Discrete Mathematics - MAD101

Chapter 1: Propositional Logic

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FPT Education

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Motivations:

- Logic is the basis of all mathematical reasoning
- Logic is foundation of automated reasoning
- Applications in: design of computer hardware (logic circuits), artificial intelligence, programming languages, machine translation,... and many other computer science fields.

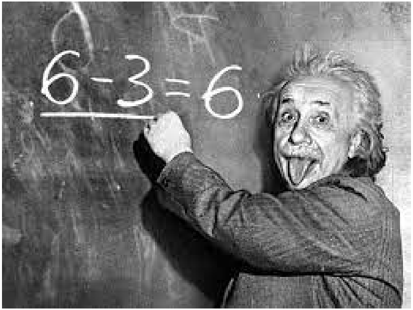
(Propositional logic helps us to understand mathematical arguments and proofs).

Propositional logic

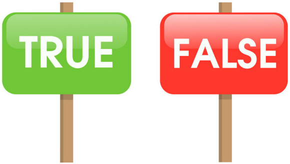
A proposition is a statement that is either true or false (but not both).

Examples

- The water boils at 100 degrees Celsius.
- There are 12 students in Discrete Mathematics Class.
- Do you have a cat?
- $x^2 + 4 \geq y$.
- She is so sexy!



Remarks



- A proposition declares a fact (either correct or incorrect).
- A proposition can be viewed as a number with value 1 (true) or 0 (false).
- Therefore, propositional logic is a mathematical system that works with 0 and 1 numbers.

Propositional Operators

1. Negation operator

- **Notation:** $\neg p$ (sometimes $\sim p$).
- **Examples:**
  1. He is rich $\longrightarrow$ He is NOT rich.
  2. The girl is pretty $\longrightarrow$ The girl is NOT pretty.
  3. Discrete math is not difficult $\longrightarrow$ Discrete math is difficult.
- **Truth table:**

p	$\neg p$
0	1
1	0

Propositional Operators

2. Conjunction operator (and)

- **Notation:** $p \wedge q$
- **Examples:**
  1. $p =$ "He is fat". $q =$ "He is super rich".
     $\longrightarrow$ $p \wedge q$ = He is fat BUT super rich.
  2. $p =$ "The school is closing". $q =$ "The kids are not happy".
     $\longrightarrow$ The school is closing AND the kids are not happy".
- **Truth table:**

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Propositional Operators

3. Disjunction operator (or)

- **Notation:** $p \vee q$
- **Examples:**
  1. p = "She is smart". q = "She is hard-working".
     $\longrightarrow$ She is smart OR hard-working (cause she passed the MAD101 course).
  2. At least one of the suspects is the murder.
- **Truth table:**

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

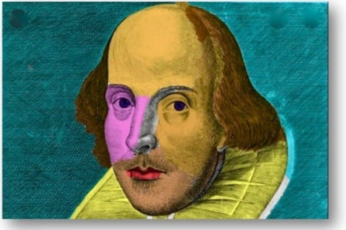
Propositional Operators

4. Exclusive-or operator (XOR)

- **Notation:** $p \oplus q$
- **Examples:**
  1. p = "She will marry Peter". q ="She will marry Tom."
     $\longrightarrow$ $p \oplus q$ = She will marry either Peter or Tom.
  2. You never lose. You either win or learn.
- **Truth table:**

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

“To be or not to be”

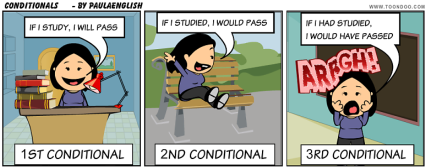


Propositional Operators

5. Implication

- **Notation:** $p \to q$
- **Examples:**
  1. p = "You don't water your plants". q ="The plants die."
     $p \to q$ = If you don't water your plants, they will die.
  2. If I am a billionaire, I will buy each of my students a Mercedes G63.
- **Truth table:**

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1



Propositional Operators

6. Bi-conditional operator

- **Notation:** $p \leftrightarrow q$
- **Examples:**
  1. $x \geq y \leftrightarrow x - y \geq 0.$
  2. Two triangles are congruent if and only if all three pairs of corresponding sides are congruent.
- **Truth table:**

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Precedence of Logical Operators

From high to low priority:

- 1. Negation
- 2. Conjunction
- 3. Disjunction
- 4. Exclusive-or
- 5. Implication
- 6. Bi-condition

Summarization (truth table):

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	1	0	0	1	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

Example 1: Evaluating Compound Proposition

Evaluate:

$$A = \neg p \to (q \oplus r \to \neg p \wedge q) \leftrightarrow s$$

in which $p = 0, q = 1, r = 0, s = 1$.
Solution: Substitute p, q, r, s to the expression:

$$\begin{aligned} A &= \neg 0 \to (1 \oplus 0 \to \neg 0 \wedge 1) \leftrightarrow 1 \\ &= 1 \to (1 \oplus 0 \to 1 \wedge 1) \leftrightarrow 1 \\ &= 1 \to (1 \oplus 0 \to 1) \leftrightarrow 1 \\ &= 1 \to (1 \to 1) \leftrightarrow 1 \\ &= 1 \to 1 \leftrightarrow 1 \\ &= 1 \leftrightarrow 1 \\ &= 1. \end{aligned}$$

Example 2: Formalize a proposition

Let p, q, and r be the propositions

- p: Grizzly bears have been seen in the area.
- q: Hiking is safe on the trail.
- r: Berries are ripe along the trail.

Write the following propositions using p, q, and r and logical connectives (including negations).

1. Berries are ripe along the trail, but grizzly bears have not been seen in the area.
2. Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
3. If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.
4. It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.
5. For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.
6. Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

Example 2 (continue)

- p: Grizzly bears have been seen in the area.
- q: Hiking is safe on the trail.
- r: Berries are ripe along the trail.

Solution

1. Berries are ripe along the trail, but grizzly bears have not been seen in the area. $r \wedge \neg p$
2. Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail. $\neg p \wedge q \wedge r$
3. If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area. $r \rightarrow (q \leftrightarrow \neg p)$
4. It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe. $\neg q \wedge \neg p \wedge r$
5. For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area. $q \rightarrow (\neg r \wedge \neg p)$
6. Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail. $p \wedge r \rightarrow \neg q$

Tautology, Contradiction and Logical Equivalences

Definitions

A **contradiction** is a statement that is always false.
A **tautology** is a statement that is always true.
If $p \leftrightarrow q$ is a tautology, then p is **logically equivalent** to q. We denote by $p \equiv q$.

Remark

- Two statements p and q are said to be logically equivalent if they have the same truth value in every model.
- A logical equivalence can be verified by comparing the two corresponding columns in the Truth Table.

Example: Prove that $p \to q$ is logically equivalent to $\neg p \vee q$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Useful logical equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

A long list, but the most important to remember are:

- **Double negation:** $\neg(\neg p) \equiv p$
- **Distribution:** $p \wedge (q \vee r) \equiv p \wedge q \vee p \wedge r$
- **De Morgan:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- **Conditional:** $p \rightarrow q \equiv \neg p \vee q$
- **Bi-conditional:** $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Quantifiers

Let $P(x)$= "$x^2 \geq 5$".

- $P(x)$ is not a proposition since its truth value is unknown.
- But if $x = x_0$ is given, then $P(x_0)$ is a proposition.

Propositional function

A **propositional function** is a mapping $P : \mathbb{X} \to \{0, 1\}$, in which $\mathbb{X}$ is a given set of parameters.

Examples

- $P(x)$: "x is the prettiest girl in our school".
- $Q(x, y)$: "x loves y".
- $R(x, y, z)$: $x^2 + y^2 + z^2 \geq 3xy + 3yz + 3xz$.

Quantifiers

Definition

- **Universal quantifiers** ($\forall$) : $\forall x$. x passes the MAD exam.
- **Existential quantifiers** ($\exists$): $\exists x$. x wins "Best Student" award.
- **Uniquely existential quantifiers** ($\exists!$): $\exists! x.\ x^3 = 27$.

Examples

Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n (n^2 \geq 0)$ **b)** $\exists n (n^2 = 2)$

c) $\forall n (n^2 \geq n)$ **d)** $\exists n (n^2 < 0)$

Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x (x^2 = 2)$ **b)** $\exists x (x^2 = -1)$

c) $\forall x (x^2 + 2 \geq 1)$ **d)** $\forall x (x^2 \neq x)$

Negation of Quantified Expressions

Rules

- $\neg\big(\forall x.P(x)\big) \equiv \exists x.\neg P(x)$.
- $\neg\big(\exists x.Q(x)\big) \equiv \forall x.\neg Q(x)$.
- Negation of universal is existential and vice versa.

Nested quatifications