Discrete Mathematics

Chapter 2: Basic Structures: Sets, Functions, Sequences and Sums

Department of Mathematics
The FPT university

2.1 Sets

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- Complement of A with respect to the universal set U: $\overline{A} = U - A$

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De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

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- Use Membership table, similar to the method of using truth table to establish propositional equivalences.

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Example. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

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Example. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then the subset $A = \{1, 3, 4, 6\}$ is represented as the bit string 10110100.

2.3 Functions

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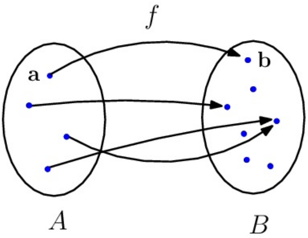

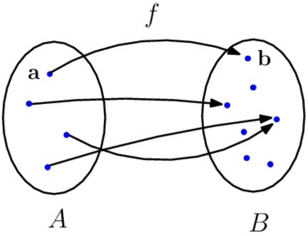
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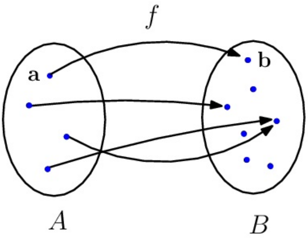

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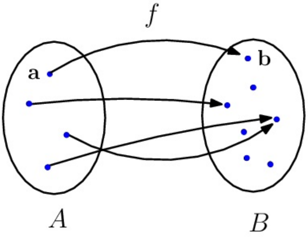

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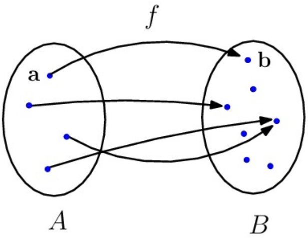

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Let S be a subset of A. The set

$$f(S) = \{b \in B | \exists a \in A(f(a) = b)\}$$

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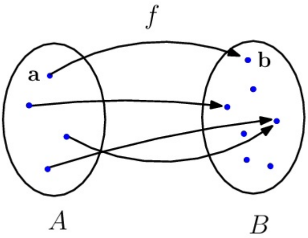

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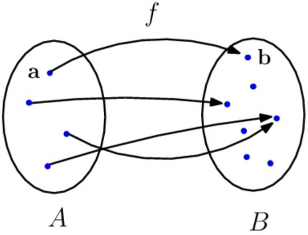
2.3 Functions



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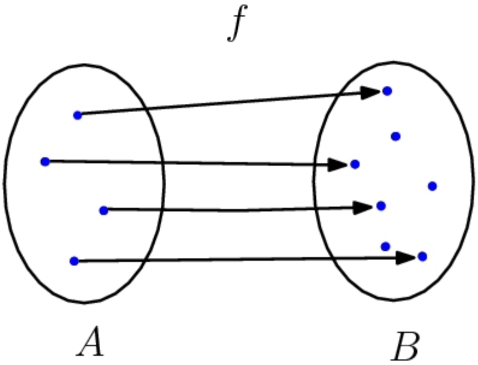

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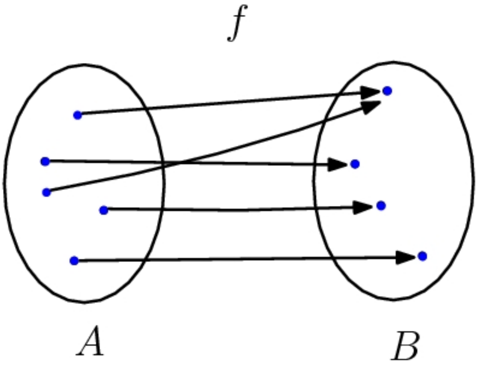

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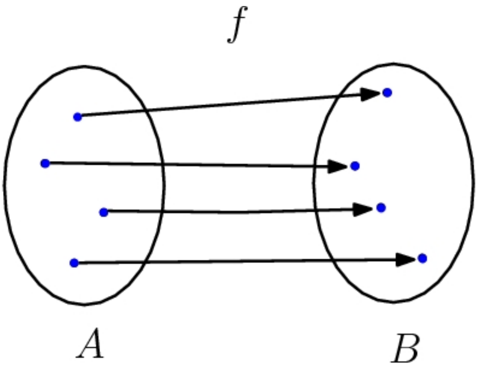

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(d) A bijection from the set of all real numbers to the set of positive real numbers?

Compositions and Inverses

Composition

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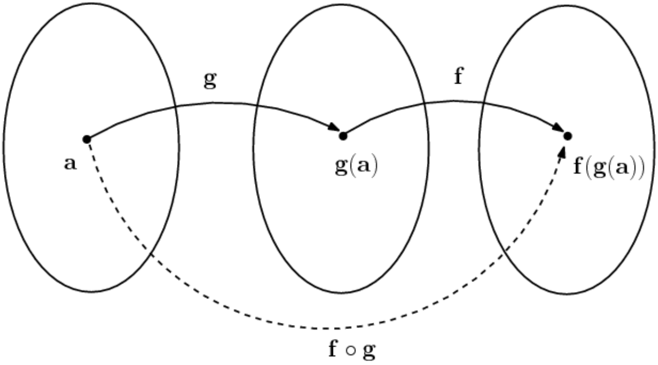

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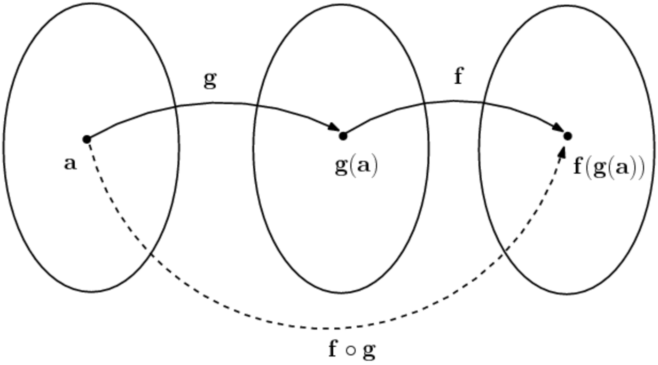

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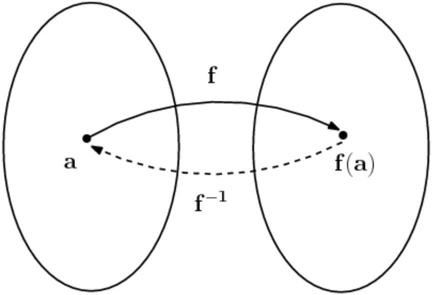

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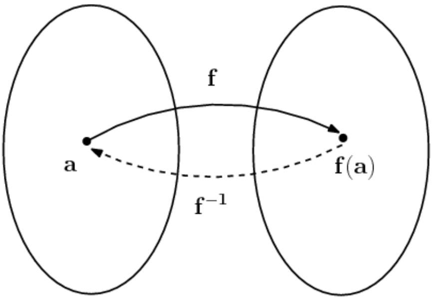
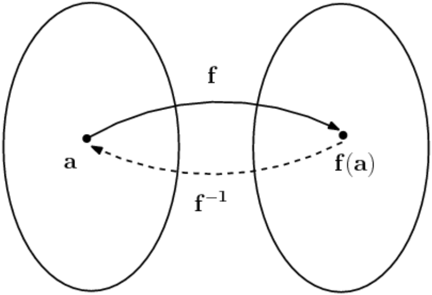


Note.

Inverse



Note. The function $f : A \to B$ has an inverse if and only if f is a bijection.

Sequences

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2.4 Sequences and Summations

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