Discrete Mathematics

Chapter 4: Induction and Recursions

Department of Mathematics
The FPT university

Topics covered:

Chapter 4: Introduction

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Topics covered:

4.1 Mathematical Induction

4.2 Strong Induction and Well-Ordering

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4.1 Mathematical Induction

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4.3 Recursive Definitions and Structural Induction

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4.4 Recursive Algorithms

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4.1 Mathematical Induction

4.2 Strong Induction and Well-Ordering

4.3 Recursive Definitions and Structural Induction

4.4 Recursive Algorithms

4.5 Program Correctness

4.1 Mathematical Induction

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Example 4. Let n be a positive integer. Prove that every checkerboard of size $2^n \times 2^n$ with one square removed can be titled by triominoes.

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Example 2. Prove that every postage of 12 cents or more can be formed using only 4-cent and 5-cent stamps.

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Any nonempty set of non-negative integers has a least element.

4.3 Recursive Definitions and Structural Induction

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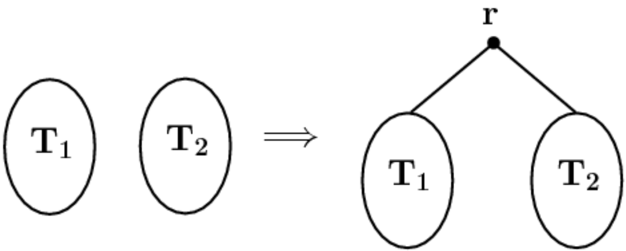

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> Basic step: Prove that P is true for elements of S defined in the basic step.
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> Recursive step: Show that if the property P is true for the elements used to construct new elements in the recursive step of the definition of S, then the property P is also true for these new elements.

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Prove that $a_{m,n} = m + n(n+1)/2$ for all $m, n \geq 0$.

4.4 Recursive Algorithms

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Procedure power (n: non-negative)
if $n = 0$ then $power(0) := 1$
else $power(n) := power(n-1) * 5$

Example 2. Write a recursive algorithm to compute $n!$.

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Example 3. Write a recursive algorithm to compute the greatest common divisor of two non-negative integers.

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Recursion and Iteration

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if $n = 0$ then $y := 0$
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  x:=0
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Procedure Fib (n)
if $n = 0$ then $Fib(0) := 0$
else if $n = 1$ then $Fib(1) := 1$
else
$Fib(n) := Fib(n-1) + Fib(n-2)$

Merge Sort Algorithm

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Procedure mergesort $(L = a_1, a_2, \ldots, a_n)$
if $n > 1$ **then**
$m := \lfloor n/2 \rfloor$
$L_1 = a_1, a_2, \ldots, a_m$
$L_2 = a_{m+1}, a_{m+2}, \ldots, a_n$
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Theorem

The number of comparisons needed to merge sort a list of n elements is $O(n \log n)$.