Chapter 3
The Fundamentals: Algorithms
The Integers

Objectives

- Algorithms
- The Growth of Functions
- Complexity of Algorithms
- The Integers and Division
- Primes and Greatest Common Divisors
- Integers and Algorithms

3.1- Algorithms

An algorithm is a finite set of precise instructions for performing a computation or for solving a problem.

Specifying an algorithm: natural language/ pseudocode

Pseudocode for Finding the Maximum Element in a Finite Sequence

Procedure max (a_1,a_2,a_3,…,a_n: integers)

max:=a_1

for i:=2 **to** n

if max < a_i **then** max:= a_i

{max is the largest element}

Properties of an algorithm

- Input
- Output
- Definiteness - tính rõ ràng — Each step of the algorithm must be clearly and unambiguously defined.
- Correctness - tính chính xác — An algorithm should produce the correct output for all valid inputs.
- Finiteness - tính hữu hạn — An algorithm must terminate after a finite number of steps.
- Effectiveness - tính hiệu quả — The steps of an algorithm must be simple enough to be carried out, in principle, by a person using only pencil and paper.
- Generality - tính tổng quát — Example: The Bubble Sort algorithm can sort any list of numbers, not just a specific list of five numbers.

The Linear Search (tìm kiếm tuyến tính)

- **Linear search/Sequential search (tìm kiếm tuần tự)** là giải thuật hỗ trợ tìm kiếm 1 phần tử trong mảng
- Tìm kiếm từ đầu mảng đến cuối mảng (hoặc ngược lại)
  - Nếu tìm thấy trả về vị trí của kết quả tìm kiếm
  - Nếu không thấy trả về 0

Procedure linear search (x: integer, $a_1,a_2,\ldots,a_n$:distinct integers)

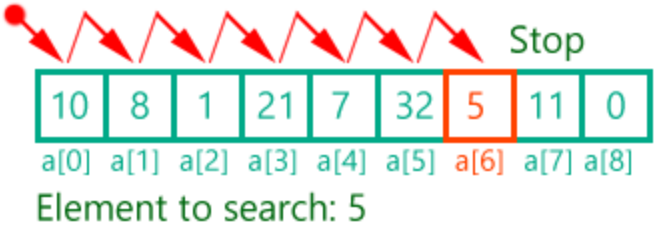
i:=1

while $i \leq n$ **and** $x \neq a_i$

i:=i+1

if $i \leq n$ **then** location:= i

else location:=0



{location is the subscript of the term that equals x, or is 0 if x is not found}

The Linear Search (tìm kiếm tuyến tính)

- Trong trường hợp tốt nhất, phần tử cần tìm nằm ở vị trí đầu tiên, thuật toán sử dụng 1 lần so sánh.
- Trong trường hợp xấu nhất, phần tử cần tìm nằm ở vị trí cuối hoặc không nằm trong mảng, thuật toán sử dụng n lần so sánh.
- Linear Search là một giải thuật đơn giản khi hiện thực và khá hiệu quả với danh sách đủ nhỏ hoặc một danh sách chưa được sắp xếp.

The Binary Search (tìm nhị phân)

- Binary search là một thuật toán chỉ sử dụng với mảng đã sắp xếp.
- Xét một mảng arr[left,...,right]
- So sánh phần tử cần tìm x với phần tử nằm ở vị trí chính giữa mảng mid=left+right/2. Nếu x=arr[mid] thì trả về vị trí và thoát khỏi vòng lặp.
- Nếu x<arr[mid] thì x nằm bên phía bên trái tức là từ arr[left,...,mid-1]
- Nếu x>arr[mid] thì x nằm bên phía bên phải tức là từ arr[mid+1,....,right]
- Tiếp tục thực hiện chia đôi các khoảng tìm kiếm tới khi nào tìm thấy được vị trí của x trong mảng hoặc khi đã duyệt hết mảng.

The Binary Search (tìm nhị phân)

procedure binary search (x:integer, $a_1,a_2,\ldots,a_n$: increasing integers)

i:=1 { i is left endpoint of search interval}

j:=n { j is right endpoint of search interval}

while i<j

begin

m:= $\lfloor (i+j)/2 \rfloor$

if x>a_m **then** i:=m+1

else j:= m

end

if x=a_i **then** location := i

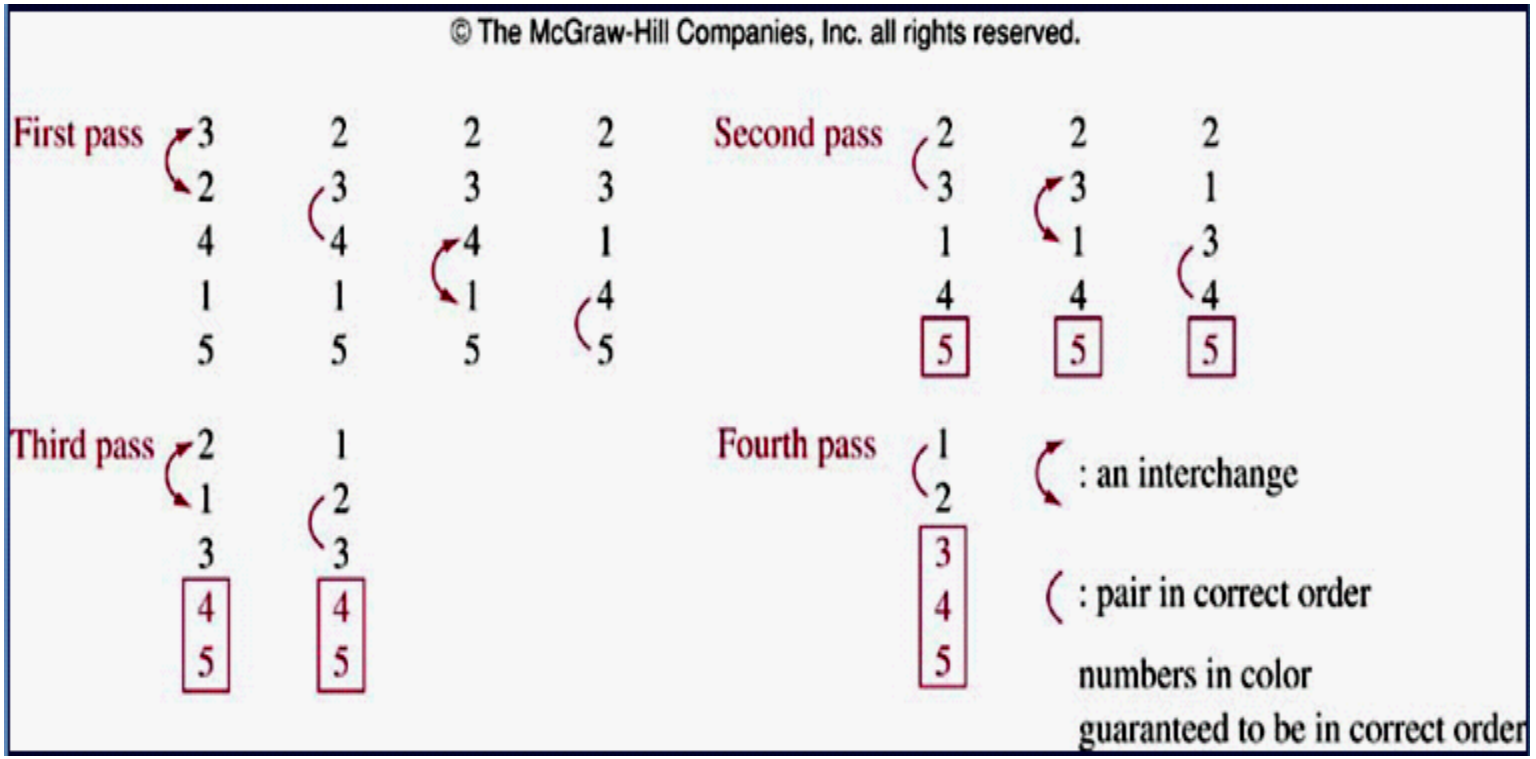
else location:= 0

{location is the subscript of the term that equals x, or is 0 if x is not found}

Sorting (sắp xếp)

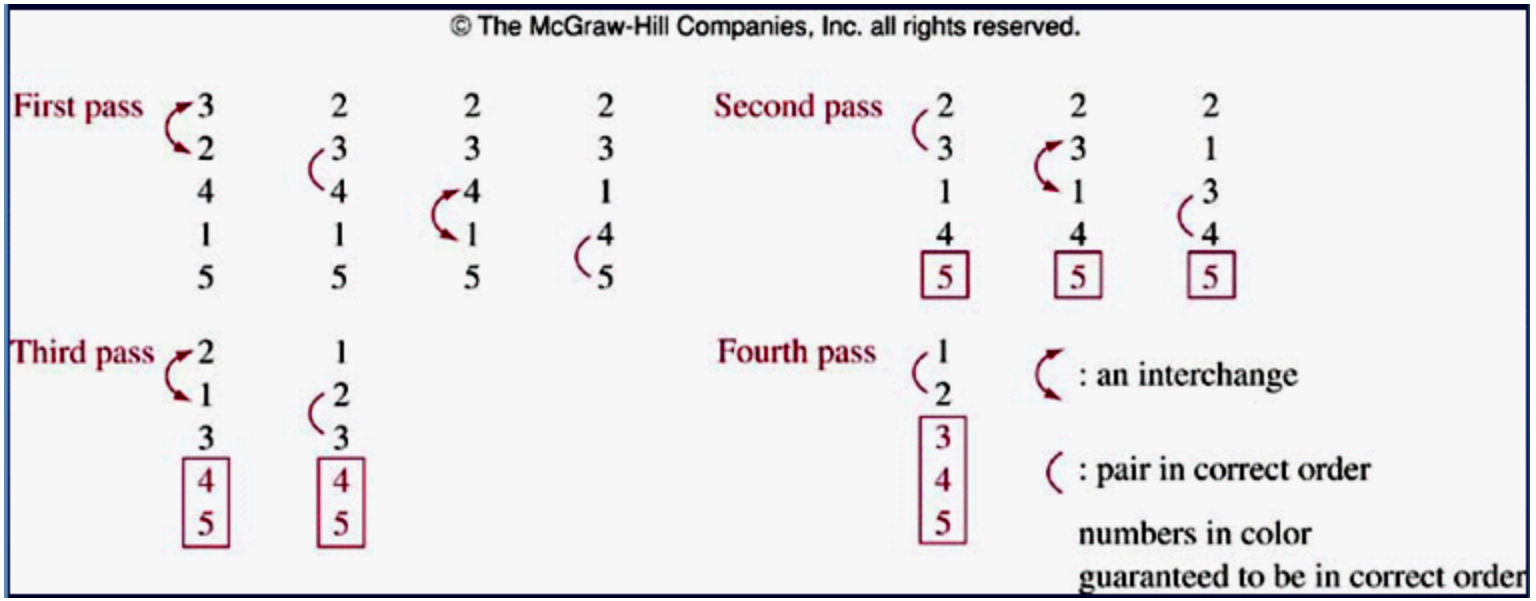
- **Putting elements into a list in which the elements are in increasing order.**
- There are some sorting algorithms
- Bubble sort
- Insertion sort
- Selection sort (exercise p. 178)
- Binary insertion sort (exercise p. 179)
- Shaker sort (exercise p.259)
- Merge sort and quick sort (section 4.4)
- Tournament sort (10.2)

Bubble Sort (nổi bọt)



Bubble sort: thực hiện sắp xếp dãy số bằng cách lặp đi lặp lại công việc đổi chỗ 2 số liên tiếp nhau nếu chúng đứng sai vị trí cho đến khi dãy số được sắp xếp.

Bubble Sort (nổi bọt)



procedure buble sort ($a_1,a_2,\ldots,a_n$:real numbers with $n \geq 2$)

for i:= 1 **to** n-1

  for j:=1 **to** n- i

    if $a_j > a_{j+1}$ **then interchange** a_j **and** a_{j+1}

{$a_1,a_2,\ldots,a_n$ are sorted}

Insertion Sort (chèn)

- Thuật toán sắp xếp chèn thực hiện sắp xếp dãy số theo cách duyệt từng phần tử và chèn từng phần tử đó vào đúng vị trí trong mảng con (dãy số từ đầu đến phần tử phía trước nó) đã sắp xếp sao cho dãy số trong mảng đã xếp đó vẫn đảm bảo tính chất của dãy số tăng dần.

Ví dụ:

6 5 3 1 8 7 2 4
5 6 3 1 8 7 2 4
3 5 6 1 8 7 2 4
1 3 5 6 8 7 2 4
1 3 5 6 8 7 2 4
1 3 5 6 7 8 2 4
1 2 3 5 6 7 8 4
1 2 3 4 5 6 7 8

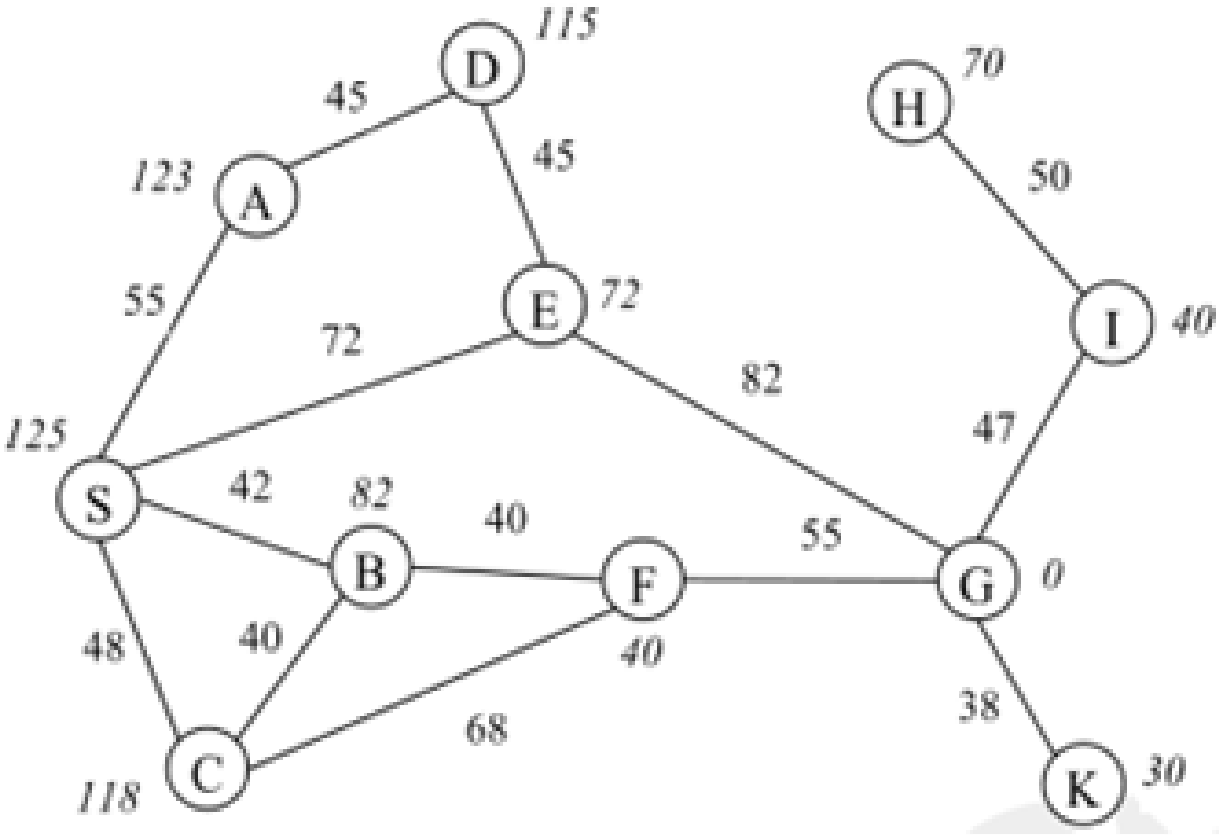
Insertion Sort (chèn)

procedure insertion sort ($a_1,a_2,\ldots,a_n$:real numbers with $n \geq 2$)
for j:= 2 **to** n { j: position of the examined element }
begin
{ finding out the right position of a_j }
i:=1
while $a_j > a_i$ i:= i+1
m:=a_j { save a_j }
{ moving j-i elements backward }
for k:=0 to j-i-1 $a_{j-k}:=a_{j-k-1}$
{move a_j to the position i}
a_i:= m
end
{$a_1,a_2,\ldots,a_n$ are sorted}

It is usually not the most efficient

Greedy Algorithm
Giải thuật tham lam

- They are usually used to solve optimization problems: Finding out a solution to the given problem that either minimizes or maximizes the value of some parameter.
- Selecting the best choice at each step, instead of considering all sequences of steps that may lead to an optimal solution.
- Some problems:
  - Finding a route between two cities with smallest total mileage (number of miles that a person passed).
  - Determining a way to encode messages using the fewest bits possible.
  - Finding a set of fiber links between network nodes using the least amount of fiber.


115
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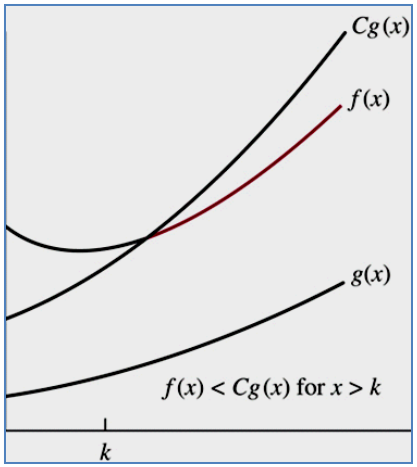
55
B
F
G
0
48
40
40
38
68
118
C
K
30

3.2- The Growth of Functions

- The complexity of an algorithm that acts on a sequence depends on the number of elements of sequence.
- The growth of a function is an approach that help selecting the right algorithm to solve a problem among some of them.
- Big-O notation is a mathematical representation of the growth of a function.

3.2.1-Big-O Notation

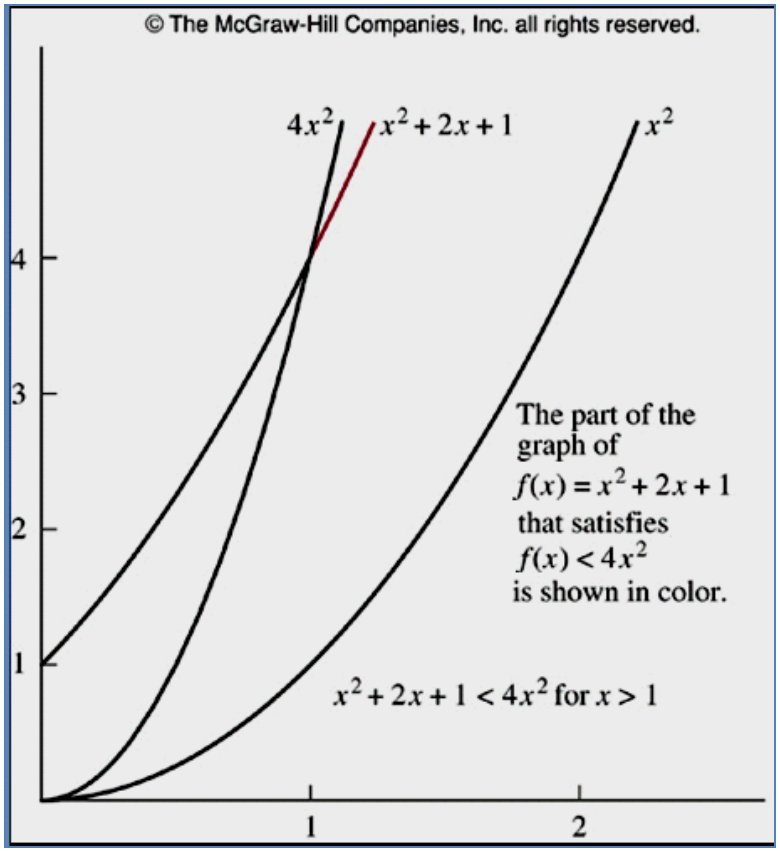


Definition:

Let *f* and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is O(g(x)) if there are constants C and k such that $|f(x)| \leq C|g(x)|$ whenever x>k

Example*:* **Show that $f(x)=x^2 + 2x +1$ is $O(x^2)$**

- Examine with x>1 ➔ $x^2 > x$
- ➔ $f(x)=x^2 + 2x +1 < x^2 + 2x^2 + x^2$
- ➔ $f(x) < 4x^2$
- ➔ Let $g(x)= x^2$
- ➔ C=4, k=1, $|f(x)| \leq C|g(x)|$
- ➔f(x) is $O(x^2)$



Big-O: Theorem 1

Let $f(x)=a_nx^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, where $a_0,a_1,\ldots,a_n$ are real number, then f(x) is $O(x^n)$

If $x>1$

$|f(x)| = | a_nx^n + a_{n-1}x^{n-1}+\ldots+a_1x+a_0|$

$\leq | a_nx^n |+|a_{n-1}x^{n-1}|+\ldots+|a_1x|+|a_0|$ { triangle inequality }

$\leq x^n (| a_n |+|a_{n-1}x^{n-1}/x^n|+\ldots+|a_1x/x^n|+|a_0/x^n|)$

$\leq x^n (| a_n |+|a_{n-1}/x|+\ldots+|a_1/x^{n-1}|+|a_0/x^n|)$

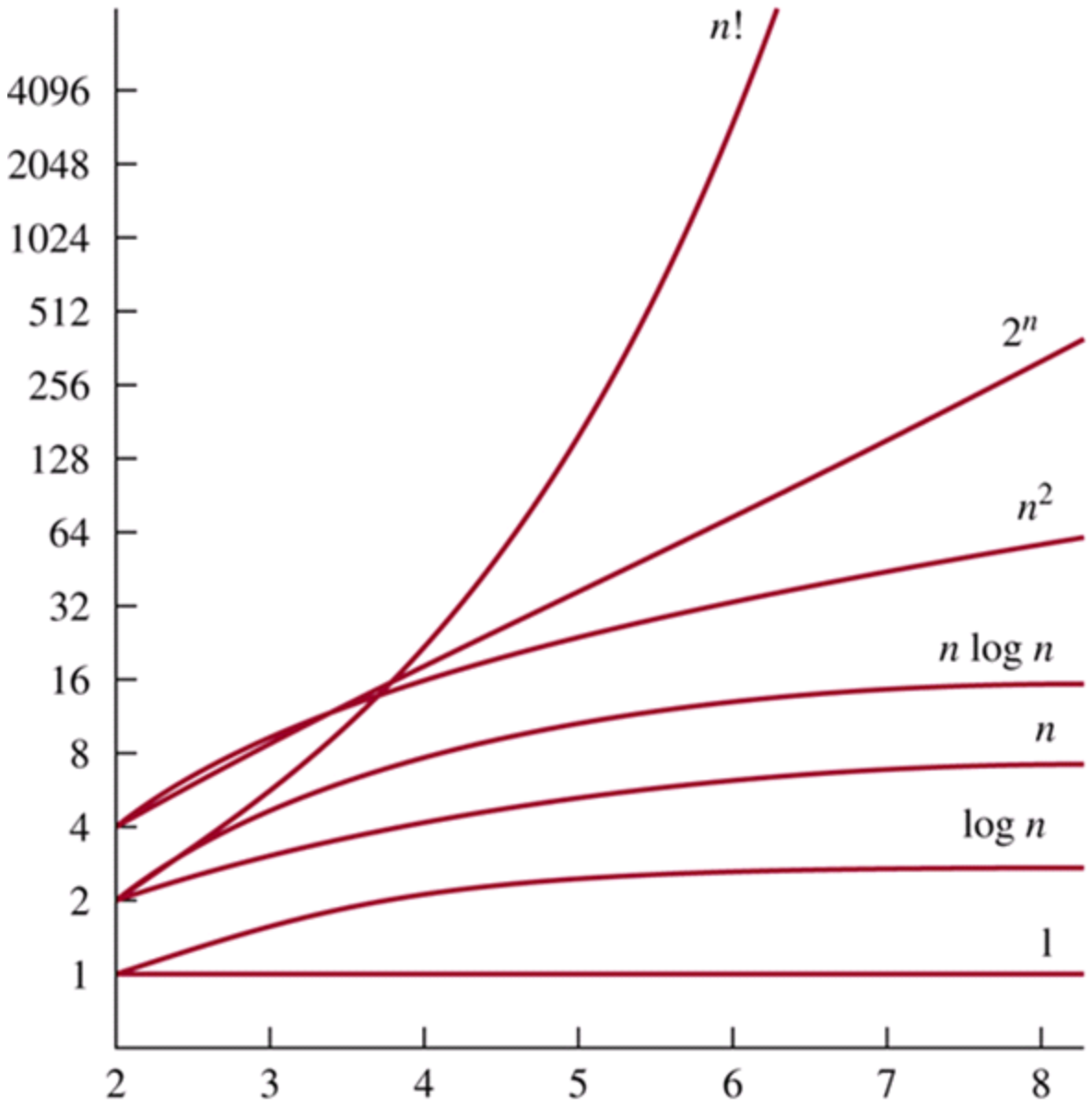
$\leq x^n (| a_n |+|a_{n-1}|+\ldots+|a_1|+|a_0|)$

Let $C= | a_n |+|a_{n-1}|+\ldots+|a_1|+|a_0|$

$|f(x)| \leq Cx^n$

➔ $\mathbf{f(x) = O(x^n)}$

The Growth of Combinations of Functions



Big-O : Theorems

Theorem 2:

$f_1(x)=O(g_1(x)) \wedge f_2(x)=O(g_2(x))$

$\rightarrow (f_1+f_2)(x) = O(\max(|g_1(x)|,|g_2(x)|))$

Theorem 3:

$f_1(x)=O(g_1(x)) \wedge f_2(x)=O(g_2(x))$

$\rightarrow (f_1 f_2)(x) = O(g_1 g_2(x)))$

Corollary (hệ quả) 1:

$f_1(x)=O(g(x)) \wedge f_2(x)=O(g(x)) \rightarrow (f_1+f_2)(x) = O(g(x))$

Big-O : Example

- Ví dụ 1: Xác định độ phức tạp của chương trình sau

```
VAR i,j,n : longint
      s1,s2: longint
Begin
{1}   readln(n);
{2}   s1:=0;
{3}   for i:=1 to n do
{4}   s1:=s1+i
{5}   s2:=0;
{6}   for j:=1 to n do
{7}   s2:=s2+j*j;
{8}   write(s1);
{9}   write(s2);
End.
```

O(n)

Big-O : Example

- Ví dụ 2: Xác định độ phức tạp của chương trình sau

```
VAR i,j,n: longint;
      c: longint;
Begin
{1} readln(n);
{2} c:=0;
{3} for i:=1 to 2*n do
{4} c:=c+1
{5} for i:=1 to n do
{6}     for j:=1 to n do
{7}     c:=c+1;
{8} writeln(c);
End.
```

O(n^2)

3.3- Complexity of Algorithms

- Computational complexity = Time complexity + space complexity.
- Time complexity can be expressed in terms of the number of operations used by the algorithm.
  - Average-case complexity
  - Worst-case complexity
- Space complexity will not be considered.

Demo 1

Describe the time complexity of the algorithm for finding the largest element in a set:

Procedure max ($a_1, a_2, \ldots, a_n$: integers)

max:= a_1

for i:=2 to n

If max < a_i **then** max:= a_i

Time Complexity is θ(n)

i	Number of comparisons
2	2
3	2
…	2
n	2
n+1	1, max< a_i is omitted

2(n-1) +1 = 2n-1 comparisions

Demo 2

Describe the average-case time complexity of the linear-search algorithm :

Procedure linear search (x: integer, $a_1,a_2,\ldots,a_n$:distinct integers)

i:=1

while ($i \leq n$ **and** $x \neq a_i$) i:=i+1

if $i \leq n$ **then** location:= i

else location:=0

i	Number of comparisons done
1	2
2	4
…	
n	2n
n+1	1, $x \neq a_i$ is omitted

Avg-Complexity= [(2+4+6+…+2n)]/n +1 +1

= [2(1+2+3+…+n) /n+2

= [2n(n+1)/2]/n + 2

=[n(n+1)]/n + 2

= n+1 + 2 = n+3

= θ(n)

See demonstrations about the worst-case complexity: Examples 5,6 pages 195, 196

Understanding the Complexity of Algorithms

Complexity	Terminology	Problem class
Θ(1)	Constant complexity	Tractable (dễ), class P
Θ(log n)	Logarithmic complexity	Class P
Θ(n)	Linear complexity	Class P
Θ(n logn)	n log n complexity	Class P
$\Theta(n^b)$, b□1, integer	Polynominal complexity	Intractable, class NP
$\Theta(b^n)$, b>1	Exponential complexity	
Θ(n!)	Factorial complexity	

NP : Non-deterministic Polynomial time

3.4- The Integers and Division

Definition: If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that b=ac.

When a divides b, we say that:

a is a factor (thừa số, divisor – số chia, ước số) of b

b is a multiple (bội số) of a

Notation: a|b : a divides b ałb : a does not divide b

Example:

3ł7 because 7/3 is not an integer

3|12 because 12/3=4 , remainder=0

Corollary 1:

$a|b \wedge a|c \rightarrow a|(mb+nc)$, m,n are integers

The Division Algorithm

Theorem 2: Division Algorithm: Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$

d: divisor , r : remainder, q: quotient (thương số)

Note: r can not be negative

Definition: $a = dq + r$

a: dividend d: divisor

q: quotient (thương số) r : remainder,

q= a div d r = a mod d

Example:

101 is divided by 11 :101 = 11.9 + 2 ➔ q=9, r=2

-11 is divided by 3 : 3(-4)+ 1 ➔ q=-4, r=1

No OK : -11 is divided by 3 : 3(-3)-2 ➔ q=-3, **r = -2**

Modular Arithmetic – Phép toán đồng dư

Definition: a, b: integers, m: positive integer.

a is called ***congruent*** (đồng dạng) to b modulo m if m|a-b

Notations:

$a \equiv b \pmod{m}$, a is congruent to b modulo m

$a \not\equiv b \pmod{m}$, a is not congruent to b mod m

Examples:

15 is congruent to 6 modulo 3 because 3|15-6

15 is **not** congruent to 7 modulo 3 because 3 ł15-7

Modular Arithmetic – Phép toán đồng dư

Theorem 3

a,b: integers, m: positive integer

$a \equiv b \pmod m \leftrightarrow a \bmod m = b \bmod m$

Proof

(1) $a \equiv b \pmod m \rightarrow a \bmod m = b \bmod m$

$a \equiv b \pmod m$ → m| a-b → a-b= km→ a=b + km

→ a mod m = (b + km) mod m

→ a mod m = b mod m { km mod m = 0 }

(2) $a \bmod m = b \bmod m \rightarrow a \equiv b \pmod m$

a = k1m + c ^ b=k2m + c → a-b = (k1-k2) m { suppose a>b)

= km { k= k1-k2}

→ m| a-b → a= b (mod m)

Modular Arithmetic…

Theorem 4

a,b: integers, m: positive integer

a and b are congruent modulo m if and only if there is an integer k such that a = b + km

Proof

(1) $a \equiv b \pmod m \rightarrow a = b + km$

$a \equiv b \pmod m \rightarrow m \mid a-b \rightarrow a-b = km$ { from definition of division}

(2) $a = b + km \rightarrow a \equiv b \pmod m$

$a = b + km \rightarrow a-b=km \rightarrow m \mid a-b \rightarrow a \equiv b \pmod m$

Modular Arithmetic…

Theorem 5

m: positive integer

$a \equiv b \pmod{m} \wedge c \equiv d \pmod{m} \rightarrow$

$a+c \equiv b+d \pmod{m} \wedge ac \equiv bd \pmod{m}$

Proof : See page 204

Corollary 2:

m : positive integer, a,b : integers

$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$

$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$

Proof: page 205

Applications of Congruences

Hashing Function: $H(k) = k \bmod m$

Using in searching data in memory.

k: data searched, m : memory block

Examples:

$H(064212848) \bmod 111 = 14$

$H(037149212) \bmod 111 = 65$

Collision: $H(k_1) = H(k_2)$. For example, $H(107405723) = 14$

Applications of Congruences

Pseudo-random Numbers $x_{n+1}=(ax_n+c) \bmod m$

a: multiplier, c: increment, x_0: seed

with $2 \leq a<m$, $0 \leq c<m$, $0 \leq x_0<m$

Examples:

m=9 ➔ random numbers: 0..8

a= 7, c=4, $x_0=3$

Result: Page 207

Applications of Congruences

Cryptography: letter 1 → letter 2

Examples: shift cipher with k – mật mã dịch k ký tự

$f(p) = (p+k) \bmod 26 \rightarrow f^{-1}(p) = (p-k) \bmod 26$

Sender: (encoding)

Message: "ABC" , k=3

Using $f(p) = (p+3) \bmod 26$ // 26 characters

ABC ➔ 0 1 2 → 3 4 5 ➔ DEF

Receiver: (decoding)

DEF ➔ 3 4 5

Using $f^{-1}(p) = (p-3) \bmod 26$

3 4 5 ➔ 0 1 2 ➔ ABC

3.5- Primes and Greatest Common Divisors

Definition 1:

A positive integer p greater than 1 is called ***prime*** *(số nguyên tố)* if the only positive factors are 1 and p

A positive integer that is greater than 1 and is *not prime* is called *composite (hợp số)*

Examples:

Primes: 2 3 5 7 11

Composites: 4 6 8 9

Finding very large prime: test for supercomputer

Primes…

Theorem 1- The fundamental theorem of arithmetic:

Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size

Examples:

Primes: 37

Composite: $100 = 2.2.5.5 = 2^2 5^2$

$999 = 3.3.3.37 = 3^3 37$

Primes…

Converting a number to prime factors

Examples: 7007

Try it to 2,3,5 : 7007 can not divided by 2,3,5

7007 : 7

1001 : 7

143: 11

13: 13

0

➔ $7007 = 7^2.11.13$

Primes…

Theorem 2: If n is a composite, then n has a prime **divisor** less than or equal to $\sqrt{n}$

Proof:

n is a composite ➔ n = ab in which a or b is a prime

If $(a > \sqrt{n} \wedge b > \sqrt{n})$ → ab>n ➔ false

➔ Prime divisor of n less than or equal to $\sqrt{n}$

Primes…

Theorem 3: There are infinite many primes

Proof: page 212

Theorem 4: The prime number theorem:

The ratio of the number of primes not exceeding x and x/ln x approaches 1 and grows with bound (ln x: natural logarithm of x)

See page 213

Example:

$x = 10^{1000}$ → The odds that an integer near 10^{1000} is prime are approximately $1/\ln 10^{1000} \sim 1/2300$

Primes: Conjecture and Open Problem
Các phỏng đoán và bài toán chưa có lời giải

See pages: 214, 215

- 3x + 1 conjecture
- Twin prime conjecture: there are infinitely many twin primes

Greatest Common Divisors and Least Common Multiples

Definition 2:

Let a, b be integers, not both zero. The largest integer d such that d|a and d|b is called the *greatest common divisor* of a and b.

Notation: gcd(a,b)

Example: gcd(24,36)=?

Divisors of 24: 2 3 4 6 8 12 = $2^3 3^1$

Divisors of 36: 2 3 4 6 9 12 18 = $2^2 3^2$

gcd(24,36)=12 = $2^2 3^1$ // Get factors having minimum power

Greatest Common Divisors and Least Common Multiples

Definition 3:

The integers a, b are *relatively prime* if their greatest common divisor is 1

Example:

gcd(3,7)=1 ➔ 3,7 are relatively prime

gcd (17,22)=1 ➔ 17,22 are relatively prime

gcd(17,34) = 17 ➔ 17, 34 are **not** relatively prime

Greatest Common Divisors and Least Common Multiples

Definition 4:

The integers $a_1, a_2, a_3, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$

Example:

7 10 11 17 23 are pairwise relatively prime

7 10 11 16 24 are **not** pairwise relatively prime

➔ Adjacent number of every composite in sequence must be a prime.

Greatest Common Divisors and Least Common Multiples

Definition 5:

The Least common multiple of the positive integer a and b is the smallest integer that is divisible by both a and b

Notation: lcm(a,b)

Example:

lcm(12,36) = 36 lcm(7,11) = 77

lcm $(2^3 3^5 7^2, 2^4 3^3) = 2^4 3^5 7^2$

$2^3 3^5 7^2, 2^4 3^3 7^0$ ➔ $2^4 3^5 7^2$ // get maximum power

Greatest Common Divisors and Least Common Multiples

Theorem 5:

Let a, b be positive integers then

ab= gcd(a,b). lcm(a,b)

Example: gcd(8, 12) = 4 lcm(8,12)=24 ➔ 8.12 = 4.24

Proof: Based on analyzing a, b to prime factors to get gcd(a,b) and lcm(a,b)

➔ ab=gcd(a,b). lcm(a,b)

3.6- Integers and Algorithms

- Representations of Integers
- Algorithms for Integer Operations
- Modular Exponentiation
- Euclid Algorithm

Representations of Integers

Theorem 1: Khai triển hệ cơ sở b của số nguyên n

Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$

Where k is a nonnegative integer, $a_0, a_1, a_2, \ldots, a_k$ are nonnegative integers less than b and $a_k \neq 0$

Proof: page 219

Example: $(245)_8 = 2.8^2 + 4.8 + 5 = 165$

Common Bases Expansions: Binary, Octal, Decimal, Hexadecimal

Finding expansion of an integer: Pages 219, 220, 221

Algorithm 1: Constructing Base b Expansions

Procedure base b expansion (n: positive integer)

q:=n

k:=0

while $q \neq 0$

begin

$a_k := q \bmod b$ dư

$q := \lfloor q/b \rfloor$ thương

k := k +1

end { The base b expansion of n is $(a_{k-1}a_{k-2}\ldots a_1a_0)$}

Algorithms for Integer Operations

Algorithm 2: Addition integers in binary format

Algorithm 3: Multiplying integers in binary format

Algorithm 4: Computing div and mod integers

Algorithm 5: Modular Exponentiation

Algorithm 2: Adding of Integers

Complexity: O (n)

```
1 1 1 0 0
  1 1 1 0 (a)
+ 1 0 1 1 (b)
-----------
1 1 0 0 1 (s)
```

procedure add (a,b: integers)

{ a: $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$ b: $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$ }

c:=0

for j:=0 to n-1

Begin

d:= $\lfloor (a_j+b_j+c)/2 \rfloor$ // next carry of next step

d là thương của (aj + bj + c) div 2

$s_j = a_j+b_j+c - 2d$ // result bit sj là (aj + bj + c) mod 2

c:=d // updating new carry to next step

End

s_n = c // rightmost bit of result

{ The binary of expansion of the sum is $(s_ns_{n-1}\ldots s_1s_0)$}

Algorithm 3: Multiplying Integers

Complexity: $O(n^2)$

```
      1 1 0 (a)
X     1 0 1 (b)
      1 1 0
+   0 0 0 0
  1 1 0 0 0
  1 1 1 1 0 (p)
```

```
procedure multiply ( a,b: integer)
{ a: (a_{n-1}a_{n-2}…a_1a_0)_2      b: (b_{n-1}b_{n-2}…b_1b_0)_2 }
for j:= 0 to n-1
begin
   if b_j =1 then c_j := a shifted j places
end
{ c_0, c_1, …, c_{n-1}  are the partial products }
p := 0
for j:= 0 to n-1
     p:=p+c_j
{p is the value of ab}
```

Complexity: $O(n)$

Algorithm 4: Computing div and mod

tính thương và dư trong hệ nhị phân

```
procedure division ( a: integer; d: positive integer)
 q:=0
 r:= |a|
while r ≥ d  {quotient= number of times  of successive subtractions}
   begin
      r:= r-d
      q := q+1
   end
If a<0 and r>0  then  {updating remainder when a<0}
   begin
      r:= d-r
     q := -(q+1)
   end
{ q = a div d is the quotient, r=a mod d is the remainder}
```

11/3 thì cuối cùng 11-3-3-3 = 2 =r và q= 0 + 1 + 1 + 1 = 3

tăng thương lên 1

Algorithm 5: Modular Exponentiation

{ $b^n \bmod m = ?$. Ex: $3^{644} \bmod 645 = 36$ }

procedure mod_ex (b: integer, $n=(a_{k-1}a_{k-2}\ldots a_1a_0)_2$, m: positive integer)

x:=1

power := b mod m power cho b^(2^0)

for i:=0 to k-1

begin

if a_i=1 then x:= (x.power) mod m

power := (power.power) mod m

end

{ x equals b^n mod m }

bản chất là biểu diễn n dưới dạng nhị phân

$$b^n = b^{a_{k-1}2^{k-1}+a_{k-2}2^{k-2}\ldots a_1 2^1+a_0}$$

$$b^n = b^{a_{k-1}2^{k-1}}.b^{a_{k-2}2^{k-2}}\ldots b^{a_1 2^1}b^{a_0}$$

bản chất là b^4 mod m = (b^2 mod m * b^2 mod m) mod m

b^8 mod m = (b^4 mod m * b^4 mod m) mod m

.....

power chính là phần b^(2^i) mod m với i chạy từ 0 đến k-1

Corollary 2: ab mod m = ((a mod m)(b mod m)) mod m

b^n mod m = result of successive $factor_i$ mod m

Algorithm 5: Modular Exponentiation

$i = 0$: Because $a_0 = 0$, we have $x = 1$ and $power = 3^2 \bmod 645 = 9 \bmod 645 = 9$;

$i = 1$: Because $a_1 = 0$, we have $x = 1$ and $power = 9^2 \bmod 645 = 81 \bmod 645 = 81$;

$i = 2$: Because $a_2 = 1$, we have $x = 1 \cdot 81 \bmod 645 = 81$ and $power = 81^2 \bmod 645 = 6561 \bmod 645 = 111$;

$i = 3$: Because $a_3 = 0$, we have $x = 81$ and $power = 111^2 \bmod 645 = 12{,}321 \bmod 645 = 66$;

$i = 4$: Because $a_4 = 0$, we have $x = 81$ and $power = 66^2 \bmod 645 = 4356 \bmod 645 = 486$;

$i = 5$: Because $a_5 = 0$, we have $x = 81$ and $power = 486^2 \bmod 645 = 236{,}196 \bmod 645 = 126$;

$i = 6$: Because $a_6 = 0$, we have $x = 81$ and $power = 126^2 \bmod 645 = 15{,}876 \bmod 645 = 396$;

$i = 7$: Because $a_7 = 1$, we find that $x = (81 \cdot 396) \bmod 645 = 471$ and $power = 396^2 \bmod 645 = 156{,}816 \bmod 645 = 81$;

$i = 8$: Because $a_8 = 0$, we have $x = 471$ and $power = 81^2 \bmod 645 = 6561 \bmod 645 = 111$;

$i = 9$: Because $a_9 = 1$, we find that $x = (471 \cdot 111) \bmod 645 = 36$.

644 = 1010000100 binary

The Euclidean Algorithm

Lemma (bổ đề): Proof: page 228

Let a= bq+r, where a, b, q, r are integers. Then gcd(a,b) = gcd(b,r)

Example: 287 = 91.3 + 14 ➔gcd(287,91) =gcd(91,14)= 7

```
procedure  gcd(a,b: positive integer)
 x:=a
 y:=b
 while y ≠ 0
  begin
    r := x mod y
    x:=y
    y:= r
   end  {gcd(a,b) is x}
```

The Euclidean Algorithm

Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Solution:

662=414.1+248

414=248.1+166

248=116.1+82

116=82.1+2

82=41.2

Gcd(414,662)=2

Summary

- Algorithms
- The Growth of Functions
- Complexity of Algorithms
- The Integers and Division
- Primes and Greatest Common Divisors
- Integers and Algorithms

Thanks