C	If f''(3)=0 then (3,f(3)) is an inflection point of f.
D	There exists a function such that $f'(x) \neq 0 \ \forall x$ and f(1)=f(0).
E	None of the above

4.6

Limit at Infinity and Horizontal Asymptotes

- If the values of $f(x)$ becomes arbitrarily close to L as x becomes sufficiently large, then
$$\lim_{x\to\infty} f(x) = L$$
- If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, then
$$\lim_{x\to-\infty} f(x) = L$$

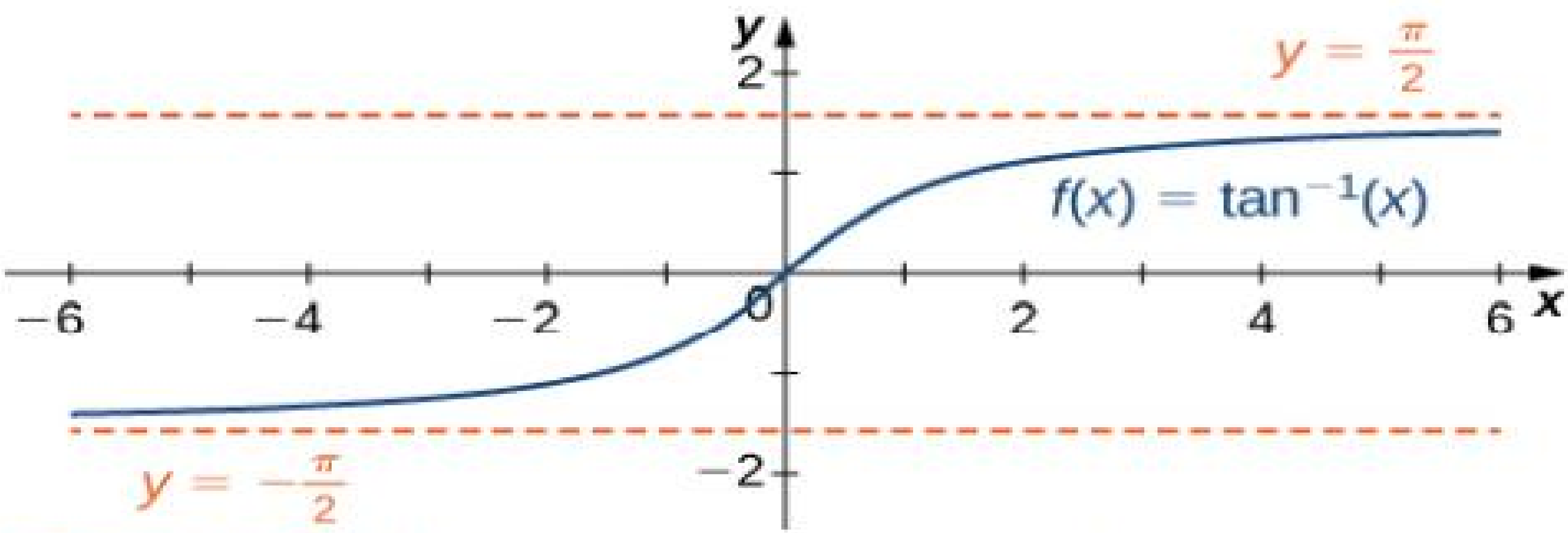


Figure 4.46 This function has two horizontal asymptotes.

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$,
the line $y = L$ is a horizontal asymptote of f

Find the horizontal asymptote(s) of the functions

$$f(x) = 5 - \frac{2}{x^2}$$

$$g(x) = \frac{\sin x}{x}$$

Compute

a. $\lim_{x \to \infty} \sin \frac{1}{x}$ 0

b. $\lim_{x \to \infty} (\sqrt{x^2+1} - x)$ 0

c. $\lim_{x \to \infty} \sin x$ Does not exist

d. $\lim_{x \to \infty} (x - x^3)$ $-\infty$

QUIZ QUESTIONS

1) Find $\lim_{x\to\infty} \cos x$

a. 0 b. infinity c. 1 d. Does not exist

2) Find $\lim_{x\to\infty} \frac{1}{x}\cos x$

a. 0 b. infinity c. 1 d. Dose not exist

QUIZ QUESTIONS

3) If $$\lim_{x\to 0} f(x) = \infty, \lim_{x\to 0} g(x) = \infty$$

Then $$\lim_{x\to 0}[f(x) - g(x)] = 0$$

a. True

b. False

4) A function can have two different horizontal asymptotes

a. True

b. False

Infinite Limits at Infinity

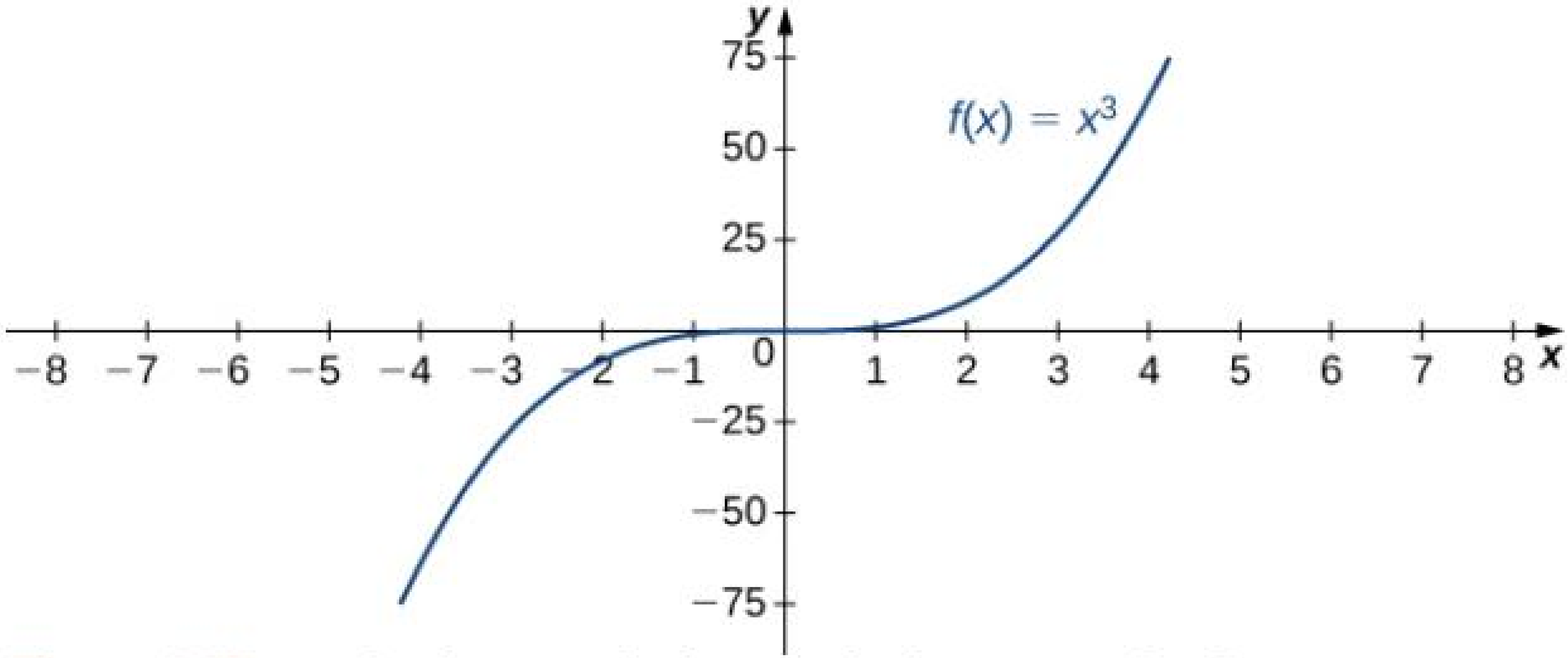


Figure 4.47 For this function, the functional values approach infinity as $x \to \pm\infty$.

$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

DRAW THE GRAPH OF A FUNCTION f

- Determine the domain of a function f and the x-intercepts and y-intercepts
- Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote.
- Determine if f has any vertical asymptotes.
- Calculate f'(critical points and determine the intervals where f is increasing/ decreasing).
- Calculate f''. Determine the intervals where f is concave up/ concave down). Verify that f has a local extremum at a critical point.

Sketch the graph of $f(x) = (x^2-1)(x+2)$; $g(x) = \frac{3x^2-5x}{x+1}$

4.7

Applied Optimization Problems

UNDERSTAND THE PROBLEM

Read the problem carefully until it is clearly understood.

- What is the **unknown**?
- What are the **given quantities**?
- What are the given **conditions**?

Example 1

A rectangular garden is to be constructed using a rock wall as one side of the garden and 2400 ft of wire fencing for the other three sides.
What is the possibly maximum area of the garden?

- The area of the garden

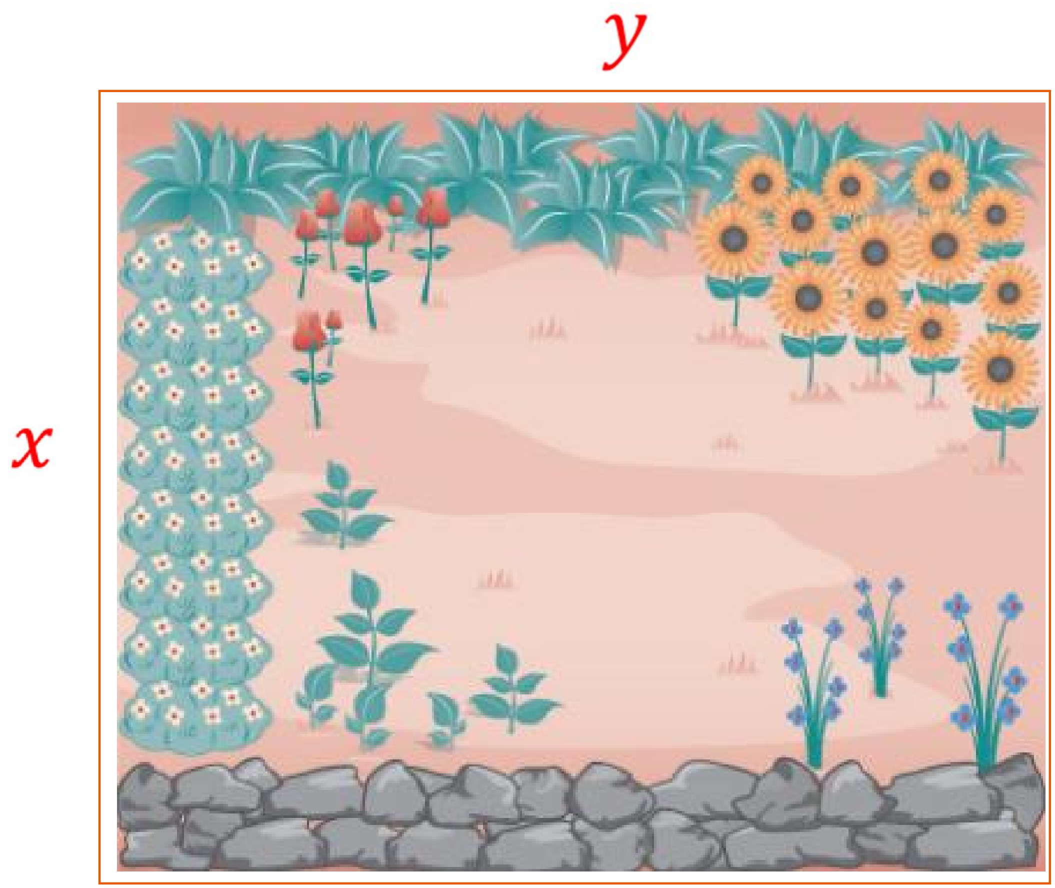
$$A = xy$$

- Constraint equation

$$2x + y = 2400$$

$\Rightarrow$ The problem becomes

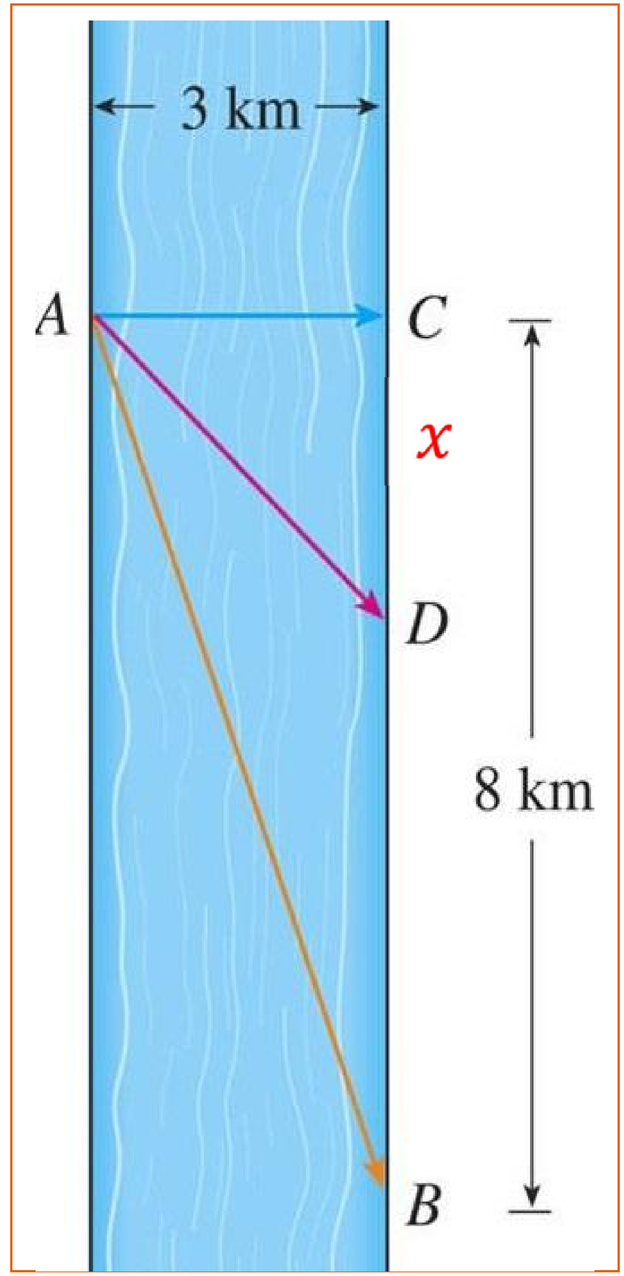
Maximize $A(x) = x(2400 - 2x)$
where $0 \leq x \leq 1200$



Example 2

A man launches his boat from point *A* on a bank of a straight river, 3 km wide, and wants to reach point *B* (8 km downstream on the opposite bank) as quickly as possible.

If he can row 6 km/h and run 8 km/h, where should he land to reach *B* as soon as possible?



Solution

Let D be the point that the man should land to reach B as soon as possible

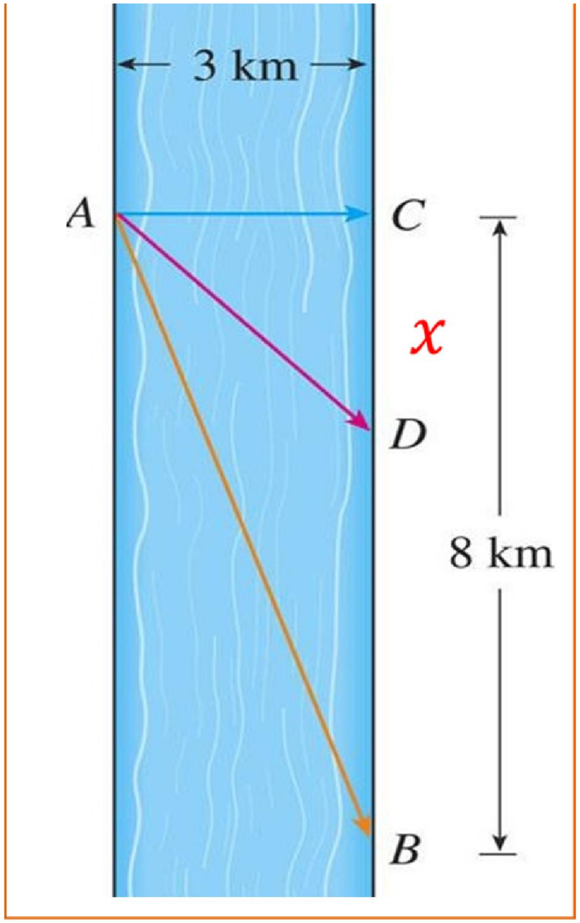
Denote $x = CD$

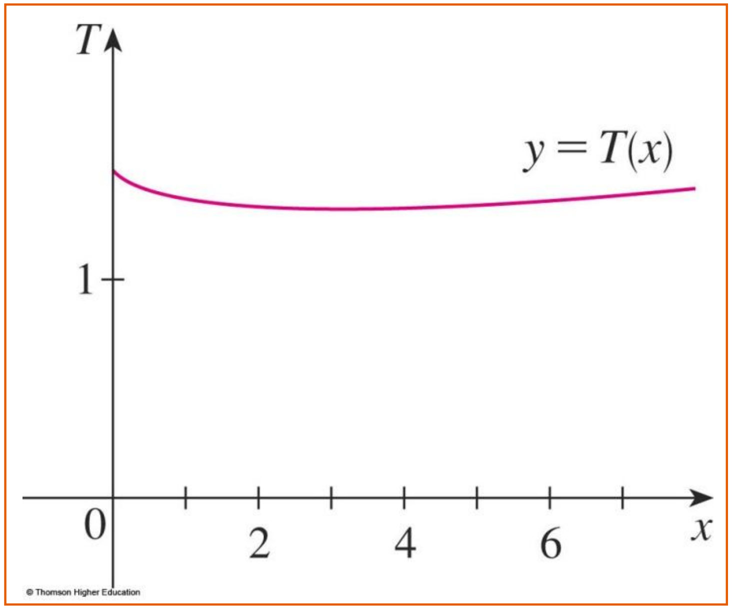
The rowing time is $\frac{\sqrt{x^2+9}}{6}$

The running time is $\frac{8-x}{8}$

The total time T is

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$





Example 3

A rectangular storage container with an open top is to have a volume of 15 m^3. The length of its base is twice the width.
Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

Example 4

Find the point on the line $y = 2x - 3$ that is closest to the origin.

Example 5

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Example 6

Find two positive numbers such that the product is 64 and the sum is the smallest.

4.9

Newton's Method

In this section, we will learn

How to solve high-degree equations

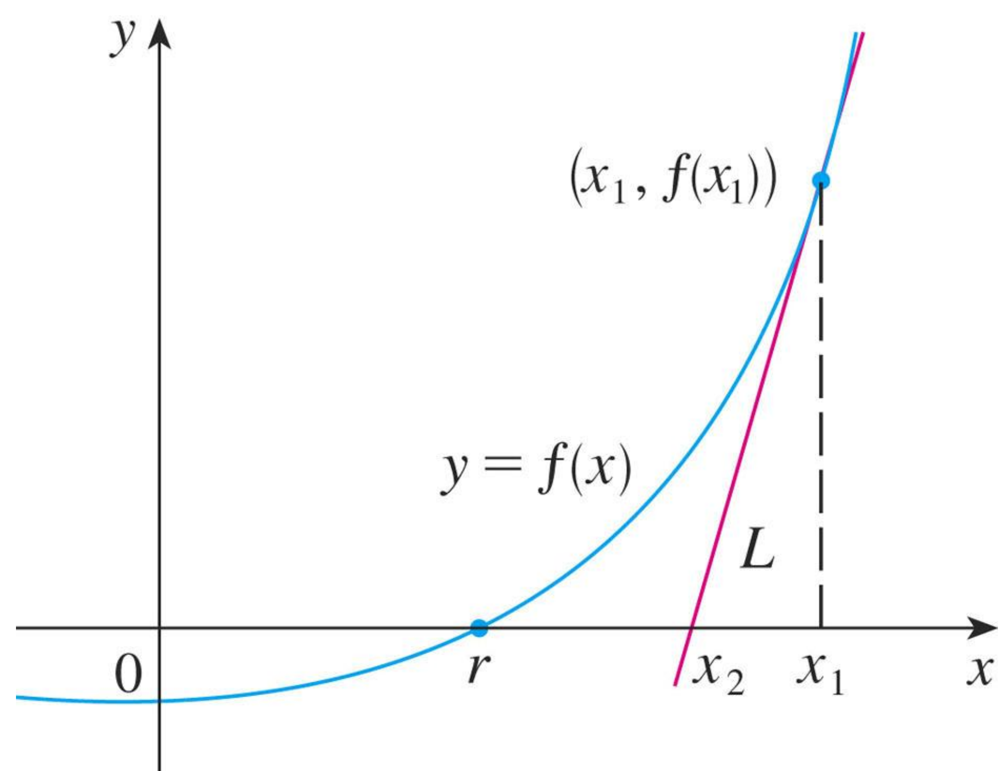
using Newton's Method.

➢ Sometimes, it is hard to find the solution of $f(x) = 0$

$\Rightarrow$ Use Newton's method to approximate the solution.

➢ We start with a first approximation x_1, which is obtained by guessing.

Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.

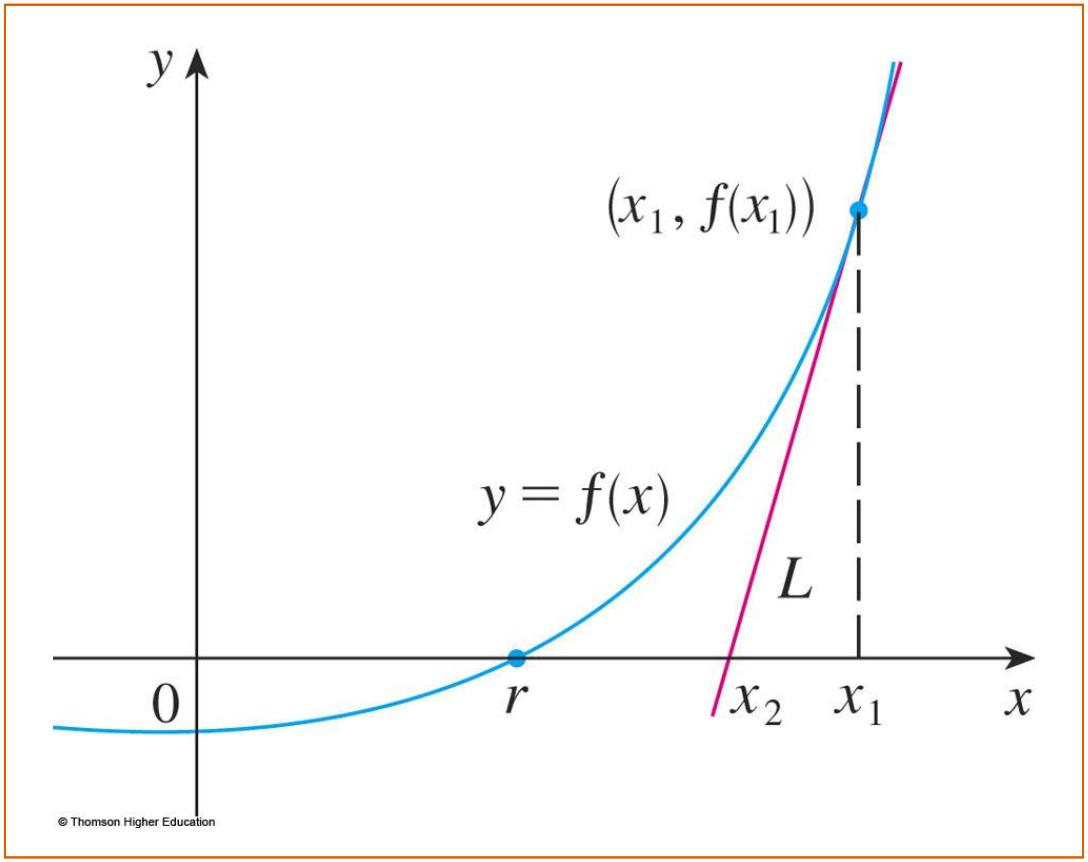


As the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

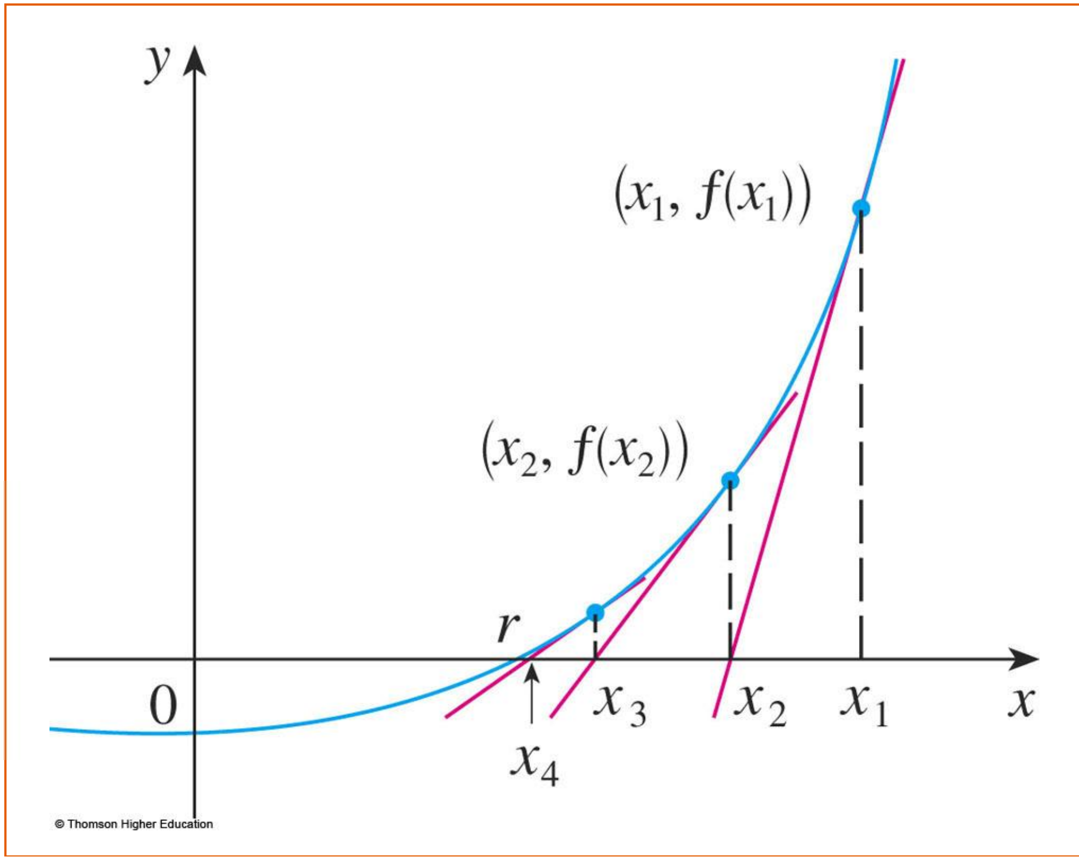


We use x_2 as a second approximation to r.

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Typically, the approximations $x_1, x_2, x_3 \ldots$ approach the actual root r.



Example: Use Newton's method to find $\sqrt[6]{2}$ (correct to eight decimal places)

- First, we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$
- We take $f(x) = x^6 - 2$
- Then, $f'(x) = 6x^5$

$$\boxed{x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}}$$

Choosing $x_1 = 1$ as the initial approximation, we obtain

$$x_2 \approx 1.16666667$$
$$x_3 \approx 1.12644368$$
$$x_4 \approx 1.12249707$$
$$x_5 \approx 1.12246205$$
$$x_6 \approx 1.12246205$$

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

As x_5 and x_6 agree to eight decimal places, we

conclude that $\sqrt[6]{2} \approx 1.12246205$ to eight decimal places.

Use Newton's method to approximate $\sqrt{3}$ by letting $x^2 - 3 = 0$ and $x_0 = 3$. Find x_1 and x_2.

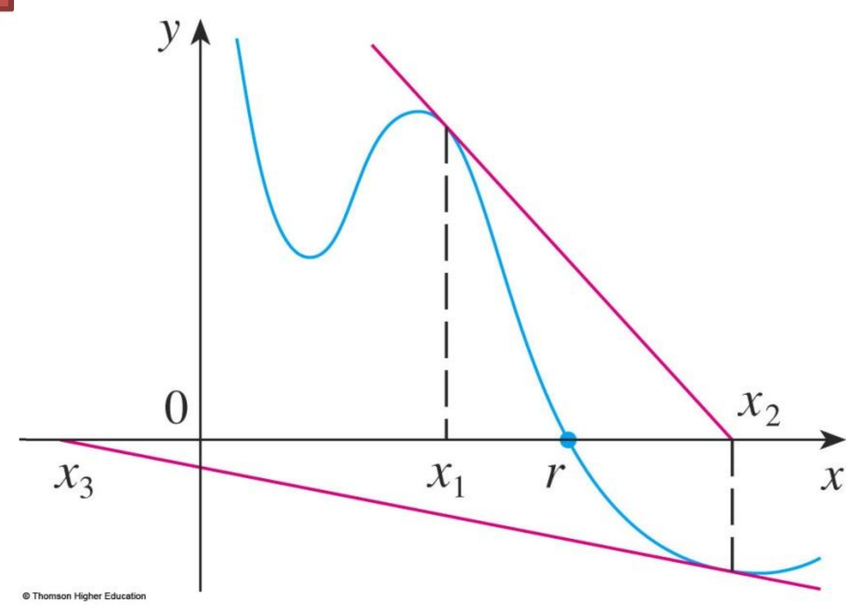
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Failures of Newton's Method

Consider the function

$$f(x) = x^3 - 2x + 2.$$

Let $x_0 = 0$. Show that the sequence $x_1, x_2, x_3 \ldots$ fails to approach a root of f.



4.10
Antiderivatives

In this section, we will learn about Antiderivatives and how they are useful in solving certain scientific problems.

DEFINITION

A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$

for all x in I.

Example

- $f(x) = x \Rightarrow F(x) = \frac{x^2}{2}$ is an antiderivative of f.
- $g(x) = \cos x \Rightarrow G(x) = \sin x$ is an antiderivative of g.

Theorem

If *F* is an antiderivative of *f* on an interval *I*, the most general antiderivative of *f* on *I* is $F(x) + C$ where *C* is an arbitrary constant.

$$\boxed{\int f(x)dx = F(x) + C}$$

1. Find all antiderivatives of the following functions

a) $f(x) = 5x^2$

b) $g(x) = \frac{1}{x}$

2. Let $f(x) = \tan x$, and $F(x)$ is an antiderivative of $f(x)$. Determine whether F is increasing or decreasing at $x = -3$ (rad)

ANTIDERIVATIVE FORMULAS

Here, we list some particular antiderivatives.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

Initial-value problem

Based on initial conditions, we can find C in the equation

$$\int f(x)dx = F(x) + C$$

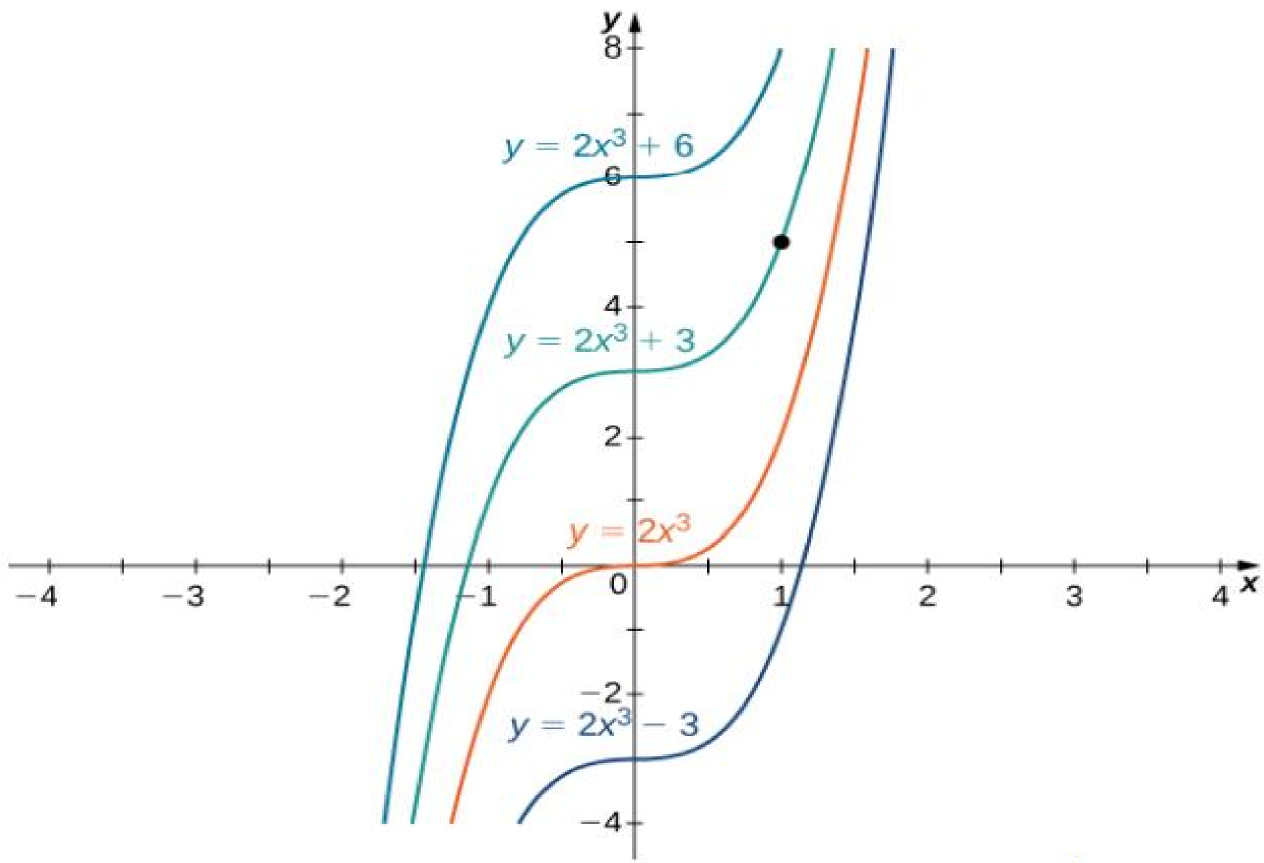


Figure 4.86 Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example

A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$.

Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm.

Find its position function $s(t)$.

1. A particle moves along the x-axis so that its velocity at time t is given by 3 sin 2t.
Assuming it starts at the origin, where is it at t = π seconds?

a. 0 b. 3/2 c. ½ d. - 1/2

2. Let $f(x) = 4 - 3x$ for all $x \geq 2$.
Select the correct one.

a. 2 is the local minimum value.
b. 2 is the local maximum value.
c. -2 is the local minimum value.
d. -2 is the maximum local value.
e. 2 is the absolute minimum value.
f. -2 is the absolute maximum value.
g. None of the above.

A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle.

How should the wire be cut for the square so that the total area (of the square and the triangle) enclosed is a minimum? Round the result to the nearest hundredth.

<u>Answer</u>

If x is the length of the square,

then the side of the triangle is $\frac{1}{3}(10-4x)$

The total area enclosed is $x^2+\frac{\sqrt{3}}{36}(10-4x)^2$

$\Rightarrow$ The problem becomes

Minimize $f(x)=x^2+\frac{\sqrt{3}}{36}(10-4x)^2$ where $0<x<2.5$

Exercises

- 1,2,3, 16-21 (p.350,351)
- 50,51, 56,57, 72,73, 78-84 (p.365,366)
- 108,109,110, 118, 132 (p.377)
- 157,161,164 (p.388)
- 251-255, 256, 257, 272 (p.436)
- 317, 318, 319, 321, 347, 349 (p.451)
- 414, 415 (p.483)
- 499-502, 509, 510 (p.497)