# Discrete Mathematics

## Chapter 10: Trees

Department of Mathematics
The FPT university

## Chapter 10: Introduction

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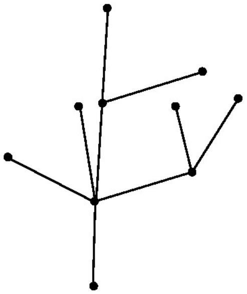

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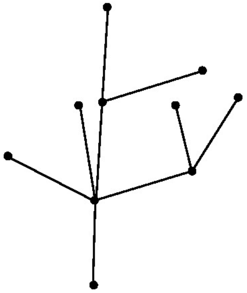

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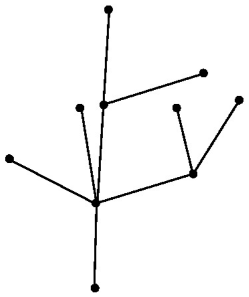

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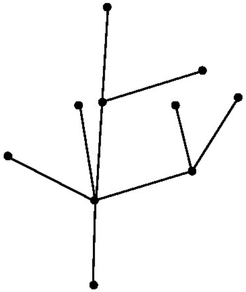

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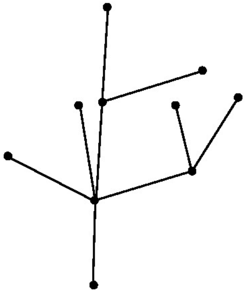

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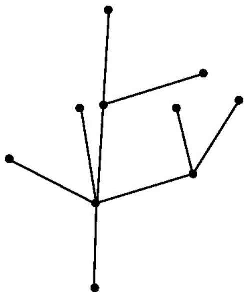

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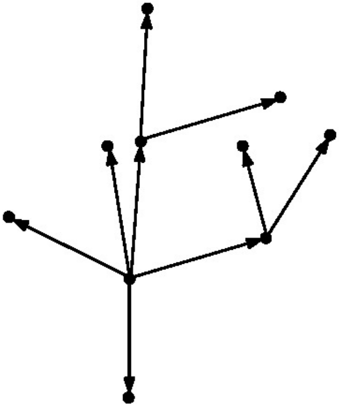

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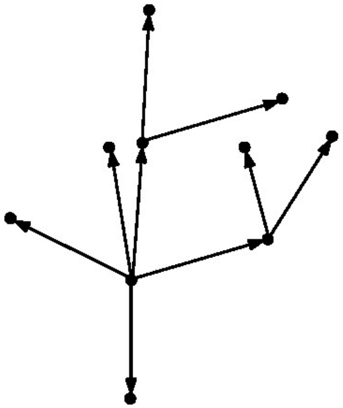

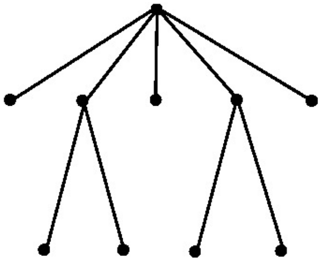

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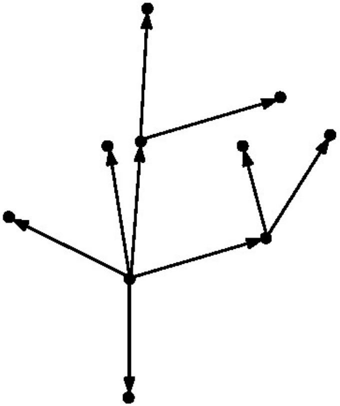

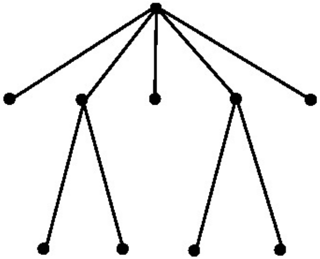

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$$\log(n!) = \log 1 + \log 2 + \cdots + \log n = \Theta(n \log n)$$

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Read textbook!

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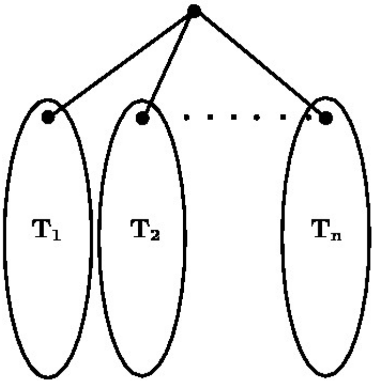

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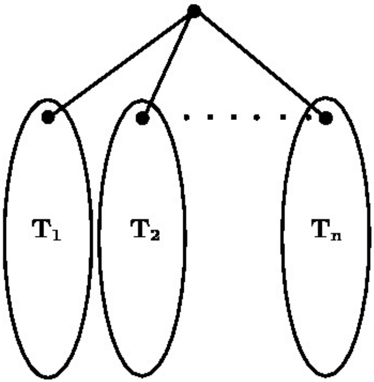

**Preorder traversal**

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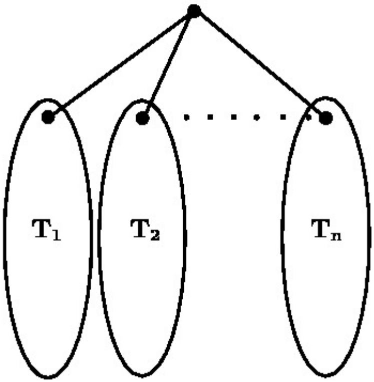

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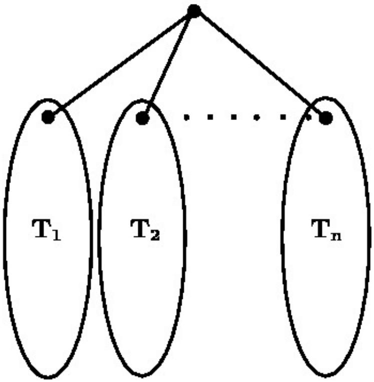

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- Visit the root
- Traverse each of $T_1, T_2, \ldots, T_n$ in preorder.

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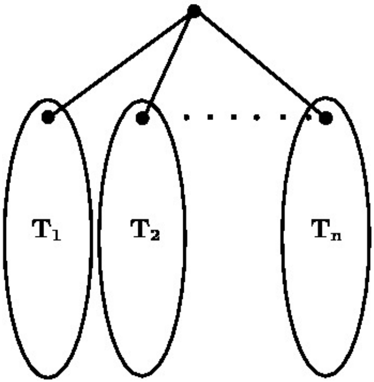

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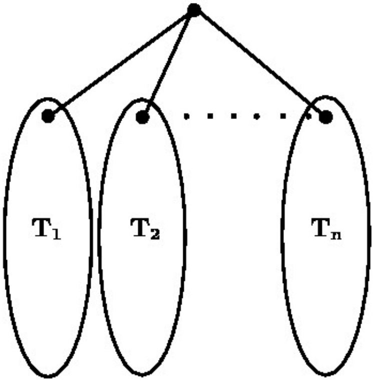
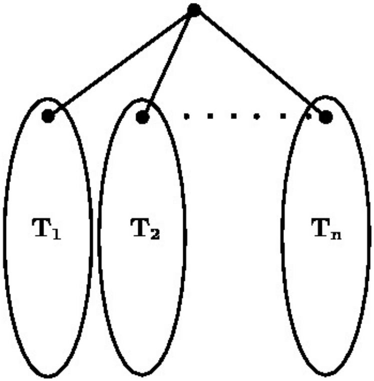
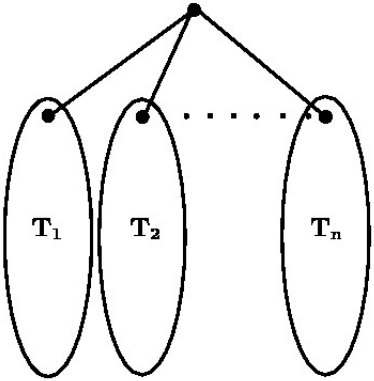

$T_1$
$T_2$
$T_n$

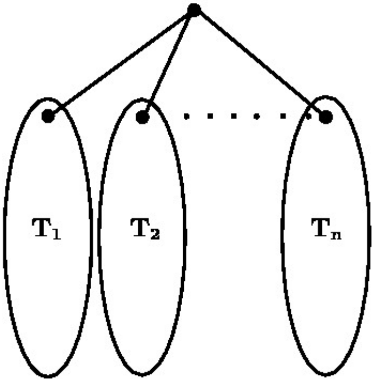


## Inorder traversal



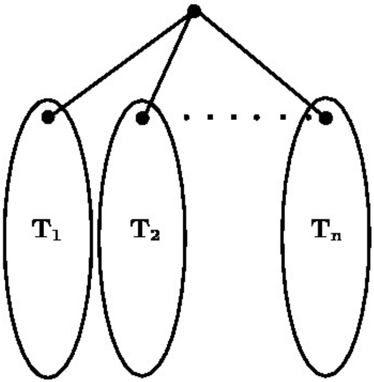
**Inorder traversal**

- Traverse $T_1$ in inorder.



**Inorder traversal**

- Traverse $T_1$ in inorder.
- Visit the root.
- Traverse each of $T_2, \ldots, T_n$ in inorder.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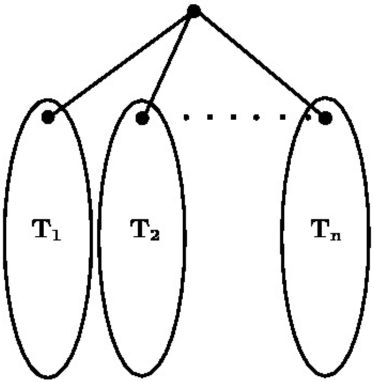
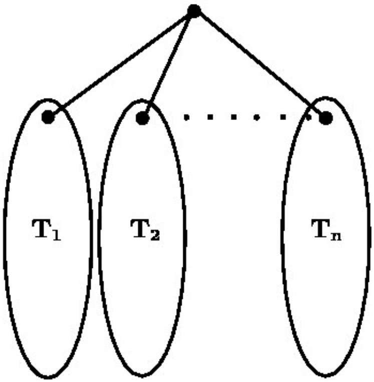


## Postorder traversal



**Postorder traversal**

- Traverse each of $T_1, T_2, \ldots, T_n$ in postorder.
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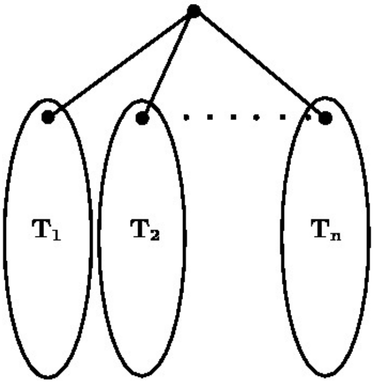

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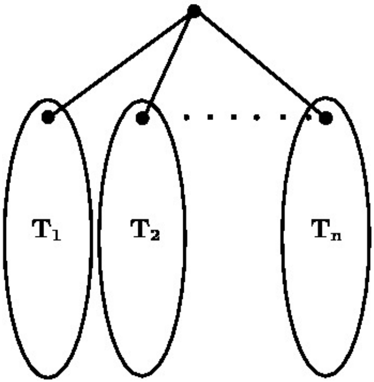

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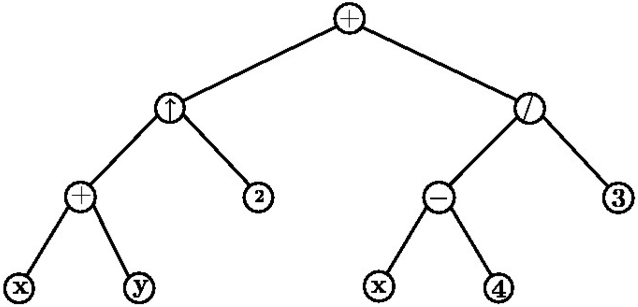
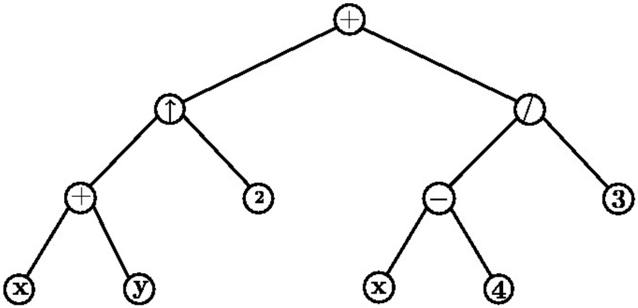
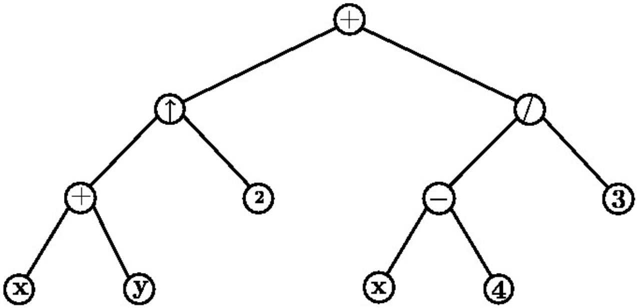



+
↑
/
+
2
−
3
x
y
x
4



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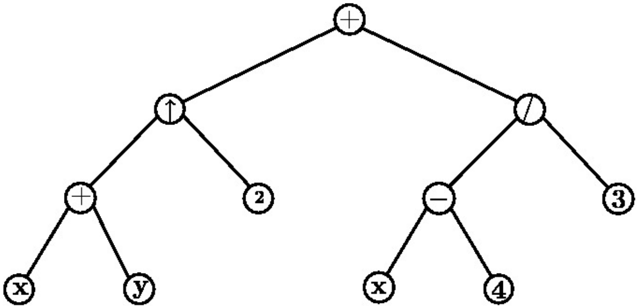

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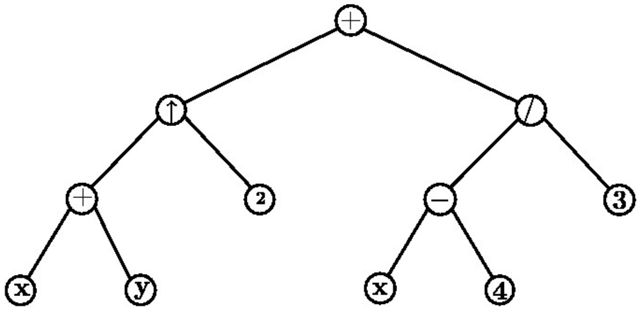

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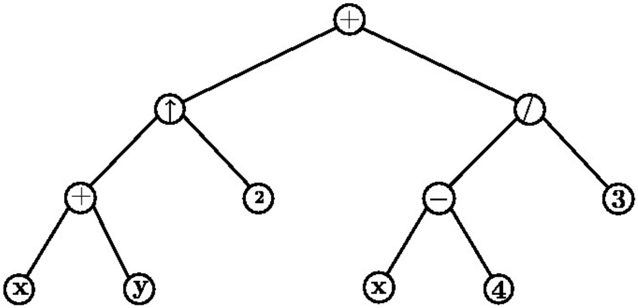

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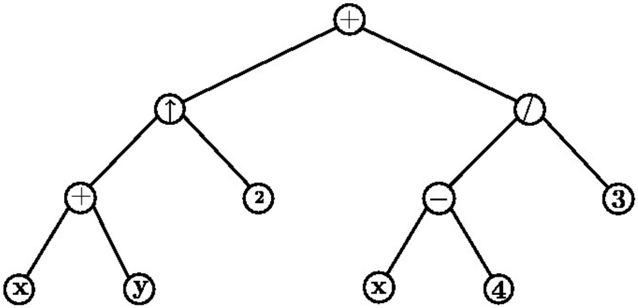

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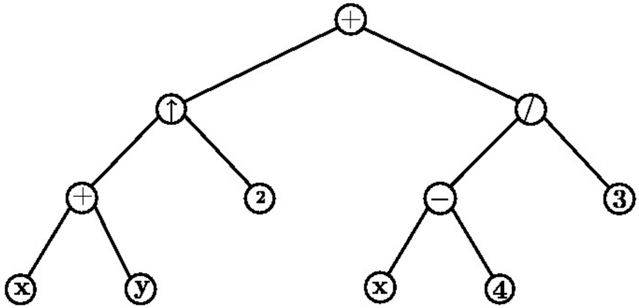

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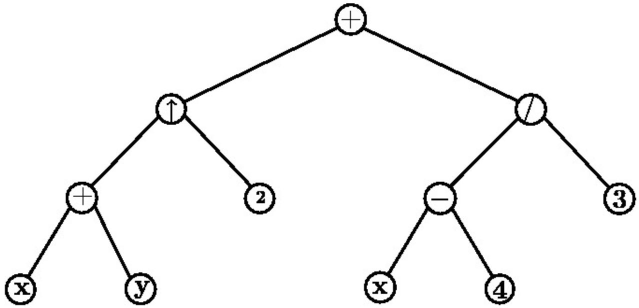

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