Introduction to Bayesian Statistics

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24 February 2016

A semester's worth of material in just a few dozen slides.

Why Bayes?

Because Bayes answers the questions we really care about.

Pr(I have disease | test +) vs Pr(test + | disease)

Pr(A better than B | data) vs Pr(extreme data | A=B)

Bayes is natural (vs interpreting a CI or a P-value)

Note: blue = Bayesian, red = frequentist

You are waiting on a subway platform for a train that is known to run on a regular schedule, only you don't know how much time is scheduled to pass between train arrivals, nor how long it's been since the last train departed.

As more time passes, do you (a) grow **more confident** that the train will arrive soon, since its eventual arrival *can only be getting closer*, not further away, or (b) grow **less confident** that the train will arrive soon, since the longer you wait, the more likely it seems that either the scheduled arrival times are far apart or else that you happened to arrive just after the last train left – or both.

If you choose (a), you're thinking like a frequentist. If you choose (b), you're thinking like a Bayesian.

from Floyd Bullard

An opaque jar contains thousands of beads (but obviously a finite number!). You know that all the beads are either red or white but you have no idea at all what fraction of them are red. You begin to draw beads out of the bin at random without replacement. You notice that all of the first several beads have been red. As you observe more and more red beads, is the conditional probability (i.e., conditional upon the previous draws' colors) of the next bead being red (a) **decreasing**, ***as it must***, since you're removing red beads from a finite population, or (b) **increasing**, because you initially didn't know that there would be so many reds, but now it seems that the jar must be mostly reds.

If you choose (a), you're thinking like a frequentist. If you choose (b), you're thinking like a Bayesian.

from Floyd Bullard

Probability

Limiting relative frequency:

$$\mathbf{P(a) = \lim_{n\to\infty}} \frac{\text{\#times a happens in n trials}}{n}$$

A nice definition mathematically, but not so great in practice. (So we often appeal to symmetry…)

What if you can’t get all the way to infinity today?

What if there is only 1 trial? E.g., P(snow tomorrow)

What if appealing to symmetry fails? E.g., I take a penny out of my pocket and spin it. What is P(H)?

Subjective probability

$$\text{Prob} = \frac{\textbf{odds}}{1+\textbf{odds}} \qquad \text{Odds} = \frac{\textbf{prob}}{1-\textbf{prob}}$$

Fair die

Event	Prob	Odds	
even #	1/2	1	[or 1:1]
X > 2	2/3	2	[or 2:1]
roll a 2	1/6	1/5	[or 1/5:1 or 1:5]

Persi Diaconis: "Probability isn't a fact about the world; probability is a fact about an observer's knowledge."

Bayes' Theorem

$P(a|b) = P(a,b)/P(b)$

$P(b|a) = P(a,b)/P(a)$ ➔ $P(a,b) = P(a)P(b|a)$

Thus, $P(a|b) = P(a,b)/P(b) = P(a)P(b|a)/P(b)$

But what is P(b)? $P(b) = \Sigma_a P(a,b) = \Sigma_a P(a)P(b|a)$

Thus, $P(a|b) = \dfrac{\mathbf{P(a)P(b\,|\,a)}}{\displaystyle\mathop{\text{å}}_{\mathbf{a*}} \mathbf{P(a*)P(b\,|\,a*)}}$

Where a* means "any value of a"

$$P(a|b) = \frac{\mathbf{P(a)P(b\,|\,a)}}{\underset{\mathbf{a\,*}}{\text{å}}\ \mathbf{P(a^*)P(b\,|\,a^*)}}$$

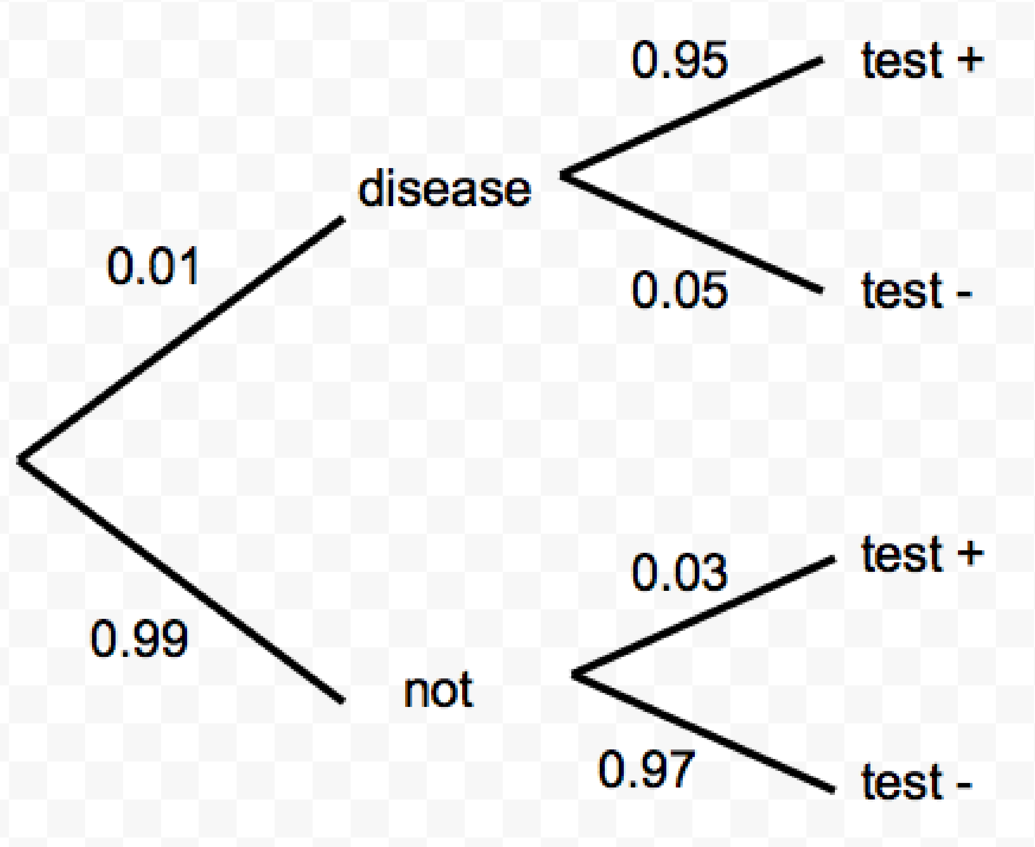
P(a|b) µ P(a) P(b|a)

posterior µ (prior) (likelihood)

Bayes’ Theorem is used to take a prior probability, update it with data (the likelihood), and get a posterior probability.

Medical test example. Suppose a test is 95% accurate when a disease is present and 97% accurate when the disease is absent. Suppose that 1% of the population has the disease.

What is P(have the disease | test +)?



$$\mathbf{p(dis\,|\,test+) = \frac{P(dis)P(test+\,|\,dis)}{P(dis)P(test+\,|\,dis) + P(\text{Ø}dis)P(test+\,|\,\text{Ø}dis)}}$$

$$= \frac{(0.01)(0.95)}{(0.01)(0.95) + (0.99)(0.03)} = \frac{0.0095}{0.0095 + 0.0297} \text{ » } 0.24$$

Typical statistics problem: There is a parameter, θ, that we want to estimate, and we have some data.

Traditional (frequentist) methods: Study and describe P(data | θ). If the data are unlikely for a given θ, then state “that value of θ is not supported by the data.” (A hyp. test asks whether a *particular* value of θ might be correct; a CI presents a range of plausible values.)

Bayesian methods: Describe the distribution P(θ | data).

A frequentist thinks of θ as fixed (but unknown) while a Bayesian thinks of θ as a random variable that has a distribution.

Bayesian reasoning is natural and easy to think about. It is becoming much more commonly used.

If Bayes is so great, why hasn’t it always been popular?

(1) Without Markov Chain Monte Carlo, it wasn’t practical.

(2) Some people distrust prior distributions, thinking that science should be objective (as if that were possible).

Bayes is becoming much more common, due to MCMC.

E.g., Three recent years worth of *J. Amer. Stat. Assoc.* “applications and case studies” papers: 46 of 84 papers used Bayesian methods (+ 4 others merely included an application of Bayes’ Theorem).

What is MCMC?

We want to find P(θ | data), which is equal to P(θ)*P(data |θ)/P(data) and is proportional to P(θ)*P(data | θ), which is "prior*likelihood”.

But to use Bayesian methods we need to be able to evaluate the denominator, which is the integral of the numerator, integrated over the parameter space. In general, this integral is very hard to evaluate.

Note: There are two special cases in which the mathematics works out: for a beta-prior with binomial-likelihood (which gives a beta posterior) and for a normal prior with normal likelihood (which gives a normal posterior).

But for general cases the mathematics is not tractable. However, MCMC has given us a way around this. We don't need to evaluate any integral, *we just sample from the distribution many times* (e.g., 50K times) and find (estimate) the posterior mean, middle 95%, etc., from that.

How does MCMC work? It is based on the Metropolis algorithm…

Note: These days we typically use Gibbs sampling, which is a variation on Metropolis.

Old software: BUGS – Bayes Using Gibbs Sampling

Newer software: JAGS – Just Another Gibbs Sampler

Newest software: Stan – uses Hamiltonian Monte Carlo

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Beta random variables

$$\mathbf{p(\theta \mid a,b) = \theta^{a-1}(1-\theta)^{b-1} / B(a,b)}$$

where

$$\mathbf{B(a,b) = \int_0^1 \theta^{a-1}(1-\theta)^{b-1}\, d\theta \;=\; \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

where

$$\mathbf{\Gamma(a) = (a-1)! = \int_0^\infty e^{-x} x^{a-1}\, dx}$$

$$\mathbf{Fact: mean = \frac{a}{a+b} \qquad mode = \omega = \frac{a-1}{a+b-2}}$$

$$\mathbf{\kappa = a + b = "concentration"}$$

Run BetaPlot.R….

Binomial (Bernoulli) data and a Beta prior

Posterior µ Prior $*$ Likelihood

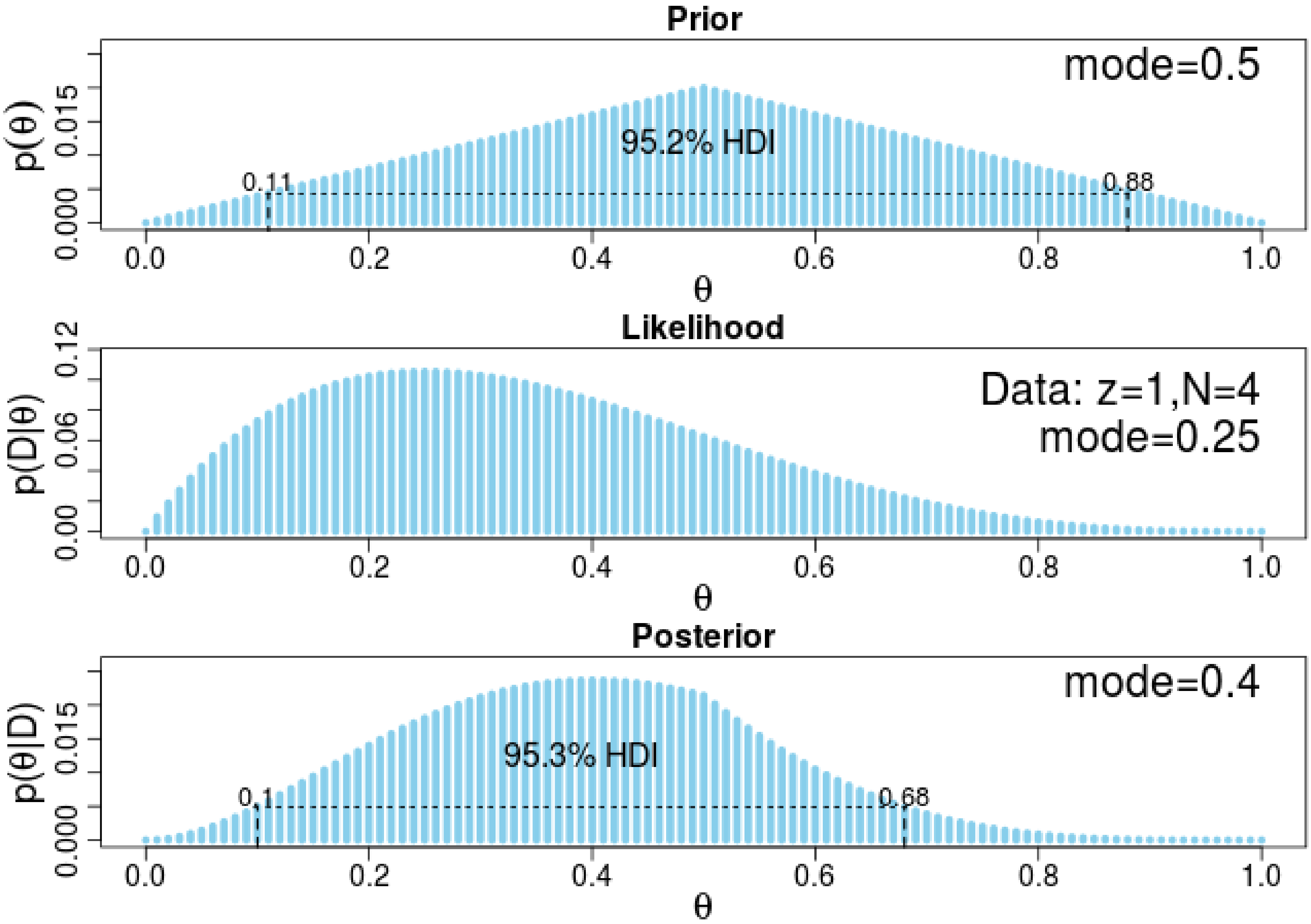
$$\propto \theta^{a-1}(1-\theta)^{b-1} * \theta^{z}(1-\theta)^{n-z}$$

if there are z successes in n trials

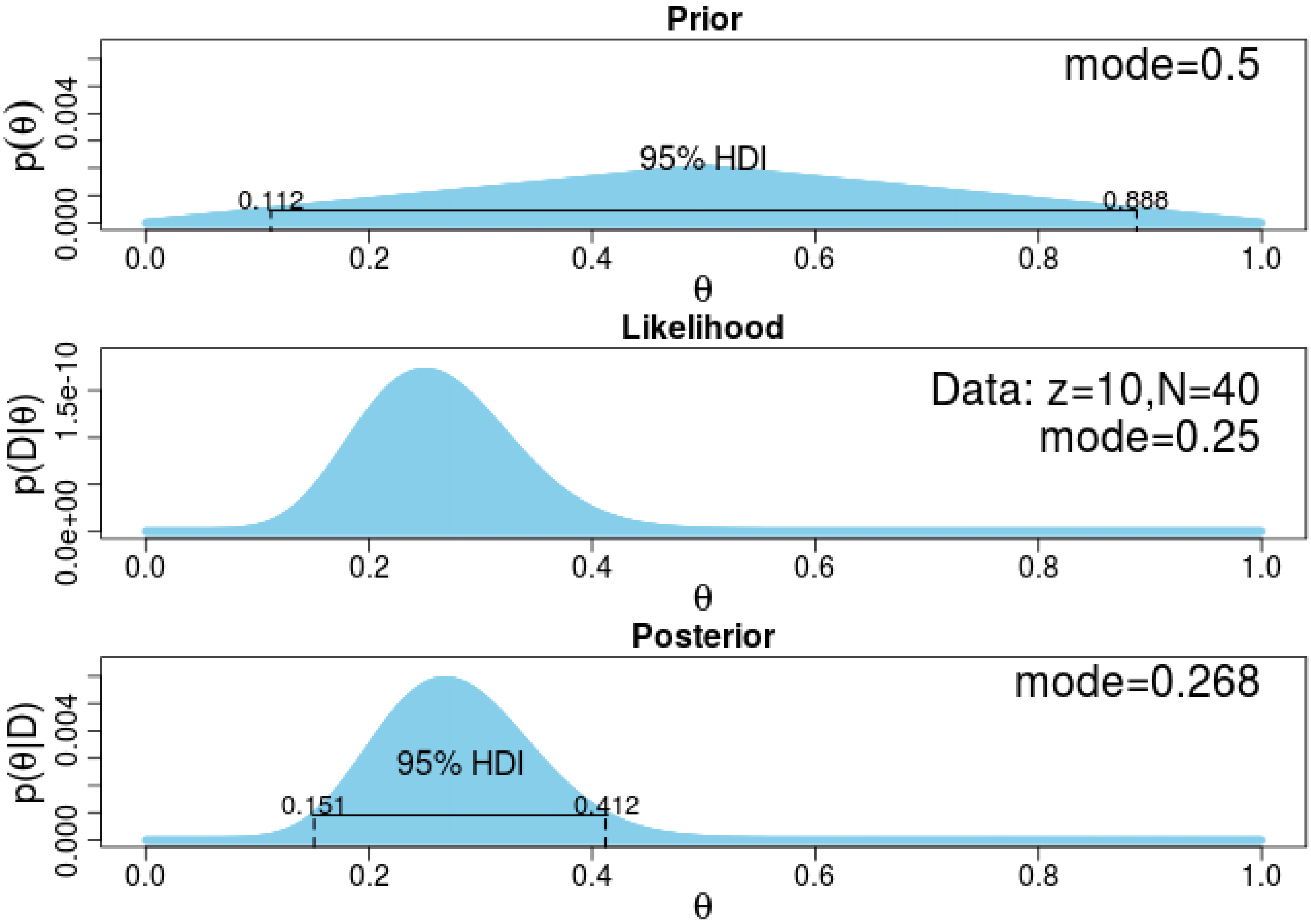
Thus post $\propto \theta^{a+z-1}(1-\theta)^{b+n-z-1}$

This is a Beta(a+z, b+n-z)

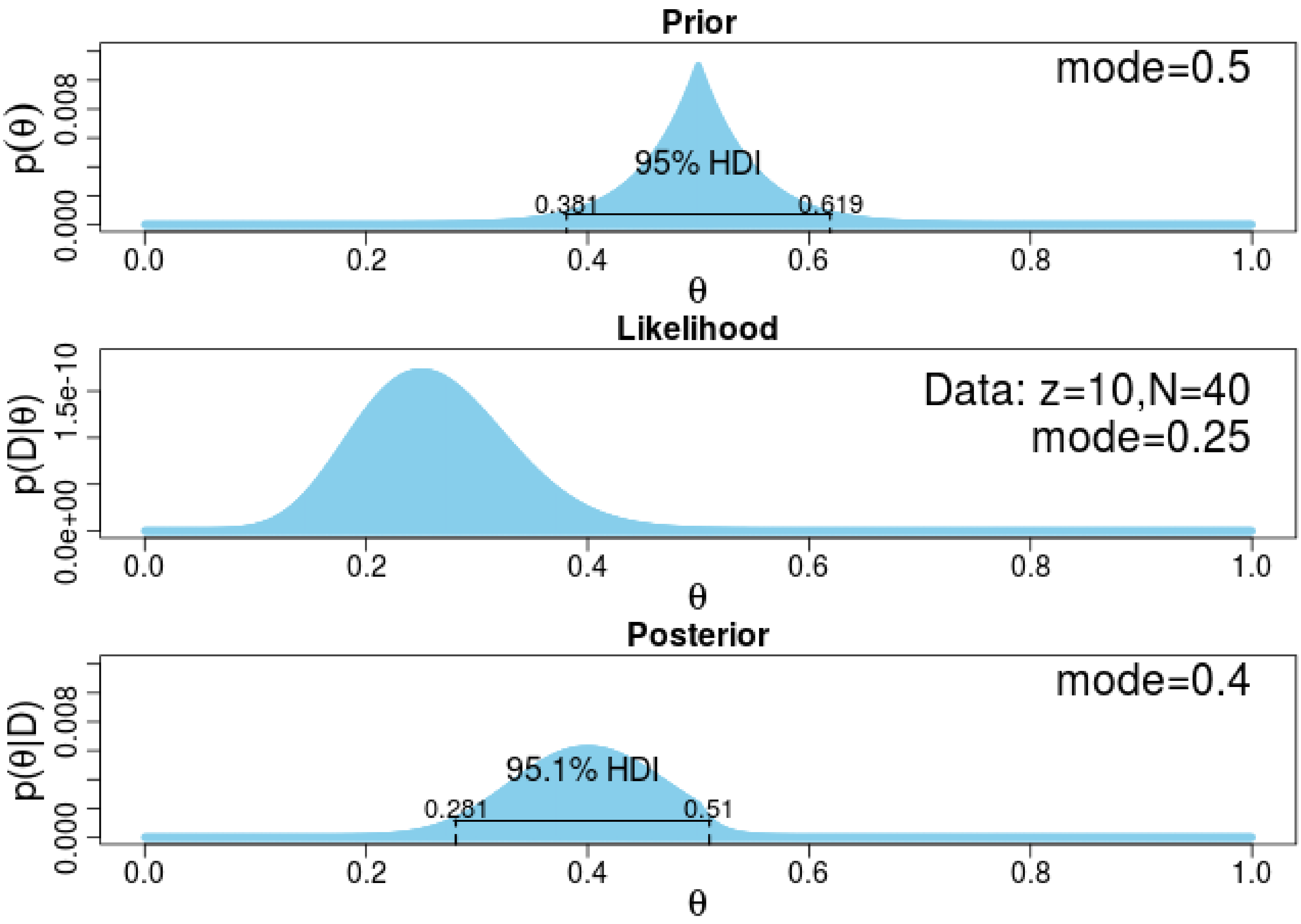
See Effect on posterior of large n.R. Data: 1 success in 4 trials.



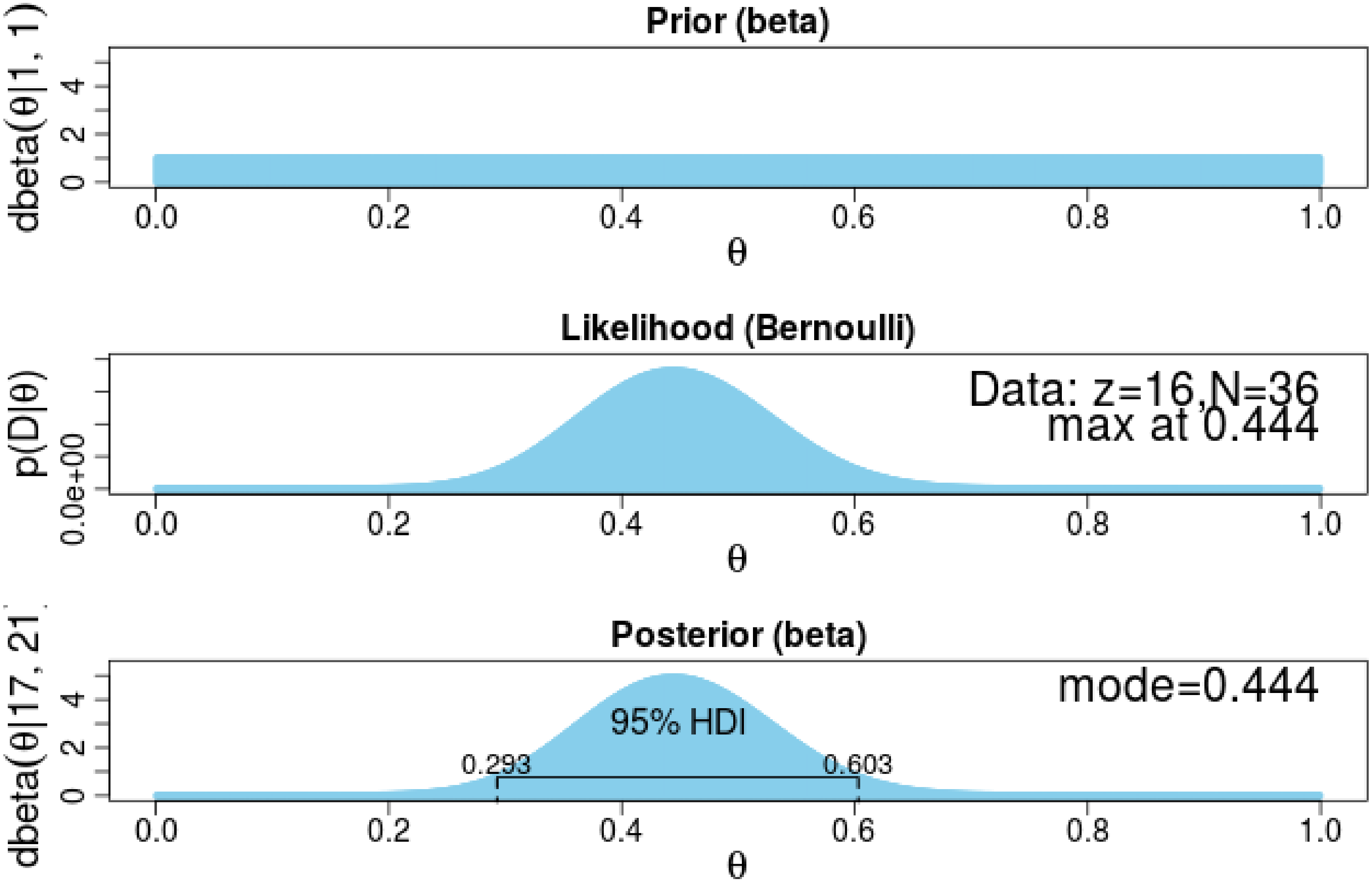
See Effect on posterior of large n.R. Data: 10/40 (vs ¼)



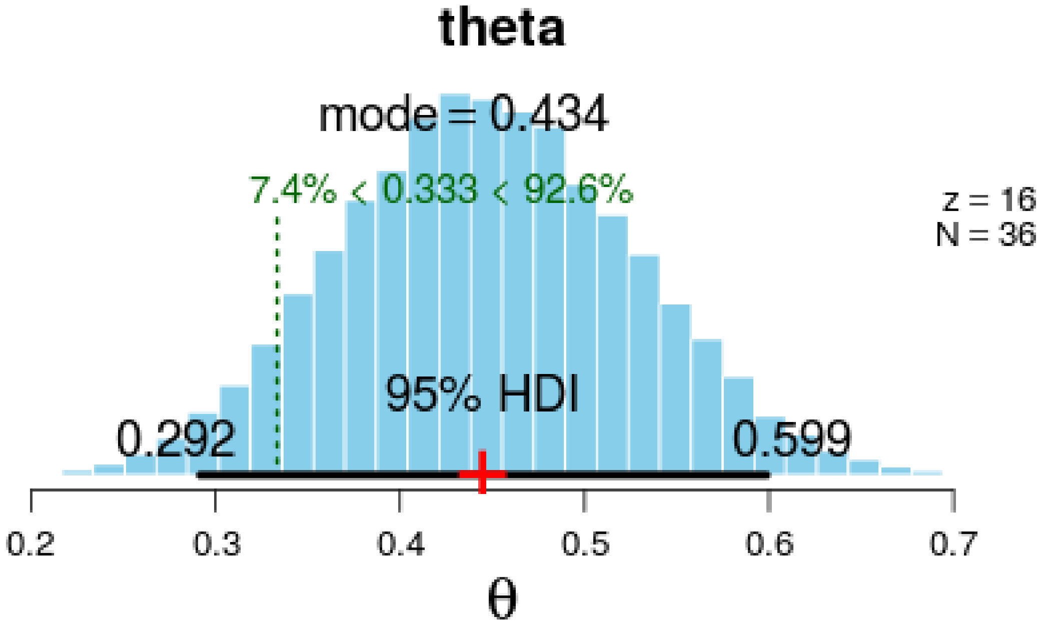
See Effect on posterior of sharp prior.R



See BernBetaExampleLoveBlindMen.R. Data: 16 of 36 men correctly identify their partner.



See Run Jags-Ydich-Xnom1subj-MbernBeta-LoveBlindMen.R which uses MCMC – and gives similar results

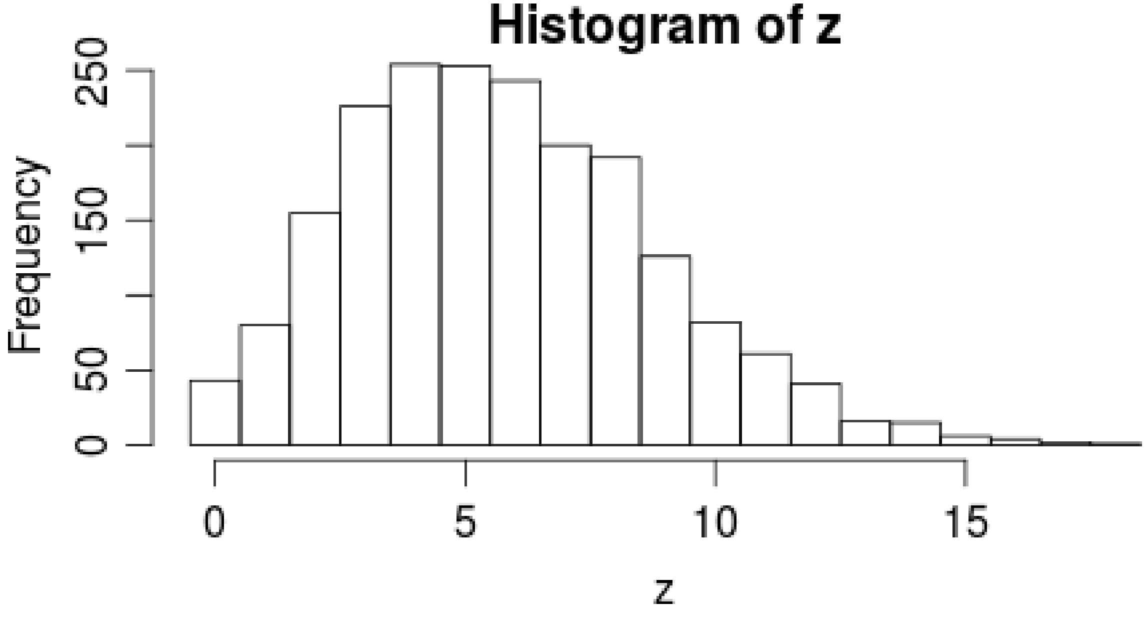


P(0.29 < θ < 0.60) = 0.95

Does the prior make sense? Does it give rise to reasonable data?

Let θ = % OC veggies. Prior: I think that θ might be around 0.20 and my prior is worth 20 observations.

This corresponds to a prior on θ that is Beta(4.6, 15.4). Does that prior give rise to reasonable data? Simulate 2000 times getting samples of n=25:



Two or more groups or subjects

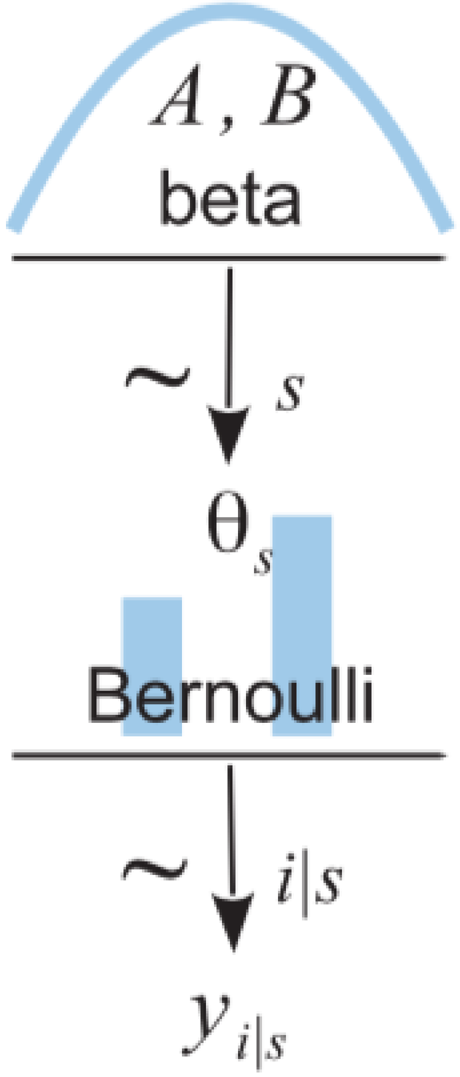
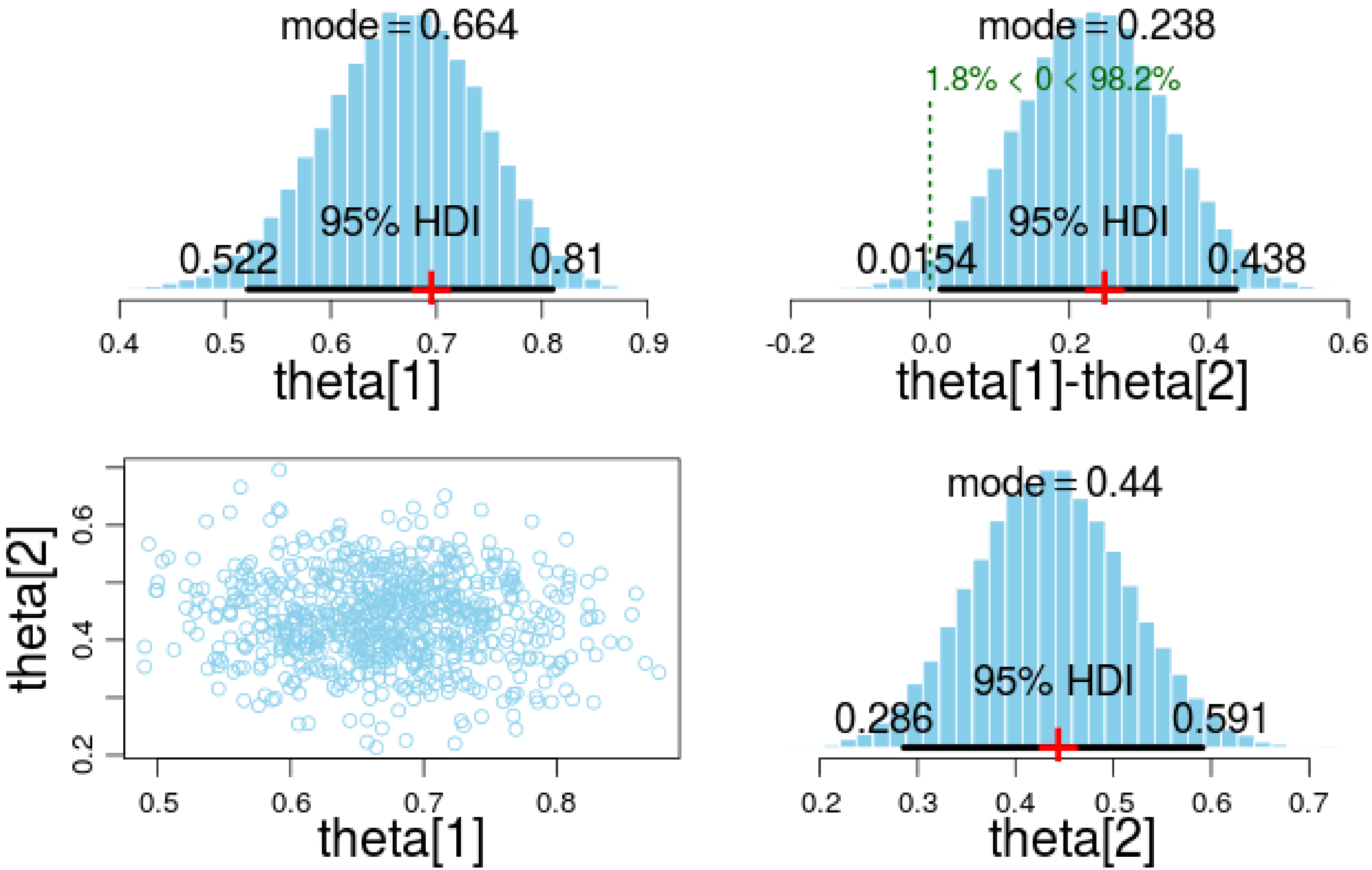


Figure 8.5: Diagram of model with Bernoulli likelihood and beta prior for multiple subjects, *s*. Notice the indices on the variables and arrows. Every arrow in the diagram has a corresponding line of code in JAGS model specification. Copyright © Kruschke, J. K. (2014). *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan. 2nd Edition.* Academic Press / Elsevier.

Comparing 2 proportions. Love is Blind data.

Men: 16/36 Women: 25/36 98% prob women better



Frequentist P-value = 0.03.

Hyperpriors…

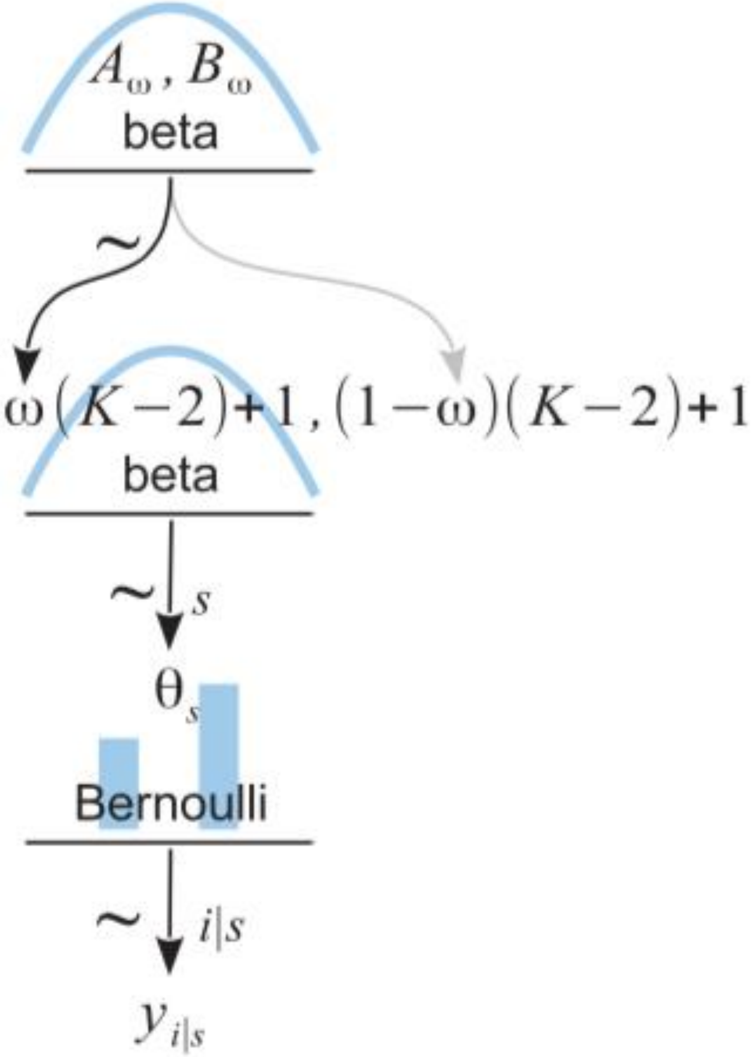


Figure 9.4: A model of hierarchical dependencies for data from several coins created independently from the same mint. A datum $y_{i|s}$, from the i^{th} flip of the s^{th} coin, depends on the value of the bias parameter θ_s for the coin. The values of θ_s depend on the value of the hyperparameter ω for the mint that created the coins. The ω parameter has a prior belief distributed as a beta distribution with shape parameters A_ω and B_ω. Copyright © Kruschke, J. K. (2014). *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan. 2nd Edition.* Academic Press / Elsevier.

Hyperprior with multiple categories

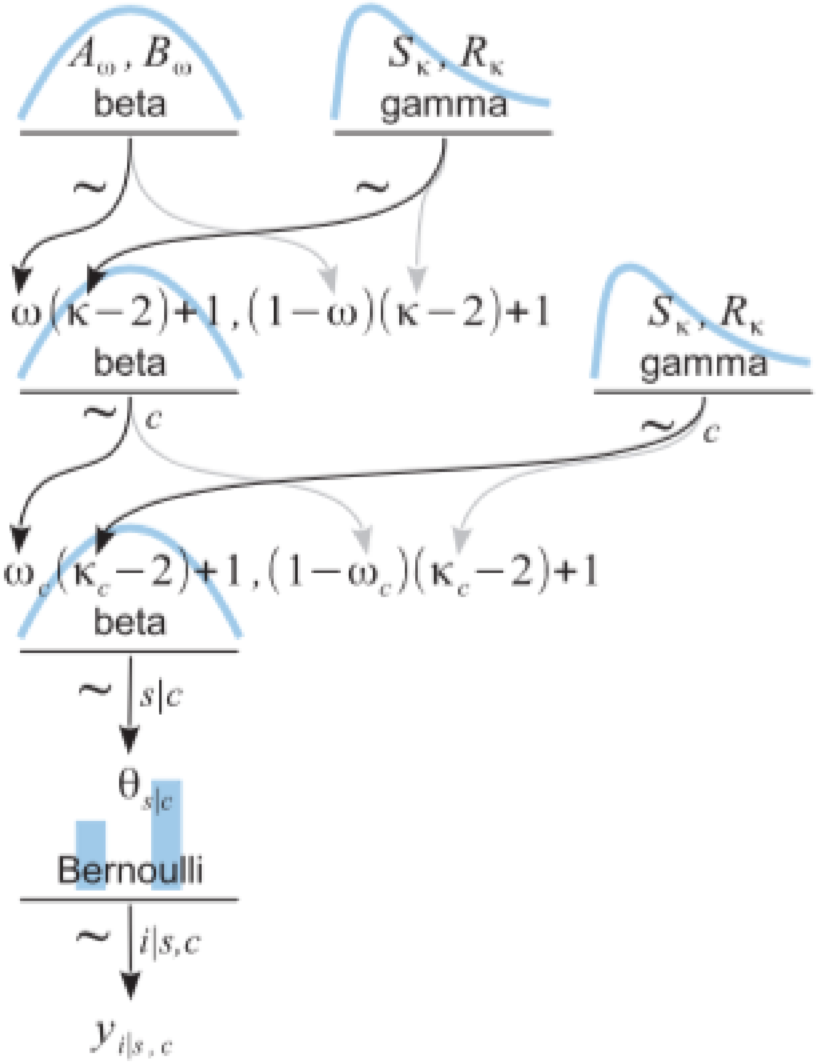
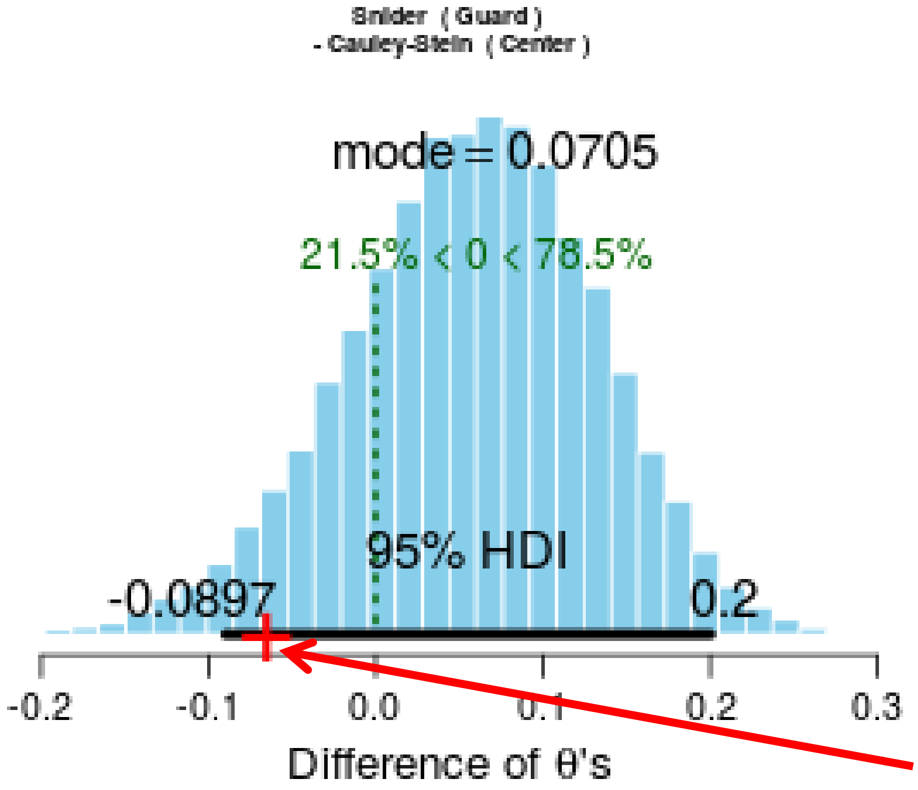


Figure 9.13: A model of hierarchical dependencies for data from several coins (indexed by subscript *s*) created by more than one category of mint (indexed by subscript *c*), with an overarching distribution across categories. Copyright © Kruschke, J. K. (2014). *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan. 2nd Edition.* Academic Press / Elsevier.

Overall, guards shoot FTs better than centers. However, in 2014-15 center Willie Cauley-Stein made 61.7% (79/128) while guard Quentin Snider made 55.3% (21/38). Is Snider really worse? A Bayesian analysis says “P(Snider is better) = 87.5%.”



So far in 2015-16 Cauley-Stein has made 58.6% (41/70) while Snider has made 71.7% (33/46).

Obs diff = 0.553 – 0.617

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basketball data comparing men and women

See

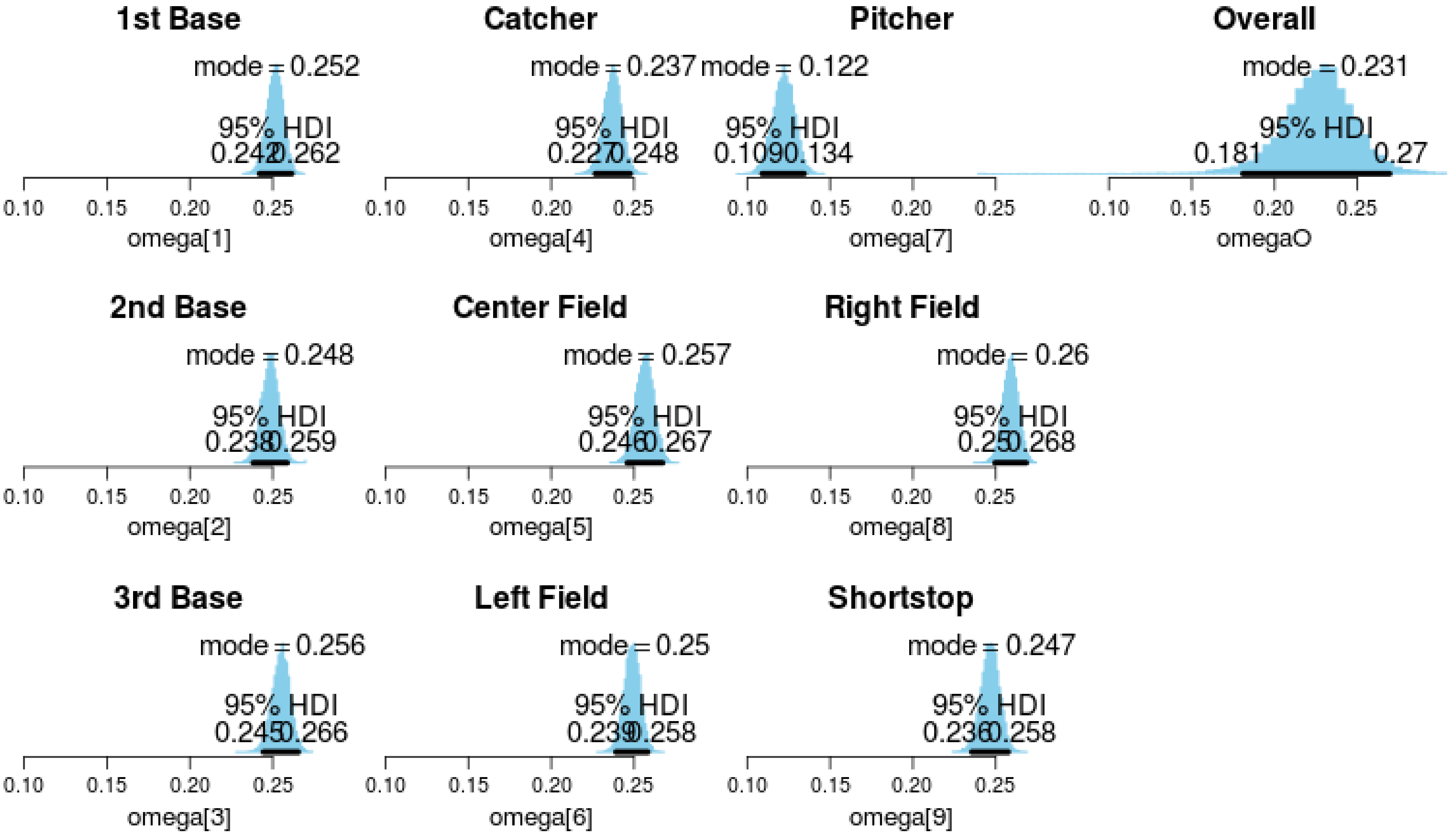
rpubs.com/jawitmer/68406

and

rpubs.com/jawitmer/99006

MCMC can handle many parameters.

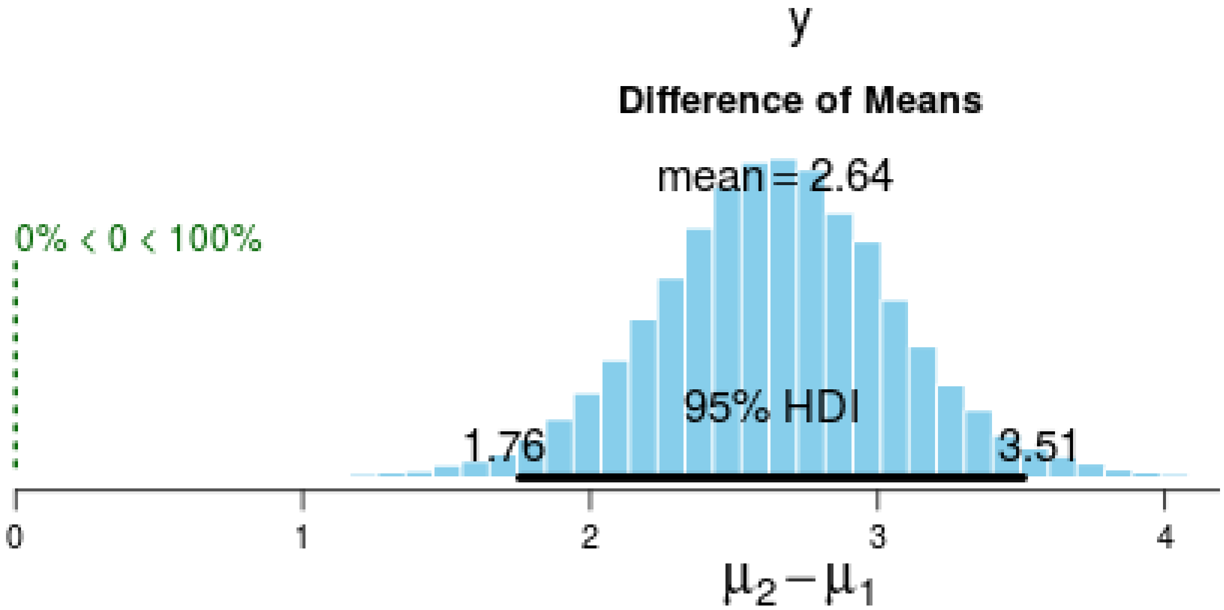
See rpubs.com/jawitmer/64891 928 parameters



Normal data – comparing 2 means

Myocardial blood flow (ml/min/g) for two groups.

Normoxia: 3.45, 3.09, 3.09, 2.65, 2.49, 2.33, 2.28, 2.24, 2.17, 1.34
Hypoxia: 6.37, 5.69, 5.58, 5.27, 5.11, 4.88, 4.68, 3.50



$P(1.76 < \mu_2 - \mu_1 < 3.51) = 0.95$.

[The standard/frequentist 95% CI is (1.85, 3.39)]

See rpubs.com/jawitmer/154977

Normal data – comparing 2 means

See the BEST website:
http://www.sumsar.net/best_online/

There is also an easy-to-use R package:
Bayesian First Aid

```
with(ExerciseHypoxia, t.test(MBF ~ Oxygen))
```

is changed to

```
with(ExerciseHypoxia, bayes.t.test(MBF ~ Oxygen))
```

t as a *likelihood*

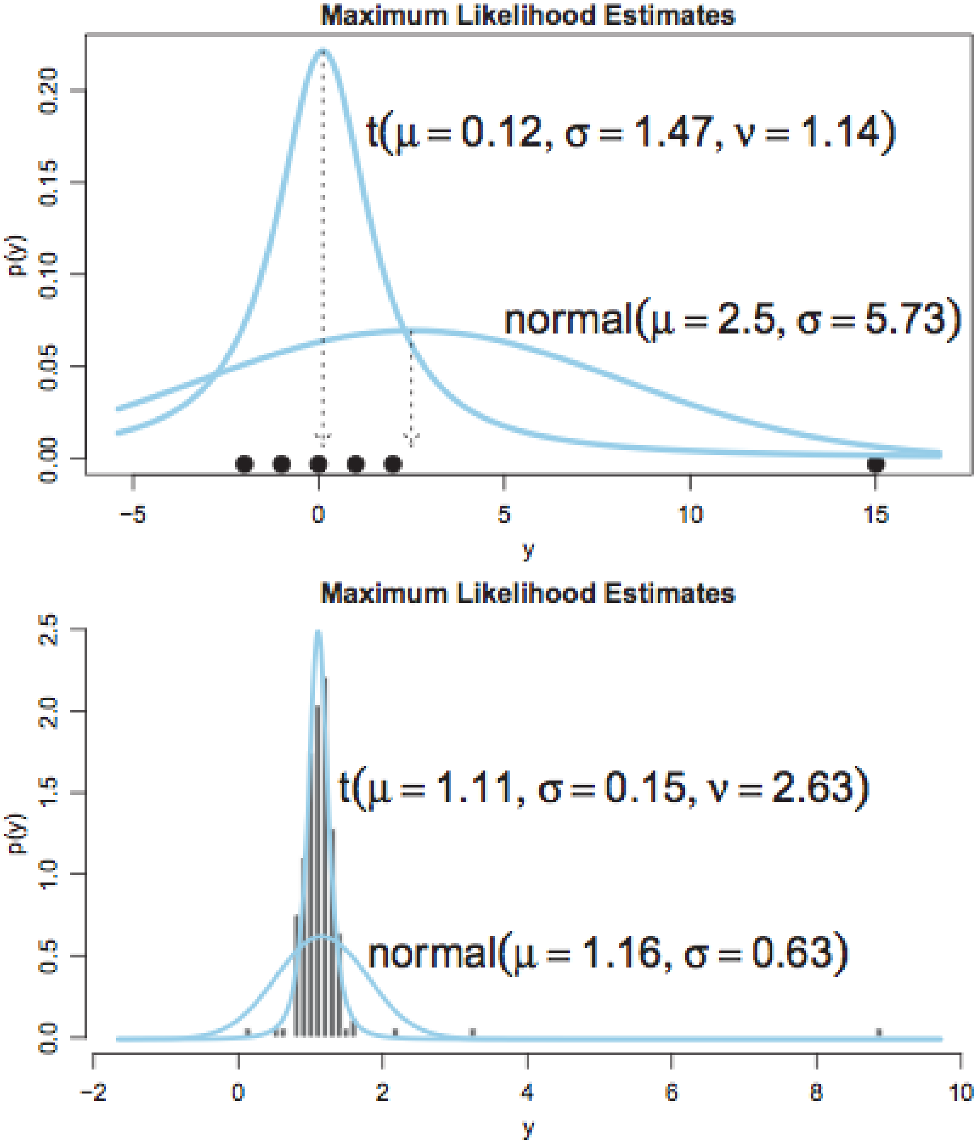
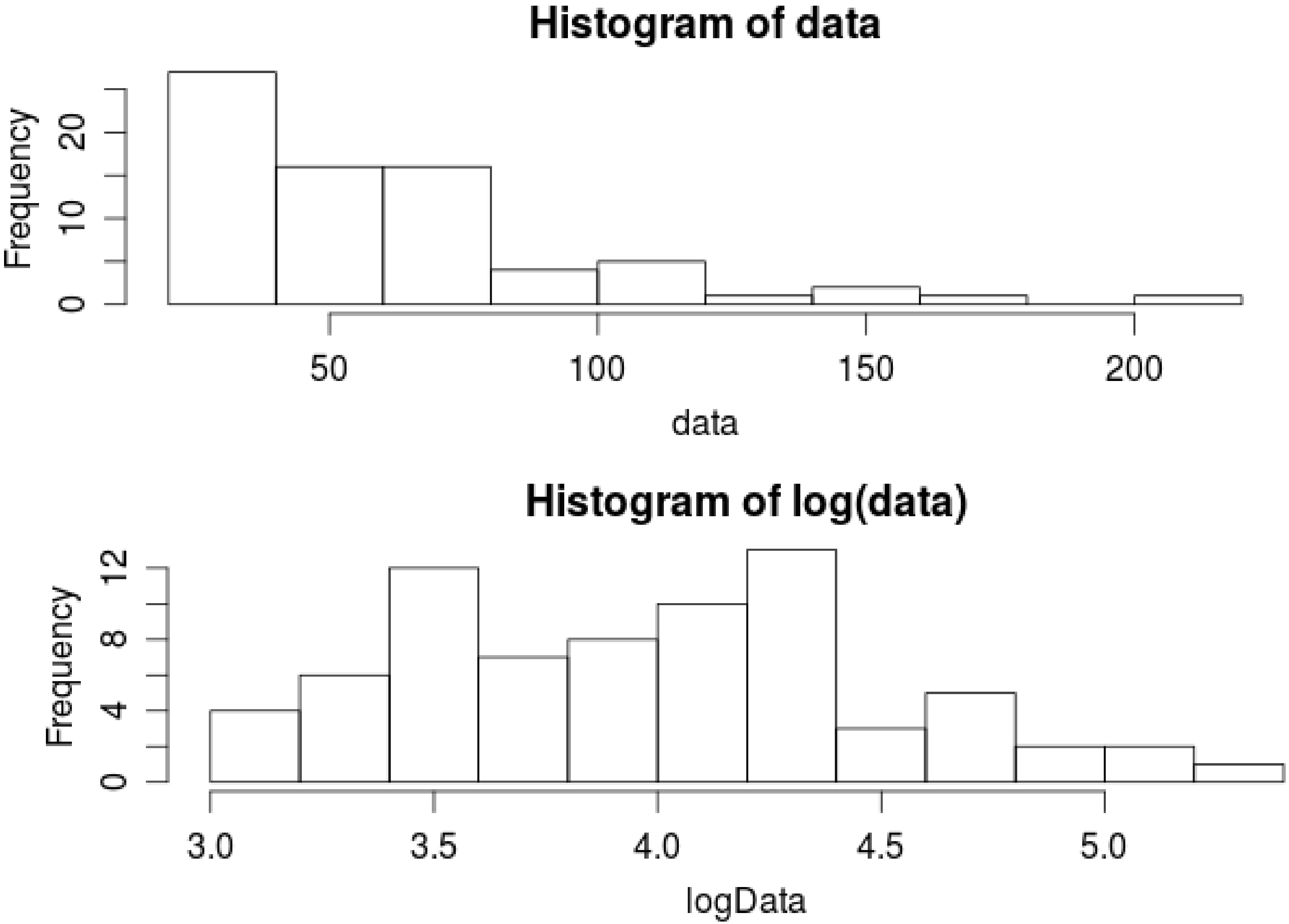


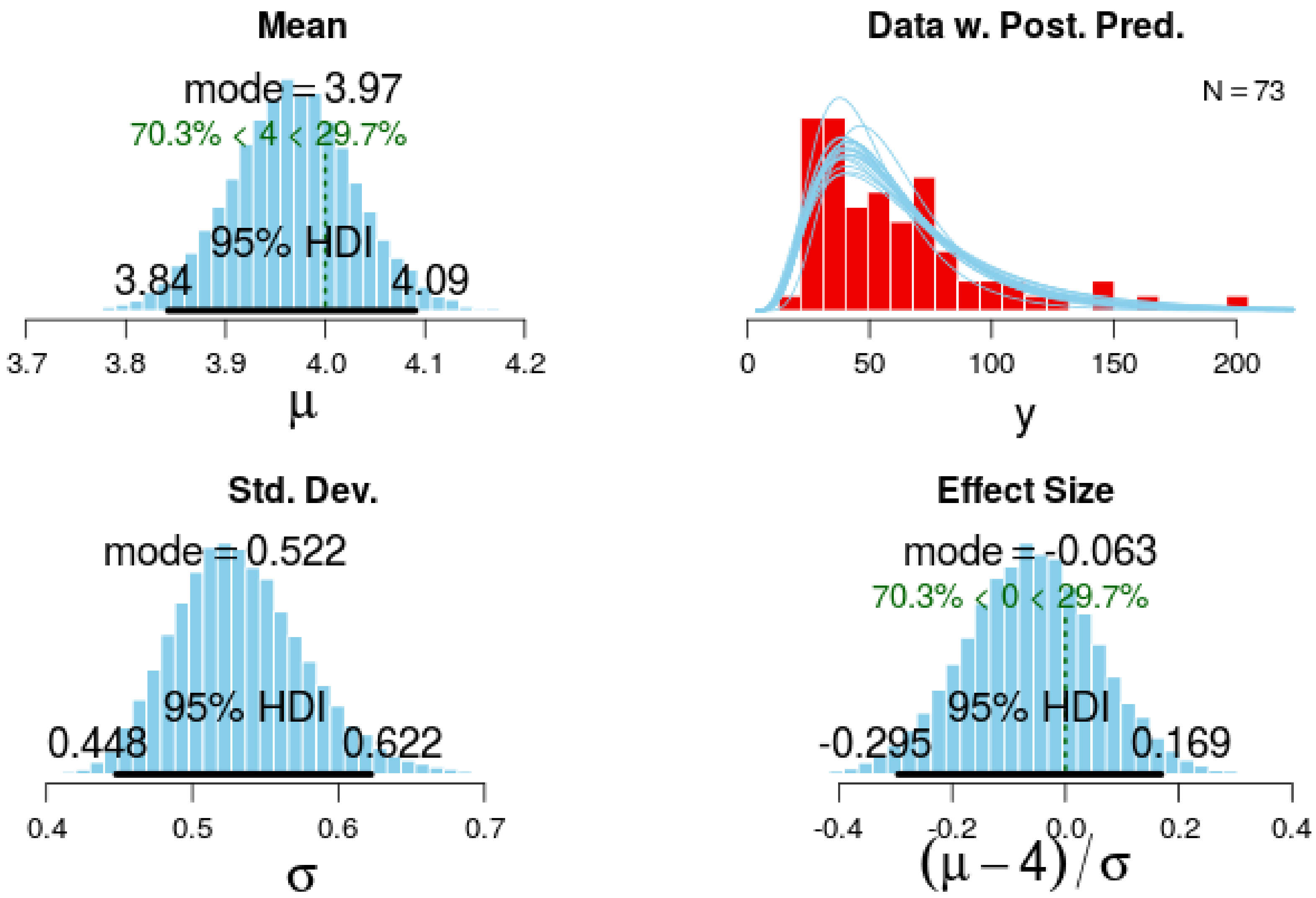
Figure 16.6: The maximum likelihood estimates of normal and *t* distributions fit to the data shown. Upper panel shows "toy" data to illustrate that the normal accommodates an outlier only by enlarging its standard deviation and, in this case, by shifting its mean. Lower panel shows actual data (Holcomb & Spalsbury, 2005) to illustrate realistic effect of outliers on estimates of the normal. Copyright © Kruschke, J.

Lognormal vs Normal

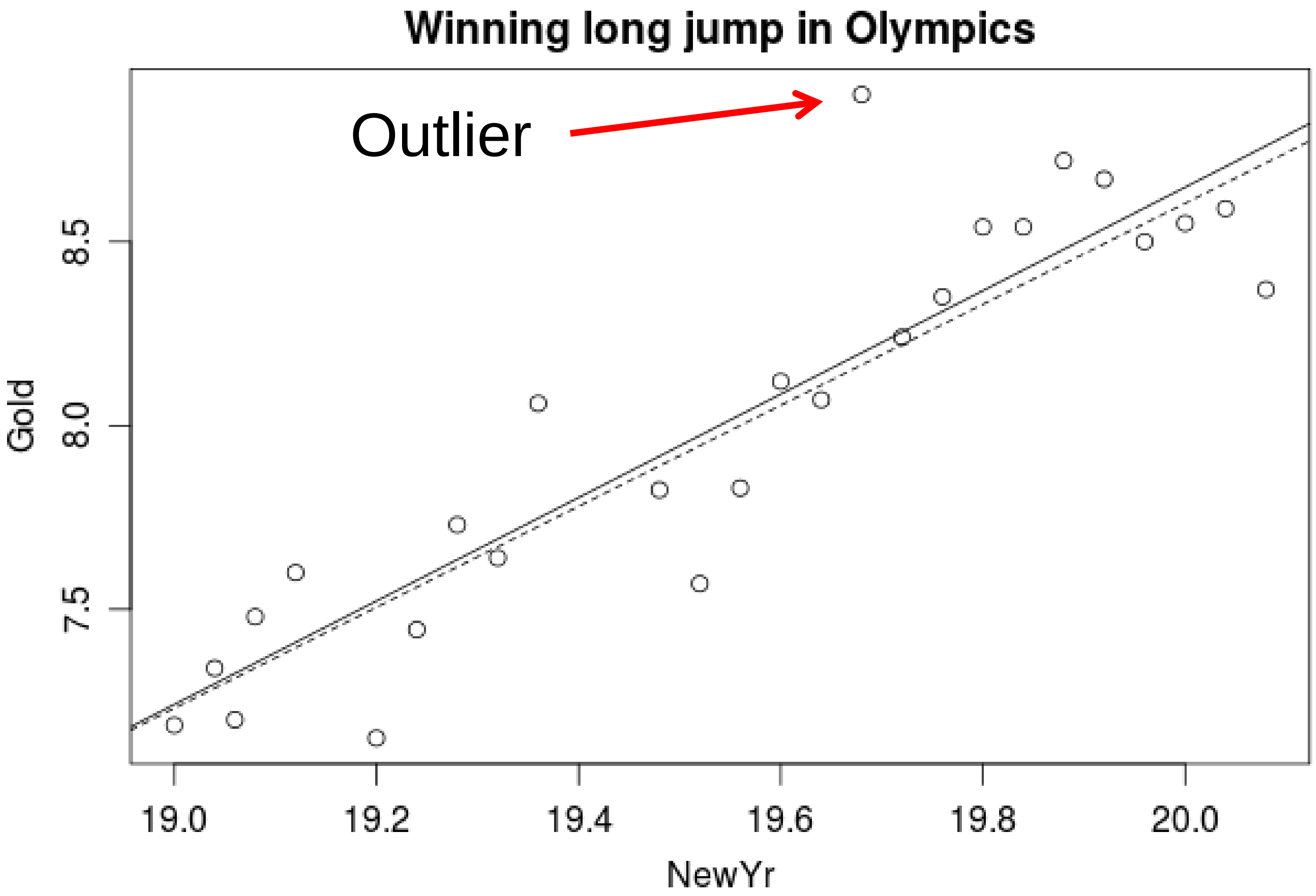


Lognormal vs Normal

With MCMC, we don’t need to assume a normality for data. Use y[i] ~dlnorm() instead of y[i] ~ dnorm().



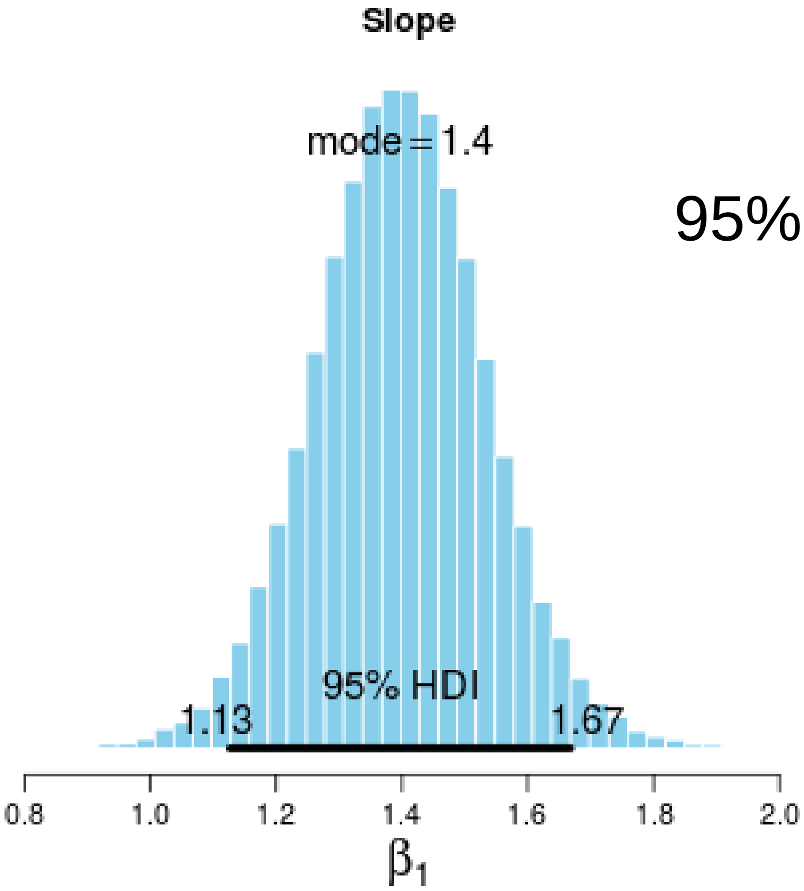
Regression – How to handle outliers…



Outlier

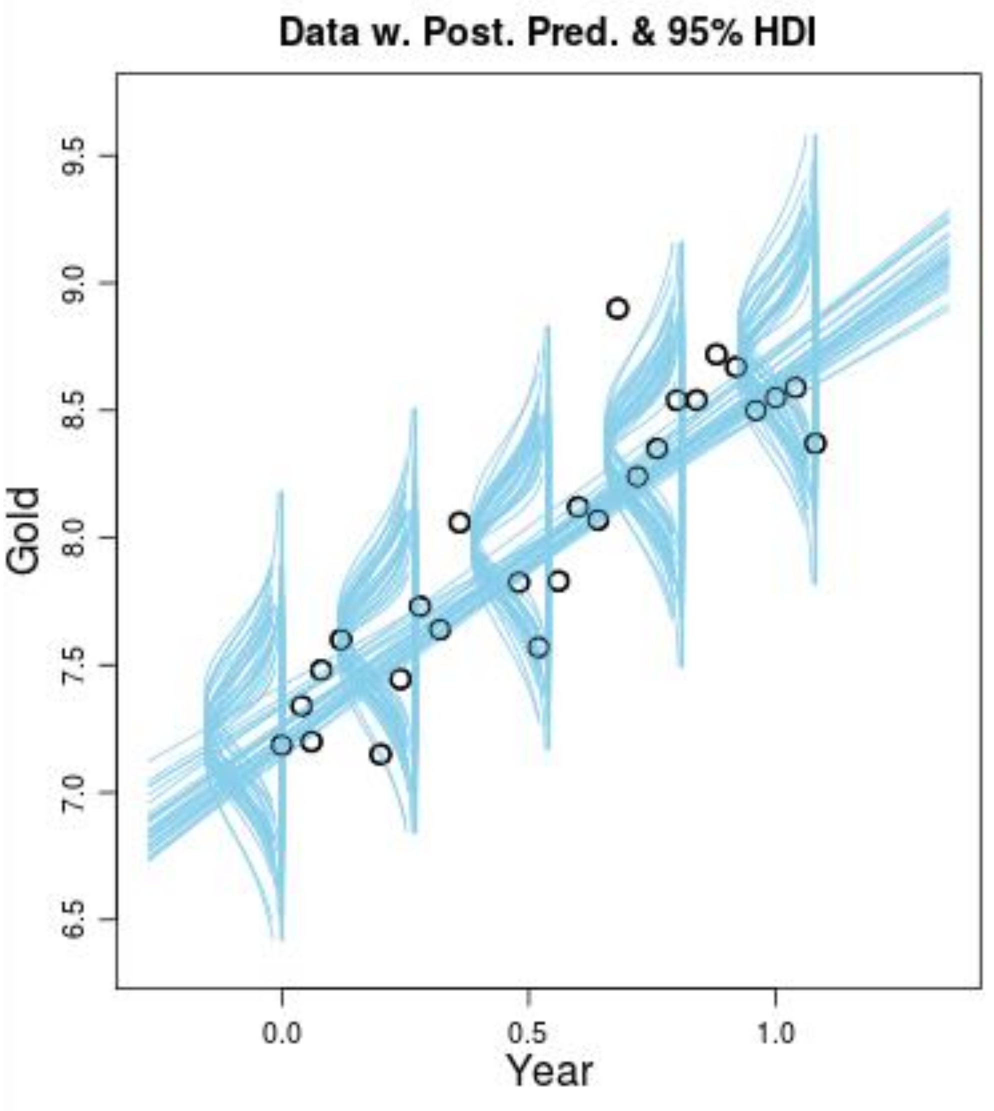
95% CI for slope: (1.12, 1.69) or (1.14, 1.60)

We can use a t as the likelihood function (vs the usual normal distribution)



95% HDI for slope: (1.13, 1.67)

A sampling of fitted models



Kruschke, John K. (2015). *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan. 2nd Ed*. Academic Press.

McGrayne, Sharon Bertsch (2011). *The Theory that Wouldn’t Die*. Yale University Press.