Bài 4.1:

a) $x(t) = e^{j\omega_0 t} u(t)$

$$\Rightarrow X(s) = \int_{-\infty}^{+\infty} e^{j\omega_0 t} u(t) \cdot e^{-st} dt.$$

$$= \int_0^{\infty} e^{(j\omega_0 - s)t} dt = \left. \frac{e^{(j\omega_0 - s)t}}{j\omega_0 - s} \right|_{t=0}^{t=+\infty}$$

$$= \frac{e^{(j\omega_0 - s)\infty} - 1}{j\omega_0 - s}$$

$\exists$ biến đổi laplace $\Leftrightarrow \begin{cases} j\omega_0 \neq s \\ e^{(j\omega - s)+\infty} = 0. \end{cases}$

Mà $e^{(j\omega_0 - s)+\infty} = e^{Re(j\omega_0 - s)+\infty} \cdot e^{j Im(j\omega_0 - s)+\infty}.$

$\Leftrightarrow Re(j\omega_0 - s) < 0$

$\Leftrightarrow Re(j\omega_0) < Re(s)$

$\Rightarrow Re(s) > 0.$

Vậy ROC: $Re(s) > 0$.

Điểm cực: $s - j\omega_0 = 0 \Leftrightarrow s = j\omega_0.$

b) $y = \sin(3t)\, u(t)$

$$\Rightarrow X(s) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} dt$$

$$= \int_0^{+\infty} \sin(3t) \cdot e^{-st} dt.$$

$$= \int_0^{+\infty} \frac{e^{j3t} - e^{-j3t}}{2j} \cdot e^{-st} dt.$$

$$= \frac{1}{2j}\int_0^{\infty} e^{(3j-s)t}\,dt - \frac{1}{2j}\int_0^{\infty} e^{-(3j+s)t}\,dt$$

$$= \frac{1}{2j(3j-s)} e^{(3j-s)t}\Big|_0^{+\infty} - \frac{1}{-2j(3j+s)} e^{-(3j+s)t}\Big|_0^{+\infty}$$

$$= \frac{-1}{2j(3j-s)} + \frac{-1}{2j(3j+s)}$$

$$= \frac{j}{2(3j-s)} + \frac{j}{2(3j+s)}.$$

$$= \frac{j(3j+s) + j(3j-s)}{2(3j-s)(3j+s)}$$

$$= \frac{-6}{2[(3j)^2 - s^2]}.$$

$$= \frac{-6}{2(-9-s^2)}$$

$$= \frac{-6}{-2(s^2+9)} = \frac{3}{s^2+9}.$$

ROC: $Re(s) > 0$.

Điểm cực : $s^2+9=0 \Leftrightarrow s^2 = \pm j3.$

e) $x(t) = e^{-2t}\cdot u(t) + e^{-3t}\cdot u(t)$.

$= x_1(t) + x_2(t)$.

$x(t) = e^{-\alpha t}\, u(t)$.

$\Rightarrow X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$.

$= \int_0^{\infty} e^{-\alpha t}\cdot e^{-st}\, dt$

$= \int_0^{\infty} e^{(-\alpha - s)t}\, dt$

$= \frac{1}{-\alpha - s}\cdot e^{(-\alpha - s)t}\Big|_0^{+\infty}$.

$= \frac{1}{\alpha + s}$ $\quad Re(-\alpha - s) < 0$

$Re(-\alpha) < Re(s)$.

$\Rightarrow Re(s) > -\alpha$.

$\Rightarrow X_1(s) = \frac{1}{s+2}$ vs ROC: $Re(s) > -2$.

$X_2(s) = \frac{1}{s+3}$ vs ROC: $Re(s) > -3$.

$\Rightarrow X(s) = X_1(s) + X_2(s) = \frac{1}{s+2} + \frac{1}{s+3}$

$= \frac{2s+5}{(s+2)(s+3)}$

$ROC = ROC[X_1(s)] \cap ROC[X_2(s)] = R(s) > -3$.

[illegible]

Điểm không: $s = \frac{\cdot}{2}$.

Điểm cực: $\begin{cases} s = -2 \\ s = -3 \end{cases}$.

Bài 4.2: Biến đổi laplace 1 phía.

a) $x(t) = e^{-t}(t-2)u(t-2)$

$$X(s) = \int_0^\infty x(t) \cdot e^{-st} dt$$

$$= \int_0^\infty e^{-t}(t-2)\,u(t-2)\,e^{-st} dt$$

$$= \int_0^\infty e^{(-1-s)t}(t-2)\,dt$$

$$= \frac{1}{1-s} e^{(-1-s)t}\, t-2 \Big|_2^\infty$$

$$- \int_2^\infty (-s-1)\, e^{(-1-s)t} dt$$

$$= \lim_{t\to\infty} \frac{-e^{(-1-s)t}(t-2) - e^{(-1-s)t}}{s+1}\Big|_2^\infty$$

+ $y(t) = e^{-t} \cdot t \cdot u(t)$

$$Y(s) = \int_0^\infty y(t) \cdot e^{-st} dt = \int_0^\infty e^{-t} \cdot t \cdot e^{-st} dt$$

$\int_0^\infty e^{(-1-s)t}$

$$= \int_0^{\infty} e^{(-1-s)t} t\, dt$$

$$= \frac{1}{-(s+1)} e^{(-1-s)t} t \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{-(s+1)} \cdot e^{-(1+s)t} dt$$

$$= \frac{-e^{(-1-s)t} t}{s+1}\Big|_0^{\infty} - \frac{1}{(s+1)^2} e^{-(1+s)t}\Big|_0^{\infty}$$

$$= \frac{1}{(s+1)^2} - \lim_{t \to +\infty} \frac{e^{(-1-s)t} t}{s+1} - \lim_{t \to +\infty} \frac{e^{-(1+s)t}}{(s+1)^2}$$

Bài 4.3

giá trị đầu: $x(0) = \lim_{s \to \infty} s \cdot X(s)$.

giá trị cuối: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} s \cdot X(s)$.

a) $X(s) = e^{-5s} \cdot \frac{-2}{s(s+2)}$

$x(0) = \lim_{s \to \infty} s \cdot X(s) = \lim_{s \to \infty} e^{-5s} \cdot \frac{-2}{s+2} = 0$.

$x(\infty) = \lim_{s \to 0} s \cdot X(s) = \lim_{s \to 0} e^{-5s} \cdot \frac{-2}{s(s+2)}$

$$= \lim_{s \to 0} e^{-Ts} \cdot \frac{-2}{s+2} = -1.$$

a) $X(s) = \dfrac{2s+3}{s^2+5s+6}$.

$$x(0) = \lim_{s \to \infty} \frac{2s^2+3s}{s^2+5s+6} = \lim_{s \to \infty} \frac{2+\frac{3}{s}}{1+\frac{5}{s}+\frac{6}{s^2}} = 2$$

$$x(\infty) = \lim_{s \to 0} \frac{2s^2+3s}{s^2+5s+6} = 0.$$

Bài 4.4:

$$x(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st} ds.$$

$$\boxed{\begin{aligned} L^{-1}\left(\frac{1}{s-a}\right) &= \begin{cases} e^{at} u(t) & \sigma > a. \\ -e^{at} u(-t) & \sigma < a \end{cases} \\ L^{-1}\left[\frac{1}{(s-a)^n}\right] &= \begin{cases} \frac{t^{n-1}}{(n-1)!} e^{at} \cdot u(t) & \sigma > a \\ -\frac{t^{n-1}}{(n-1)!} e^{at} \cdot u(-t) & \sigma < a. \end{cases} \end{aligned}}$$

a): $X(s) = \dfrac{-5s-7}{(s+1)(s-1)(s+2)}$

$$\Rightarrow X(s) = \frac{1}{s+1} - \frac{2}{s-1} + \frac{1}{s+2}.$$

$$\Rightarrow L^{-1}[X(s)] = L^{-1}\left(\frac{1}{s+1}\right) - L^{-2}\left(\frac{2}{s-1}\right) + L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow \begin{cases} L^{-1}\left(\dfrac{1}{s+1}\right) = \begin{cases} e^{-t}\cdot u(t) & \text{nếu } \sigma > -1 \quad (1) \\ -e^{-t}\cdot u(-t) & \text{nếu } \sigma < -1 \quad (2) \end{cases} \\ L^{-1}\left(\dfrac{2}{s-1}\right) = \begin{cases} 2\cdot e^{t}\cdot u(t) & \text{nếu } \sigma > 1 \quad (3) \\ -2\cdot e^{t}\cdot u(-t) & \text{nếu } \sigma < 1 \quad (4) \end{cases} \\ L^{-1}\left(\dfrac{1}{s+2}\right) = \begin{cases} e^{-2t}\cdot u(t) & \text{nếu } \sigma > -2 \quad (5) \\ -e^{-2t}\cdot u(-t) & \text{nếu } \sigma < -2 \quad (6) \end{cases} \end{cases}$$

$$\underbrace{(2)-(4)+(6)}_{}\;\Big|_{-2}\;\underbrace{(2)-(4)+(5)}_{}\;\Big|_{-1}\;\underbrace{(1)-(4)+(5)}_{}\;\Big|_{1}\;\underbrace{(1)-(3)+(5)}_{}$$

$$\Rightarrow x(t) \begin{cases} (2)-(4)+(6), & -\infty < \sigma < -2 \\ (2)-(4)+(5), & -2 < \sigma < -1 \\ (1)-(4)+(5), & -1 < \sigma < 1 \\ (1)-(3)+(5), & 1 < \sigma < \infty \end{cases}$$

b) $$X(s) = \frac{s}{s^2+5s+6}$$

$$\Rightarrow X(s) = \frac{-2}{s+2} + \frac{3}{s+3}$$

$$\begin{cases} L^{-1}\left(\dfrac{-2}{s+2}\right) = \begin{cases} -2\cdot e^{-2t}u(t) & \sigma > -2 \\ 2\cdot e^{-2t}u(-t) & \sigma < -2 \end{cases} \\ L^{-1}\left(\dfrac{3}{s+3}\right) = \begin{cases} 3\cdot e^{-3t}\cdot u(t) & \sigma > -3 \quad (3) \\ -3\cdot e^{-3t}\cdot u(-t) & \sigma < -3 \quad (4) \end{cases} \end{cases}$$

$$(1)+(4)\;\Big|_{-3}\;(1)+(3)\;\Big|_{-2}\;(2)+(3) \rightarrow \sigma$$

c) $X(s) = \frac{s^2+s-3}{s^2+3s+2} = \frac{s^2+3s+2-2s-5}{s^2+3s+2} = 1 - \frac{2s+5}{(s+2)(s+1)}$.

$\Rightarrow L^{-1}[X(s)] = \delta(t) - L^{-1}\left[\frac{2s+5}{(s+2)(s+1)}\right]$

Xét: $\frac{2s+5}{(s+2)(s+1)} = \frac{3}{s+1} - \frac{1}{s+2}$.

$\Rightarrow L^{-1}\left[\frac{2s+5}{(s+2)(s+1)}\right] = 3L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)$

$L^{-1}\left[\frac{1}{s-(-1)}\right] = \begin{cases} e^{-t}\cdot u(t) & \sigma > -1 \\ -e^{-t}\cdot u(-t) & \sigma < -1 \end{cases}$.

Và $L^{-1}\left[\frac{1}{s-(-2)}\right] = \begin{cases} e^{-2t}\cdot u(t) & \sigma > -2 \\ -e^{-2t}\cdot u(t) & \sigma < -2 \end{cases}$.

d) $X(s) = \frac{4s^2+6}{s^3+s^2-2}$

$L^{-1}\left[\frac{s+a}{(s-a)^2+b^2}\right] = \cos(bt)e^{at}\cdot u(t)$ $\sigma > \alpha$.

$L^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = \sin(bt)\cdot e^{at}\cdot u(t)$ $\sigma > \alpha$.

$X(s) = \frac{4s^2+6}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B}{s^2+3s+2}$.

$= \frac{2}{s-1} + \frac{2s-2}{(s+2)^2+1}$.

$\Rightarrow X(s) = \frac{2}{s-1} + 2\frac{s+1}{(s+1)^2+1} - 4\cdot\frac{1}{(s+1)^2+1}$

$L^{-1}\left(\frac{2}{s-1}\right) = \begin{cases} 2\cdot e^{t}\cdot u(t) & \sigma > 1 \end{cases}$

$-2\cdot e^{t}\cdot u(-t)\quad \sigma<1$

$$L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]=\cos(t)\cdot e^{-t}\cdot u(t)\ \text{vs}\ \sigma>1$$

$$L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=\sin(t)\cdot e^{-t}\cdot u(t)\ \text{vs}\ \sigma>1$$

Bài 45:

a) $$\frac{dy(t)}{dt}+3y(t)=4x(t)\quad\begin{cases}x(t)=\cos(2t)\,u(t)\\ y(0)=-2.\end{cases}$$

laplace 1 phía 2 vế p.trình:

$$s\cdot Y(s)-y(0)+3Y(s)=4X(s)$$

$$\Rightarrow Y^1(s)(s+3)=4X^1(s)+y(0)$$

$$\Rightarrow Y^1(s)=\frac{4X^1(s)}{s+3}+\frac{y(0)}{s+3}\ \rightarrow t=0$$

trạng thái 0: $Y_s^1(s)$ $\qquad Y_0^1(s)$

Mà $X^1(s)=L^1[\cos(2t)\cdot u(t)]=\dfrac{s}{s^2+4}$

$$\Rightarrow Y^1(s)=\frac{4s}{\ldots}+\frac{-2}{\ldots}$$

$$(s+3)(s^2+4) \qquad s+3$$

$$\Rightarrow * \quad y_s(t) = \mathcal{L}^{-1}\left[\frac{4s}{4(s+3)(s^2+4)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{-12/13}{s+3} + \frac{\frac{12}{13}s + \frac{16}{13}}{s^2+4}\right]$$

$$\mathcal{L}^{-1}\left[\frac{-12/13}{s+3}\right] = \frac{-12}{13} \cdot e^{-3} \cdot u(t).$$

$$\mathcal{L}^{-1}\left[\frac{\frac{12}{13}s + \frac{16}{13}}{s^2+4}\right] = \frac{12}{13}\cos(2t) \cdot e^{0t} \cdot u(t) + \frac{8}{13} \cdot \sin(2t) \cdot e^{0t} \cdot u(t).$$

$$\Rightarrow y_0(t) = \mathcal{L}^{-1}\left[\frac{-2}{s+3}\right] = -2 \cdot e^{-3t} \cdot u(t).$$

b) $$\frac{d^2 y(t)}{dt^2} + 4y(t) = 8x(t) \quad \begin{cases} x(t) = u(t) \\ y(0) = 1 \\ \left.\frac{dy(t)}{dt}\right|_{t=0} = 2. \end{cases}$$

laplace 1 vế 2pt:

$$\mathcal{L}\left[\frac{d^2 y(t)}{dt^2}\right] = s^2 \cdot Y(s) - s\,y(0) - \left.\frac{dy(t)}{dt}\right|_{t=0}.$$

$$= s^2 \cdot Y(s) - s \cdot 1 - 2$$

$$= s^2 \cdot Y(s) - (s+2).$$

$$L^{-1}[u(t)] = \frac{1}{s}.$$

$$\Rightarrow s^2 y(s) - (s+2) + 4y(s) = \frac{8}{s}$$

$$\Rightarrow y(s)(s^2+4) = \frac{8}{s} + (s+2).$$

$$\Rightarrow y(s) = \underbrace{\frac{8}{s(s^2+4)}}_{y_s(s)} + \underbrace{\frac{s+2}{s^2+4}}_{y_0(s)}.$$

$$\Rightarrow \text{④ } y_s(t) = L^{-1}\left[\frac{1}{s(s^2+4}\right] = L^{-1}\left[\frac{2}{s} + \frac{-2s}{s^2+4}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right] - 2 \cdot L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= 2u(t) - 2\cos(2t)\,u(t)$$

$$\text{④ } y_0(t) = L^{-1}\left[\frac{s+2}{s^2+4}\right]$$

$$= L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= \cos(2t) \cdot e^{0t} \cdot u(t) + \sin(2t) \cdot e^{0t} \cdot u(t).$$

$$= [\cos(2t) + \sin(2t)] \cdot u(t)$$