Discrete Mathematics

Chapter 1:
The Foundations: Logic and Proofs

Chapter 1: Introduction

Topics covered:

logic mệnh đề và logic vị từ

1.1 Propositional Logic

A proposition is a declarative sentence that is either true or false.

Chapter 1: Introduction

Example. Which of the following sentences are propositions?

- Great!
- Tokyo is the capital of Japan
- What time is it?
- It is now 3pm
- 1+7=9
- x+1=3

Compound Propositions

Let p, q be propositions.

6 compound proposition

- Negation.
  $\neg p$ = not p = proposition that is true if p is false, and is false if p is true.
- Conjunction.
  $p \wedge q$ = "p and q" = proposition that is true when both p and q are true, and is false otherwise.
- Disjunction.
  $p \vee q$ = "p or q" = proposition that is false when both p and q are false, and is true otherwise.
- Exclusive or.
  $p \oplus q$ = "only p or only q" = proposition that is true when exactly one of p and q is true and is false otherwise.

Compound Propositions

- Conditional statement.
  $p \to q$ = proposition that is false when p is true and q is false, and is true otherwise.

(*) Note: There are several ways to express the conditional statement $p \to q$

  - If p then q q là đk cần để có p
  - q if p
  - p is sufficient for q
  - q is a necessary condition for p
  - p only if q

- Biconditional statement.
  $p \leftrightarrow q$ = proposition that is true when p and q have the same truth values, and is false otherwise.

Translating Sentences into Logical Expressions

Example 1. I watch soccer only if Arsenal play or I have no homework.

p="I watch soccer"

q="Arsenal play"

r="I have homework"

Example 2. (a) You can not pass this class if you miss more than 20% of lectures.

p="You pass this class"

q="You miss more than 20% of lectures"

(b) You can not pass this class if you miss more than 20% of lectures unless you provide reasonable excuses.

r="You provide reasonable excuses"

Logic and Bit Operations

Computers represent information using bits. A bit is a symbol of two possible values, 0 and 1. A bit can represent a truth value, that is, 1 represents T (true) and 0 represents F (false). Information is often represented using bit strings, and operations on bit strings can be used to manipulate this information.

Logic and Bit Operations

Example. $1001100 \wedge 0011001 = 0001000$.

Note. Other notation for $\wedge, \vee, \oplus$ are AND, OR, XOR.

tương đương mệnh đề

1.2 Propositional Equivalences

- A compound proposition is called a tautology if it is always true regardless of the truth values of the propositions that occur in it. A compound proposition is called a contradiction if it is always false. A compound proposition that is neither tautology nor contradiction is called a contingency.
- Two propositions p and q are logically equivalent if the biconditional statement $p \leftrightarrow q$ is a tautology. In this case we use notation $p \equiv q$.

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p và q có cùng giá trị chân lí

*** Two methods for proving logical equivalences:

- Use truth table
- Use other logical equivalences.

Some logical equivalences

dấu này đọc là "tương đương logic"

Double negation law	$\neg(\neg p) \equiv p$
Identity laws luật đồng nhất	$p \wedge T \equiv p$ $p \vee F \equiv p$ p lại thành p
Domination laws luật thống trị	$p \vee T \equiv T$ $p \wedge F \equiv F$
Negation laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$
Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Commutative laws giao hoán	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative laws kết hợp	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive laws phân phối	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Some logical equivalences

Note:

nhớ 3 cái này

$$p \to q \equiv \neg p \vee q.$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \oplus q \equiv \neg(p \leftrightarrow q)$$

Example 1. Prove that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Example 2. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Some logical equivalences

- Tautology

a) $(p \wedge q) \Rightarrow p$

b) $p \Rightarrow (p \vee q)$

c) $\neg p \Rightarrow (p \Rightarrow q)$

d) $(p \wedge q) \Rightarrow (p \Rightarrow q)$

e) $\neg(p \Rightarrow q) \Rightarrow p$

f) $\neg(p \Rightarrow q) \Rightarrow \neg q$

g) $\neg p \wedge (p \vee q) \Rightarrow q$

h) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

i) $(p \wedge (p \Rightarrow q)) \Rightarrow q$

j) $((p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r$

Some logical equivalences

1) $(p \Leftrightarrow q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

2) $\neg p \Leftrightarrow q \equiv p \Leftrightarrow \neg q$

3) $\neg(p \Leftrightarrow q) \equiv \neg p \Leftrightarrow q$

1.3 Predicates and Quantifiers – Vị từ và lượng từ

Predicate.

The statement "$x > 3$" is not a proposition. It will become a proposition when a value is assigned to x.

The statement "$x > 3$" is called a propositional function, denoted by $P(x)$. Then:

P(0)=F, P(5)=T

x is called a variable, "> 3" is the predicate

A propositional function can be multi-variable.

Example. $R(x, y, z) =$ "$x + y < z$" is a propositional function with variables x, y, z and R is the predicate.

1.3 Predicates and Quantifiers – Vị từ và lượng tử

Quantifiers $\forall$, $\exists$.

Let $P(x)$ be a propositional function where x gets values in a particular domain.

The universal quantification $\forall x P(x)$= For all values of x in the domain, $P(x)$ is true

The existential quantification $\exists x P(x)$= There is at least a value of x in the domain such that $P(x)$ is true.

Example. Let x represent a real number. Determine the truth value of the following propositions

(a) $\forall x((x > 0) \rightarrow (x^2 \geq x))$ F

(b) $\forall x((x > 0) \wedge (x^2 \geq x))$ F

(c) $\forall x((x > 0) \vee (x^2 \geq x))$ T

(d) $\exists x((x > 0) \rightarrow (x^2 \geq x))$ T

(e) $\exists x((x > 0) \wedge (x^2 \geq x))$ T

(f) $\exists x((x > 0) \vee (x^2 \geq x))$ T

1.3 Predicates and Quantifiers – Vị từ và lượng từ

Negating Quantified Expressions

$$\neg\forall x P(x) = \exists x \neg P(x) \qquad \neg\exists x P(x) = \forall x \neg P(x)$$

Tại sao sử dụng logic vị từ

- Logic mệnh đề chỉ xử lý trên các sự kiện, là các khẳng định có giá trị đúng hoặc sai
- Logic vị từ (logic bậc nhất) cho phép chúng ta nói về các đối tượng, tính chất của chúng, quan hệ giữa chúng, phát biểu về một hay tất cả các đối tượng nào đó mà không cần liệt kê trong chương trình
- Các câu không thể biểu diễn bằng logic mệnh đề nhưng có thể biểu diễn bằng logic vị từ
  - Socrates là người nên socrates phải chết

Logic mệnh đề: Biểu diễn và xác định tính đúng/sai của các mệnh đề.

Logic vị từ: Mở rộng logic mệnh đề, giúp phát biểu về tính chất của một hoặc nhiều đối tượng mà không cần liệt kê từng đối tượng.

Translating Sentences into Logical Expressions

Example 1. "Every students of class SE0000 passed Calculus"

(a) If domain consists of all students of SE0000

Put $P(x)$="x passed Calculus"

(b) If domain consists of all students of the university

We need $Q(x)$="x is in SE0000"

Example 2. "Each student of SE0000 has visited Canada or Mexico"

Example 3. "Some student of SE0000 has visited Canada or Mexico"

1.4 Nested Quantifiers

$\forall x \forall y P(x,y)$ = For all x and for all y, $P(x,y)$ is true

$\forall x \exists y P(x,y)$ = For all x there is y such that $P(x,y)$ is true

$\exists x \forall y P(x,y)$ = There exists x such that for all y, $P(x,y)$ is true

$\exists x \exists y P(x,y)$ = There exist x and y such that $P(x,y)$ is true

Note. The order of the quantifiers is important!

trong 1 câu có nhiều lượng tử, lượng tử cùng loại đc phép đổi chỗ ,lượng tử khác loại không được phép đổi chỗ.

1.4 Nested Quantifiers

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$\exists x \exists y P(x, y)$ = There exist x and y such that $P(x, y)$ is true

Note. The order of the quantifiers is important!

Example. Determine the truth values of the following propositions on the set of real numbers.

$\forall x \forall y (x + y = 1)$ F

$\forall x \exists y (x + y = 1)$ T

$\exists x \forall y (x + y = 1)$ T

$\exists x \exists y (x + y = 1)$ T

Translate Logical Expressions into Sentences

Example 1. $\forall x \forall y[(x > 0) \wedge (y > 0) \rightarrow (xy > 0)]$

where x, y are real numbers.

Example 2. Let x, y represent students in a university, and

$C(x) =$ " x has a laptop" $\quad F(x, y) =$ " x and y are friends"

Translate the logical expression $\forall x[C(x) \vee \exists y(C(y) \wedge F(x, y))]$

Example 3. Let x, y represent students in a university, and

$F(x, y) =$ " x and y are friends"

Translate the logical expression

$$\exists x \forall y \forall z[(F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z)]$$

Negating Nested Quantifiers

$$\neg(\forall x \forall y P(x,y)) = \exists x \exists y \neg P(x,y) \quad \neg(\forall x \exists y P(x,y)) = \exists x \forall y \neg P(x,y)$$

$$\neg(\exists x \forall y P(x,y)) = \forall x \exists y \neg P(x,y) \quad \neg(\exists x \exists y P(x,y)) = \forall x \forall y \neg P(x,y)$$

argument = premises + conclusion

1.5 Rules of Inference/Quy tắc suy diễn

- An ***argument/lập luận*** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called ***premises/tiên đề*** and the final proposition is called the ***conclusion/kết luận***. An argument is ***valid/vững chắc*** if the truth of all its premises implies that the conclusion is true.
- An ***argument form/*** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is ***valid*** no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

1.5 Rules of Inference/Quy tắc suy diễn

- The argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid – chắc chắn, when $(p_1 \land p_2 \land \cdot \cdot \cdot \land p_n) \to q$ is a tautology – hằng đúng.

- suy diễn(argument) là đưa ra kết luận từ các tiền đề đã cho bằng phép kéo theo.
- suy diễn đó hợp logic (valid) nếu phép kéo theo đó là hằng đúng.
- suy diễn đó đúng nếu nó hợp logic và các tiền đề , kết luận đều đúng

Name	Rule of Inference	Tautology
Addition bổ sung	p $\therefore \overline{p \vee q}$	$p \rightarrow (p \vee q)$
Simplification giản lược	$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$
Modus ponens	p $\underline{p \rightarrow q}$ $\therefore q$	$p \wedge (p \rightarrow q) \rightarrow q$
Modus tollens luật gián tiếp	$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$(\neg q) \wedge (p \rightarrow q) \rightarrow \neg p$
Hypothetical syllogism gián tiếp	$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
Disjunctive syllogism loại trừ	$\neg p$ $\underline{p \vee q}$ $\therefore q$	$(p \vee q) \wedge (\neg p) \rightarrow q$

1.5 Rules of Inference/Quy tắc suy diễn

KB:

$\neg A \vee \neg B \vee P$ (1)

$\neg C \vee \neg D \vee P$ (2)

$\neg E \vee C$ (3)

A (4)

E (5)

D (6)

Cần chứng minh: $KB \vdash P$

Some fallacies

- Fallacy of affirming the conclusion: $[(p \to q) \land q] \to p$
- Fallacy of denying the hypothesis: $[(p \to q) \land \neg p] \to \neg q$

Rules of Inference for Quantified Statements

Name	Rule of Inference
Universal instantiation	$\dfrac{\forall x P(x)}{\therefore P(c),\ c \text{ is arbitrary}}$
Universal generalization	$\dfrac{P(c),\ c \text{ is arbitrary}}{\therefore \forall x P(x)}$
Existential instantiation	$\dfrac{\exists x P(x)}{\therefore P(c),\ \text{for some } c}$
Existential generalization	$\dfrac{P(c),\ \text{for some } c}{\therefore \exists x P(x)}$

$\forall x P(x) \Rightarrow P(c)$

If $\forall x$ P(x) is true, then P(c) is true for any specific constant c.

If P(c) is true for every arbitrary constant c, then $\forall x$ P(x) is true.

If $\exists x$ P(x) is true, then P(c) is true for some specific constant c

If P(c) is true for some specific constant c, then $\exists x$ P(x) is true.

Rules of Inference for Quantified Statements

- "All student are in this class had taken the course MAD"
- "Trinh is in this class"
- "Had Trinh taken course MAD?"

Rules of Inference for Quantified Statements

- "All student are in this class had taken the course MAD"
- "Trinh is in this class"
- "Had Trinh taken course MAD?"
- $\forall x(P(x) \rightarrow Q(x))$ ← Premises
- $P(Trinh) \rightarrow Q(Trinh)$ ← Universal Instantiation
- $P(Trinh)$ ← Premises
- $Q(Trinh)$ ← Modus ponens

← conclusion

Rules of Inference for Quantified Statements

- Cho các câu sau

1. Mọi bé trai đều thích chơi bóng đá
2. Ai thích chơi bóng đá đều có giày đá bóng
3. Nam là một bé trai

Câu hỏi

a) Biểu diễn các câu trên ở dạng logic vị từ
b) Chuyển các câu logic vị từ vừa viết về dạng chuẩn tắc hội
c) Viết câu truy vấn "Nam có giày đá bóng" dưới dạng logic vị từ và chứng minh

a) **Biểu diễn các câu trên ở dạng logic vị từ**

- $\forall x(Boy(x) \rightarrow LikesFootball(x))$
- $\forall x(LikesFootball(x) \rightarrow HasFootballShoes(x))$
- $Boy(Nam)$

b) **Chuyển các câu logic vị từ vừa viết về dạng chuẩn tắc hội**

- $\neg Boy(x) \vee LikesFootball(x)$
- $\neg LikesFootball(x) \vee HasFootballShoes(x)$
- $Boy(Nam)$

c) **Viết câu truy vấn "Nam có giày đá bóng" dưới dạng logic vị từ và chứng minh**

- Câu truy vấn: $HasFootballShoes(Nam)$
- Chứng minh:
    1. $\forall x(Boy(x) \rightarrow LikesFootball(x))$
    2. $\forall x(LikesFootball(x) \rightarrow HasFootballShoes(x))$
    3. $Boy(Nam)$
    4. $Boy(Nam) \rightarrow LikesFootball(Nam)$ (từ 1)
    5. $LikesFootball(Nam) \rightarrow HasFootballShoes(Nam)$ (từ 2)
    6. $LikesFootball(Nam)$ (từ 3 và 4)
    7. $HasFootballShoes(Nam)$ (từ 5 và 6)