Discrete Mathematics

Chapter 3: The Fundamentals: Algorithms, the Integers

Department of Mathematics
The FPT university

Topics covered:

Topics covered:

3.1 Algorithms

Chapter 3: Introduction

Topics covered:

3.1 Algorithms

3.2 The Growth of Functions

Topics covered:

3.1 Algorithms
3.2 The Growth of Functions
3.3 Complexity of Algorithms

Chapter 3: Introduction

Topics covered:

3.1 Algorithms

3.2 The Growth of Functions

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3.4 The Integers and Division

Chapter 3: Introduction

Topics covered:

3.1 Algorithms

3.2 The Growth of Functions

3.3 Complexity of Algorithms

3.4 The Integers and Division

3.5 Primes and Greatest Common Divisors

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3.1 Algorithms

3.2 The Growth of Functions

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3.6 Integers and Algorithms

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Input. a, b, c : integers (coefficients)
Output. Solutions if they exist.

Step 1. If $a = 0$ then Print (This is not a quadratic equation).
Step 2. Compute $\Delta = b^2 - 4ac$
Step 3. If $\Delta < 0$ then Print (No solution).
Step 4. If $\Delta = 0$, compute $x = -b/2a$
Step 5. If $\Delta > 0$, compute

$$x_1 = (-b + \sqrt{\Delta})/(2a), \;\; x_2 = (-b - \sqrt{\Delta})/(2a)$$

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- **Generality:** Algorithm should be applicable for all problems of the desired form, not just a particular set of input values.

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- *Step 4.* Stop the algorithm when there are no integers left. The temporary maximum at this point is the maximum of the sequence.

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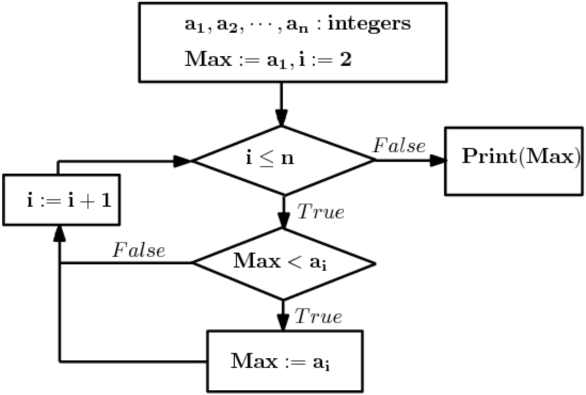

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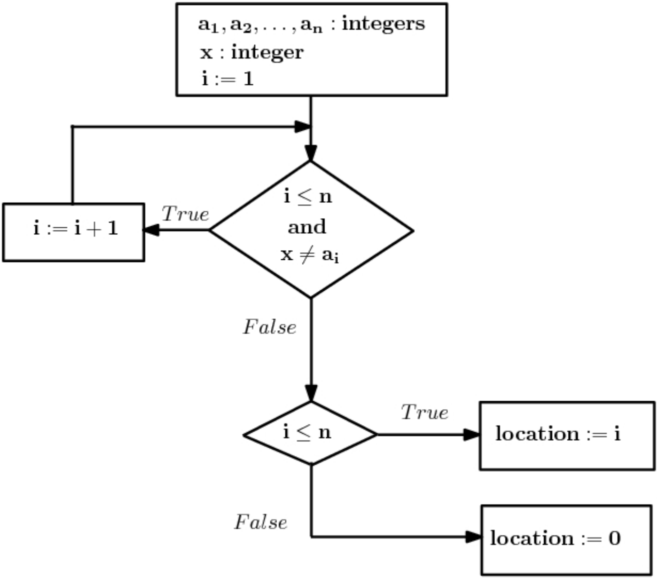
$a_1, a_2, \ldots, a_n$: integers
x : integer
$i := 1$
$i \leq n$
and
$x \neq a_i$
True
$i := i + 1$
False
$i \leq n$
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location := i
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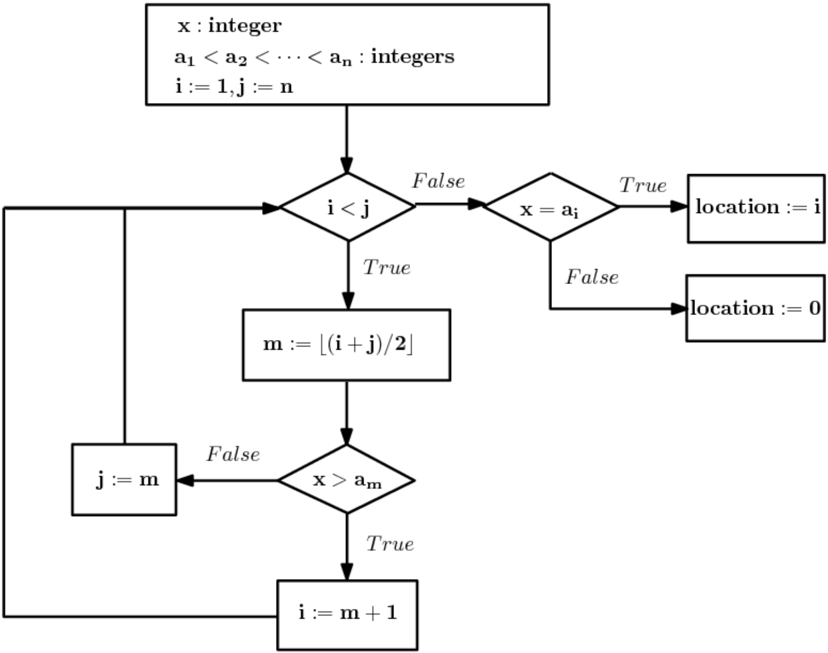
x : integer
$a_1 < a_2 < \cdots < a_n$: integers
$i := 1, j := n$
$i < j$
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Bubble Sort

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Algorithm:

1. Successively comparing two consecutive elements of the list to push the largest element to the bottom of the list.

Input: A sequence of integers $a_1, a_2, \ldots, a_n$

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1. Successively comparing two consecutive elements of the list to push the largest element to the bottom of the list.
2. Repeat the above step for the first $n-1$ elements of the list.

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Example.

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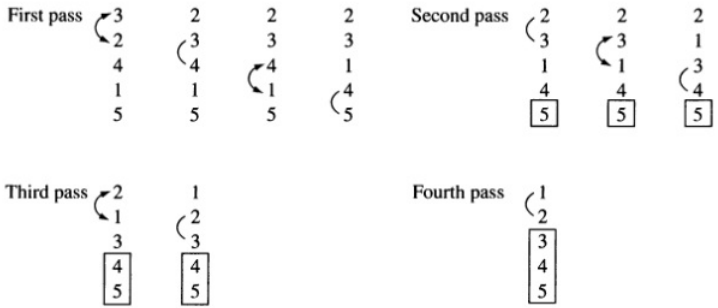

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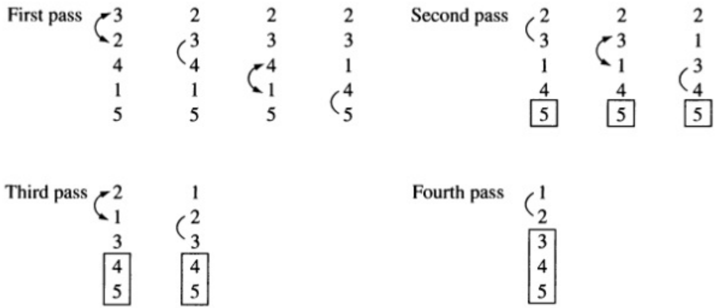

Procedure BubbleSort($a_1, a_2, \ldots, a_n$: integers)

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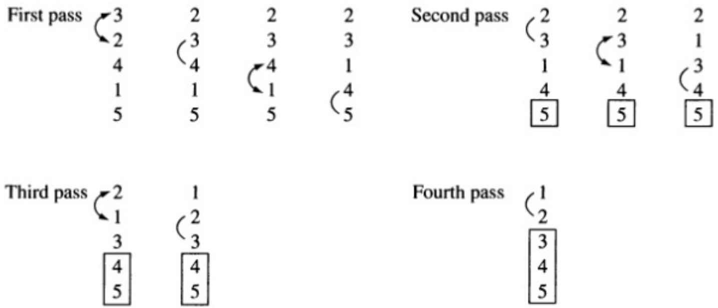

Procedure BubbleSort($a_1, a_2, \ldots, a_n$: integers)
for $i := 1$ **to** $n - 1$

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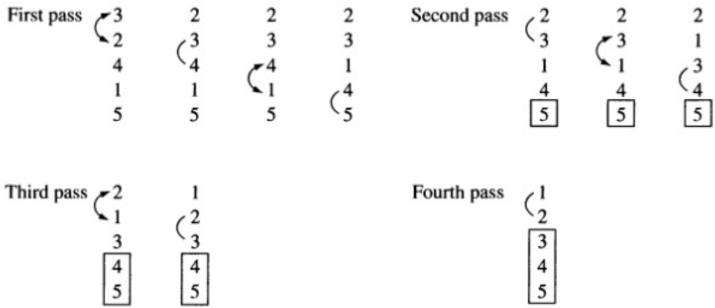

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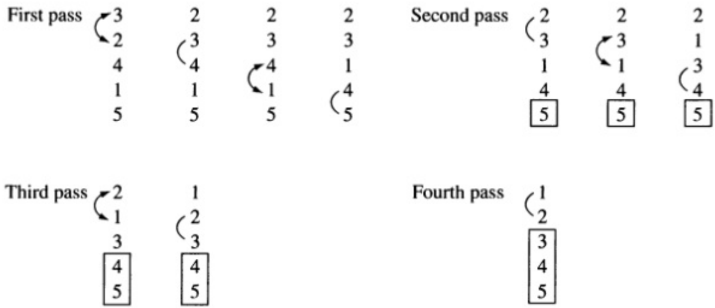
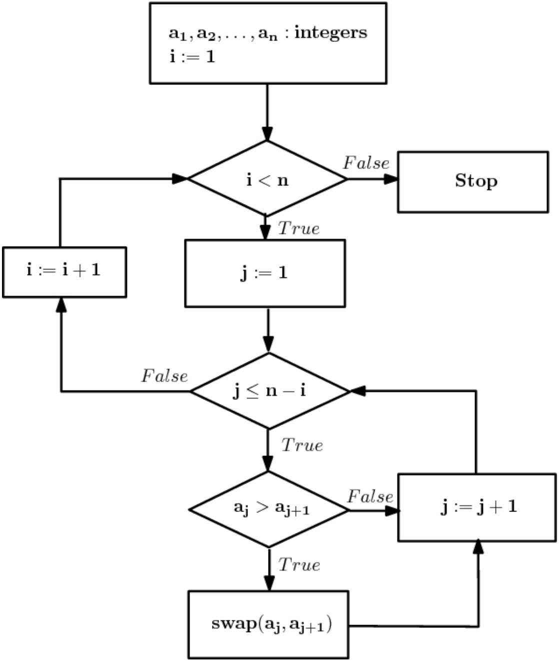


Procedure BubbleSort($a_1, a_2, \ldots, a_n$: integers)
for $i := 1$ **to** $n - 1$
  for $j := 1$ **to** $n - i$
    if $a_j > a_{j+1}$ **then** swap(a_j, a_{j+1})


$a_1, a_2, \ldots, a_n$: integers
$i := 1$
$i < n$
False
Stop
True
$i := i + 1$
$j := 1$
False
$j \leq n - i$
True
$a_j > a_{j+1}$
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$j := j + 1$
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$swap(a_j, a_{j+1})$

Insertion Sort

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Algorithm:

1 Sort the first two elements of the list

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Algorithm:

1. Sort the first two elements of the list
2. Insert the third element to the list of the first two elements to get a list of 3 elements of increasing order.

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Algorithm:

1. Sort the first two elements of the list
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3. Insert the fourth element to the list of the first three elements to get a list of 4 elements of increasing order.

Insertion Sort

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Algorithm:

1. Sort the first two elements of the list
2. Insert the third element to the list of the first two elements to get a list of 3 elements of increasing order.
3. Insert the fourth element to the list of the first three elements to get a list of 4 elements of increasing order.

Insertion Sort

Input: Sequence of integers $a_1, a_2, \ldots, a_n$

Output: The sequence in the increasing order

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Greedy Change-Making Algorithm

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Algorithm: Read textbook!

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(c) Show that $f(x) = \lfloor x/2 \rfloor$ is $\Theta(x)$

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In this lecture we analyze time complexity of some algorithms studied in previous sections.

Complexity of Algorithm of Finding Maximum Element

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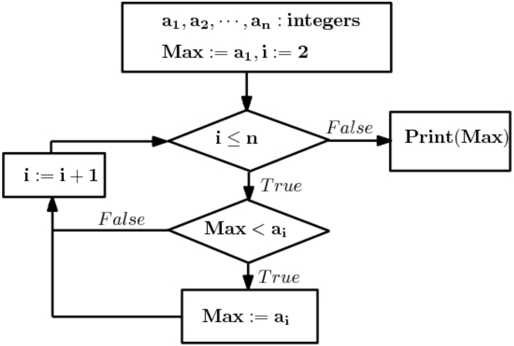

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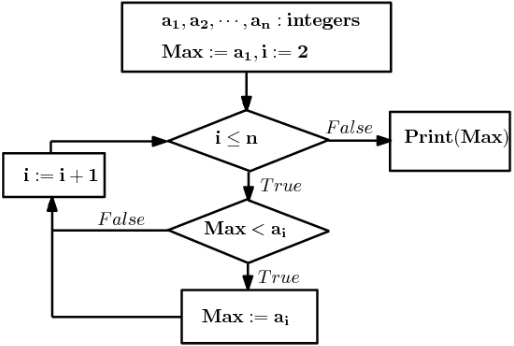

Number of loops: $n - 1$

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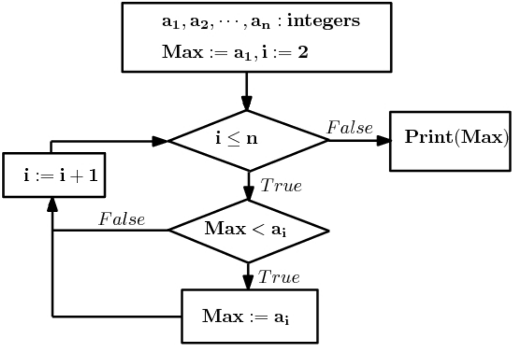

Number of loops: $n - 1$
Number of comparisons in each loop: 2

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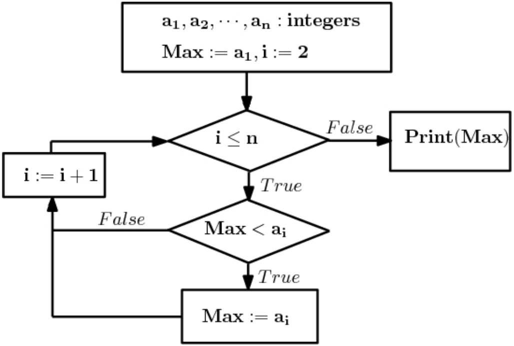
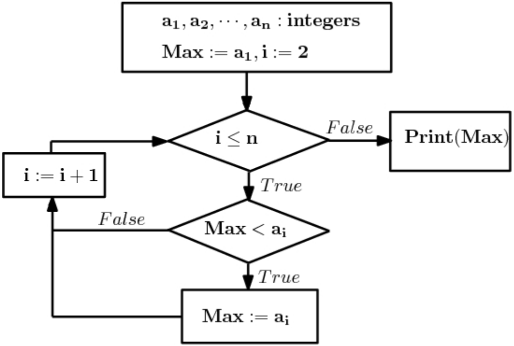


Number of loops: $n - 1$
Number of comparisons in each loop: 2
Number of comparisons to exit the loop: 1



Number of loops: $n - 1$
Number of comparisons in each loop: 2
Number of comparisons to exit the loop: 1
Total number of comparisons: $2(n-1) + 1 = 2n - 1 = O(n)$

Complexity of Linear Search Algorithm

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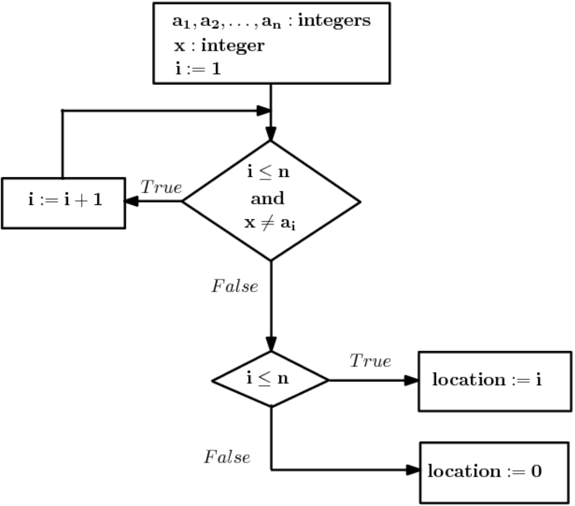

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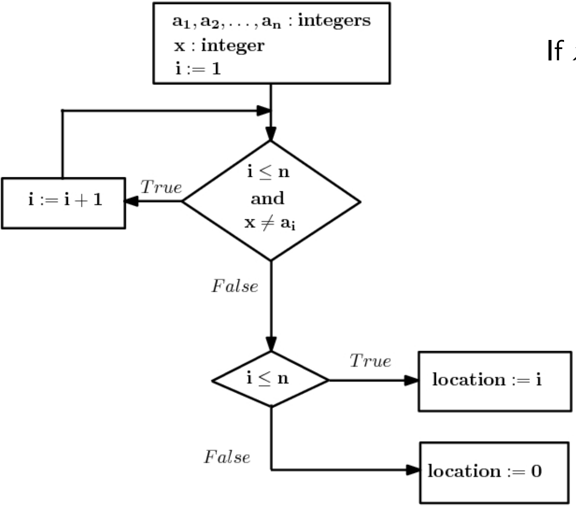

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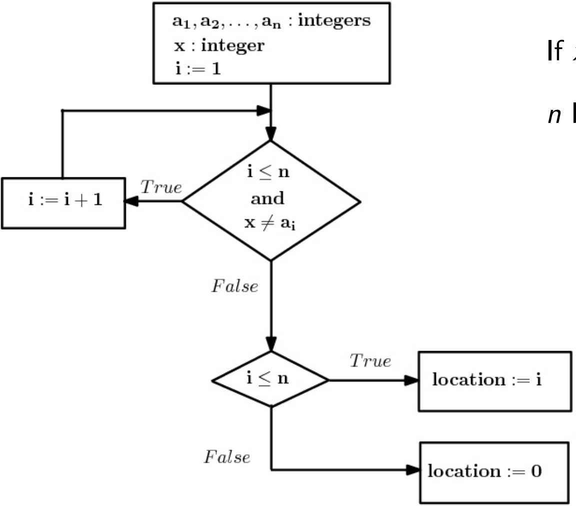

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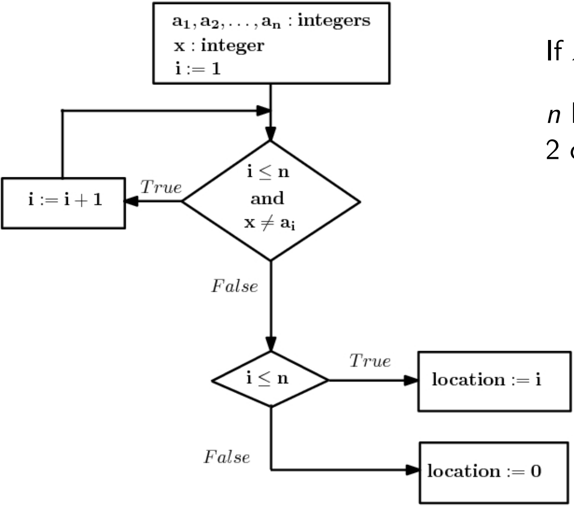

If x do not appear in the list:

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2 comparisons in each loop

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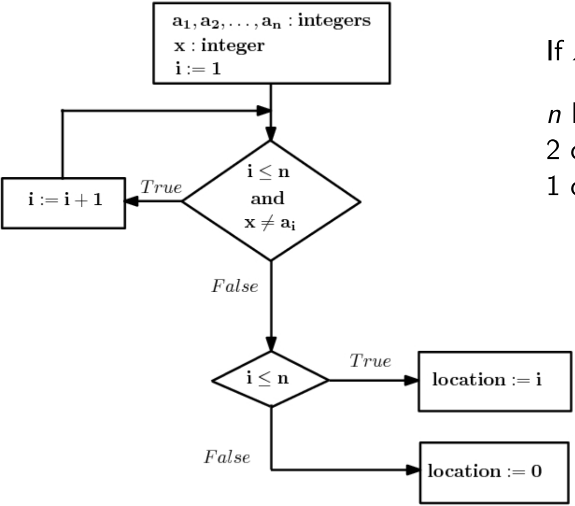

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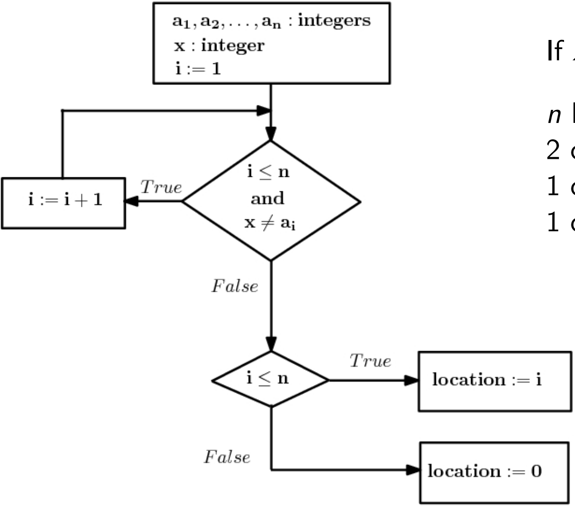

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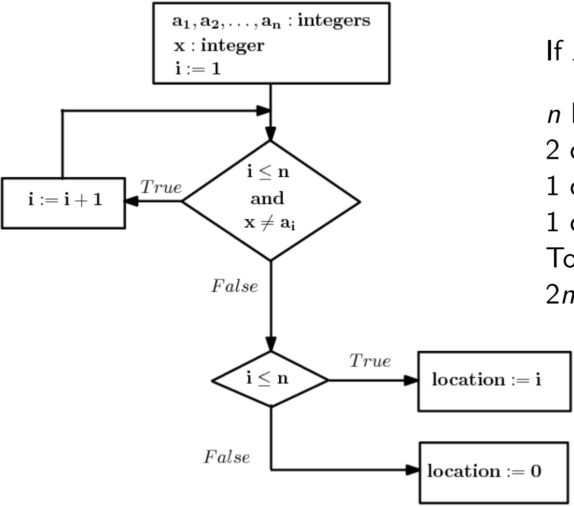
1 comparison outside the loop

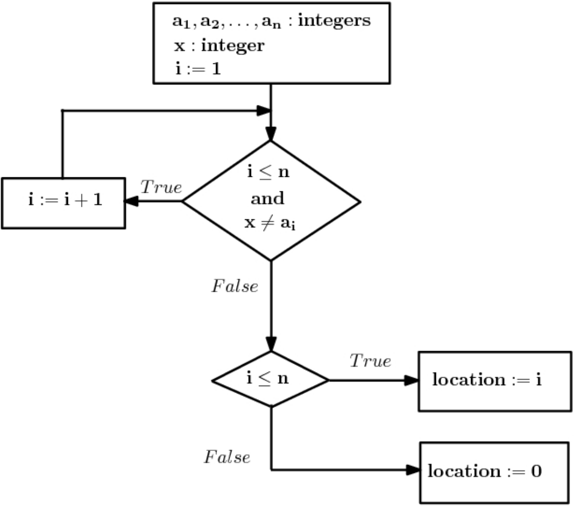
Complexity of Linear Search Algorithm



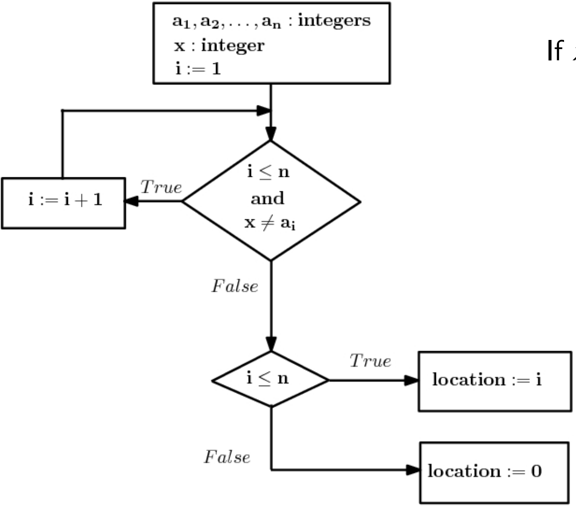
If x do not appear in the list:

n loops
2 comparisons in each loop
1 comparison to exit the loop
1 comparison outside the loop
Total number of comparisons:
$2n + 1 + 1 = 2n + 2 = O(n)$

Complexity of Linear Search Algorithm

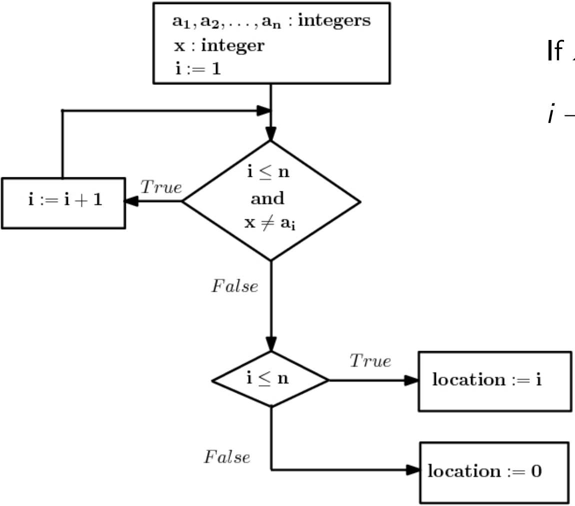


$a_1, a_2, \ldots, a_n$: integers
x : integer
$i := 1$
$i \le n$
and
$x \ne a_i$
True
$i := i + 1$
False
$i \le n$
True
location := i
False
location := 0



If x is at the ith location:

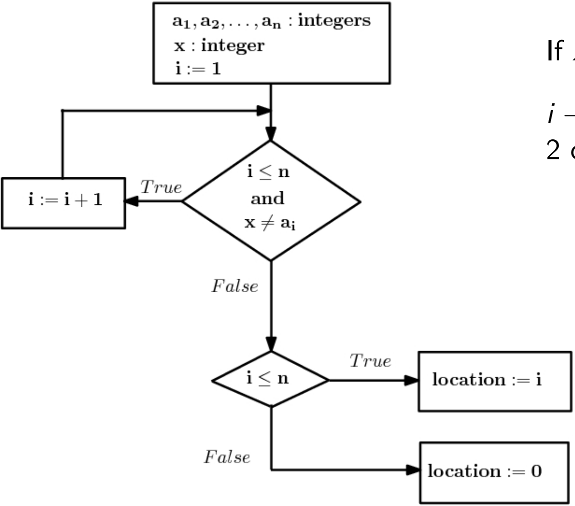
Complexity of Linear Search Algorithm



If x is at the ith location:

$i - 1$ loops

Complexity of Linear Search Algorithm



If x is at the ith location:

$i - 1$ loops
2 comparisons in each loop

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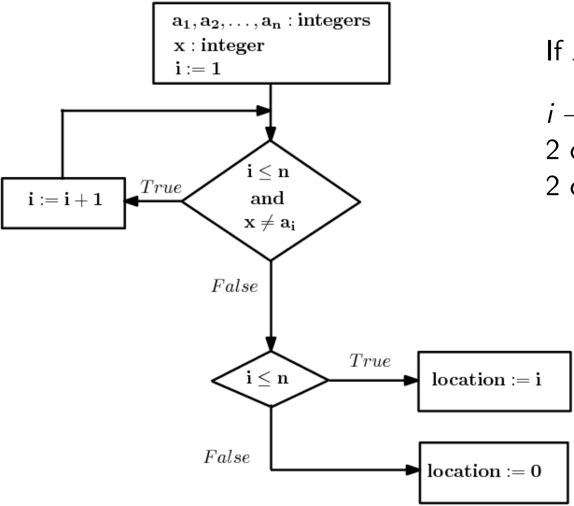

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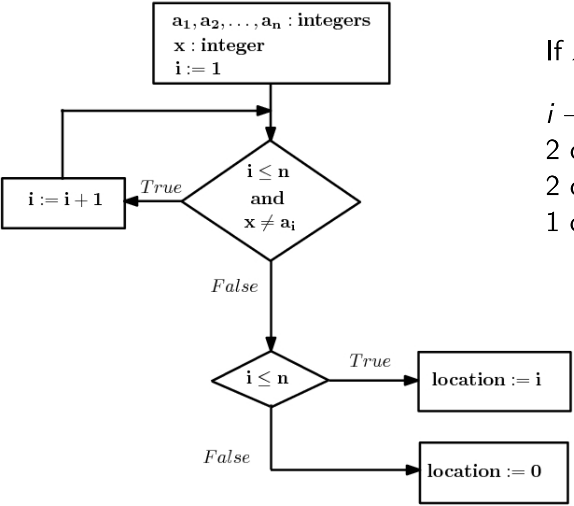

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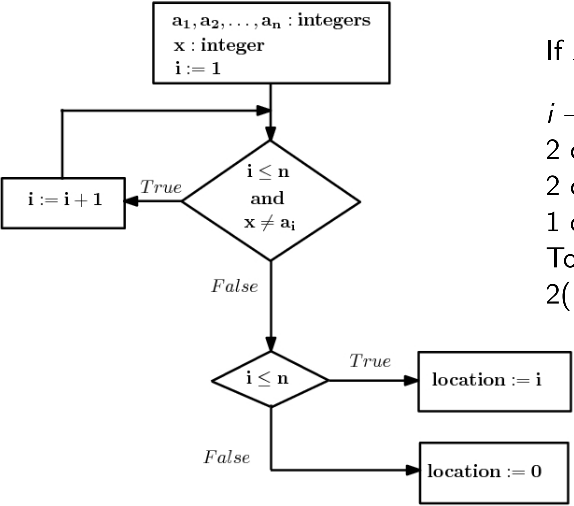

If x is at the ith location:

$i - 1$ loops
2 comparisons in each loop
2 comparisons to exit the loop
1 outside the loop
Total number of comparisons:
$2(i-1) + 2 + 1 = 2i + 1$

Complexity of Binary Search Algorithm

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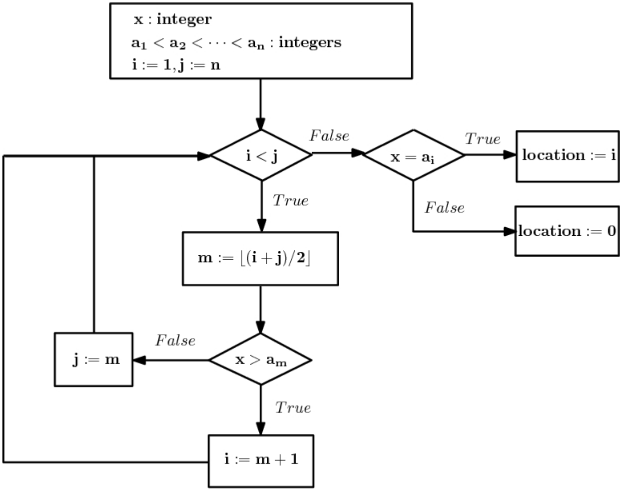

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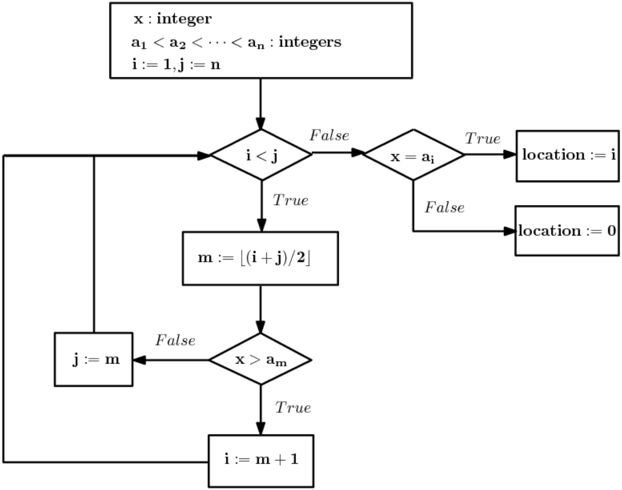

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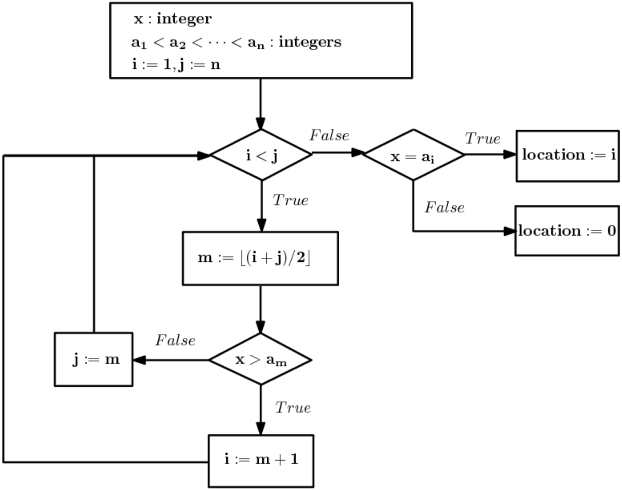

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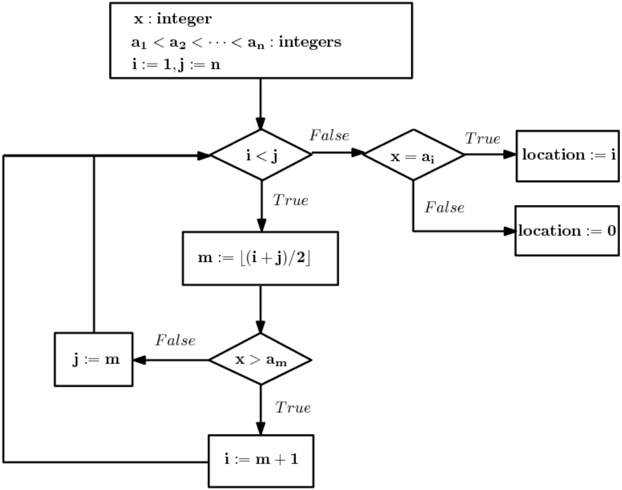

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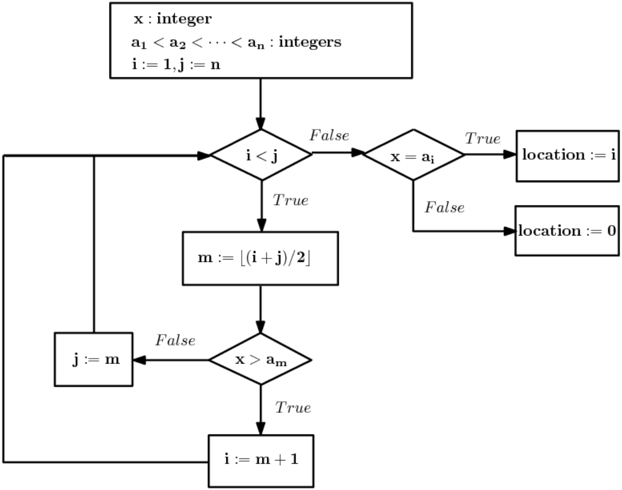

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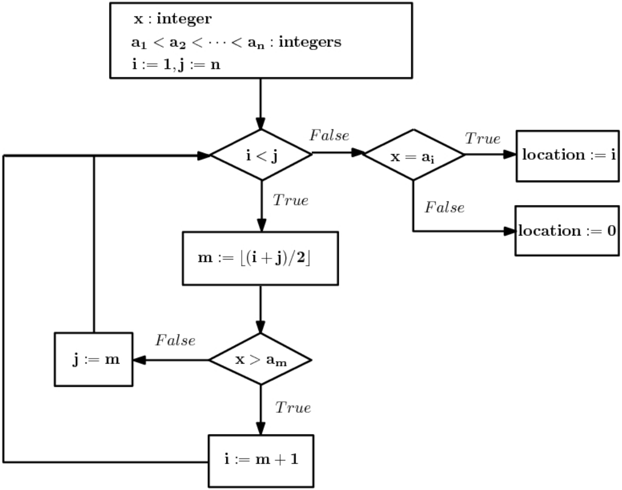

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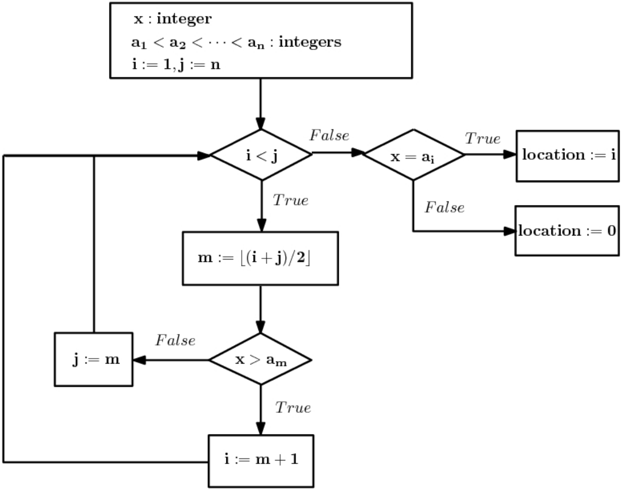

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$2k + 1 + 1 = 2k + 2$
$= 2\lceil \log n \rceil + 2$
$= O(\log n)$

Exercise.

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Analyze the complexity of Bubble sort and Insertion sort algorithms.

Commonly used terminologies for the complexity of algorithms	
Complexity	Terminology
$O(1)$	Constant complexity
$O(\log n)$	Logarithmic complexity
$O(n)$	Linear complexity
$O(n \log n)$	$n \log n$ complexity
$O(n^k)$	Polynomial complexity
$O(b^n)$, $b > 1$	Exponential complexity
$O(n!)$	Factorial complexity

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(b) -125 is divided by 11

Modular Arithmetic

Let a, b be integers and m a positive integer. We say a is congruent to b modulo m is they have the same remainders when being divided by d. We use notation $a \equiv b \mod m$. If they are not congruent we write $a \not\equiv b \mod m$.

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- If $a \equiv b \mod m$ and $c \equiv d \mod m$ then $a + c \equiv b + d \mod m$ and $ac \equiv bd \mod m$

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m is called the modulus, a is the multiplier, c is the increment and x_0 is the seed.

Cryptography.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Caesar's cipher

$f(p) = p + 3 \mod 26$

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(b) $(14, 23, 35, 61)$

3.6 Integers and Algorithms

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Let b be an integer greater than 1. Let n be a positive integer. Then n can be expressed uniquely in the form

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(b) Find the hexadecimal expansion of $(132)_5$

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Hexadecimal, Octal and binary expansions of integers from 0 though 15

Hexadecimal, Octal and binary expansions of integers from 0 though 15

Decimal	0	1	2	3	4	5	6	7
Hexadecimal	0	1	2	3	4	5	6	7
Octal	0	1	2	3	4	5	6	7
Binary	0	1	10	11	100	101	110	111
Decimal	8	9	10	11	12	13	14	15
Hexadecimal	8	9	A	B	C	D	E	F
Octal	10	11	12	13	14	15	16	17
Binary	1000	1001	1010	1011	1100	1101	1110	1111

Algorithms for Integer Operations

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Procedure Addition (a, b)
$c := 0$
for $j := 0$ **to** $n-1$
    $d := \lfloor (a_j + b_j + c)/2 \rfloor$
    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$

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Procedure Multiplication (a, b)
for $j := 0$ **to** $n - 1$
  if $b_j = 1$ **then** $c_j := a$ shifted j places
  else $c_j := 0$
$p := 0$
for $j := 0$ **to** $n - 1$
  $p := p + c_j$

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Procedure GCD(a, b: positive integers)

$x := a$

$y := b$

while $y \neq 0$

$\quad r := x \mod y$

$\quad x := y$

$\quad y := r$

Print(x)

Modular Exponentiation

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Procedure ModExp(b, m: positive integers, $n = (a_k \ldots a_0)_2$)
$x := 1$
$power := b \mod m$
for $i := 0$ **to** k

  if $a_i = 1$ **then** $x := (x * power) \mod m$

  $power := (power * power) \mod m$

Print(x)