Sai Phanindra Korlepara — **24142** — M.Tech CSA Dept.

Problem 1. *JE Book, Chapter H, Problem 2*

(a). Give a linear-programming formulation of the bipartite maximum matching problem. The input is a bipartite graph $G = (U \cup V; E)$, where $E \subseteq U \times V$; the output is the largest matching in G. Your linear program should have one variable for each edge.

(b). Now dualize the linear program from part (a). What do the dual variables represent? What does the objective function represent? What problem is this!?

Solution.

(a). To the given graph, we add two new vertices s, t and add a edge from s to each vertex in the set U & similarly add a edge from each vertex in set V to t. Set all the edge weights to 1. With this we have essentially converted the bipartite maximum matching problem to the max-flow / min-cut problem.

This is a valid conversion because, in bipartite maximum matching problem we have the following conditions.

- In a bipartite graph there are no edges among the vertices inside either the set U or set V. This is unaffected by the transformation because we did not add any edges that are going among the vertices of sets U or V.
- In a matching we are allowed to choose only one of the two edges that share a vertex. Our transformation enforces this rule by setting the flow going into set U or out of set V to 1. This forces the max-flow algorithm to choose only one of the edges going out of vertices coming into set U or only one of the edges coming into the vertices of set V.
- To make this matching maximum, Finding the max-flow, finds the maximum flow between the sets U and V. Which is essentially choosing maximum possible number of edges which follow the constraints of the maximum matching bipartite problem.

Hence the max flow in the transformed graph is the size of the maximum bipartite matching problem.

As discussed in class we can convert this max-flow problem into a linear program which can be described as follows, let the variable f_{xy} be defined as the flow from vertex x to y & c_{xy} be defined as the weight of the edge from vertex x to y

$$maximize \sum_{w \in U} f_{sw}$$

here s is the source vertex and w is any vertex where there is an edge $s \rightarrow w$ in the graph. subjected to the constraints as

$$\sum_{w} f_{vw} - \sum_{x} f_{xv} = 0 \;\; \forall \; v \in (U \cup V), \;\; x, w \in (\{s,t\} \cup U \cup V)$$

$$f_{yz} < c_{yz} \ \forall \ edges \ y \to z \ in \ graph$$

$$f_{yz} \geq 0 \quad \forall \ edges \ y \to z \ in \ graph$$

The question states that there needs to be one variable for each edge in the graph, which this LP formulation does have. The question does not specify restrictions on modifications in the graph, hence it is being assumed that this modification is valid as we are still meeting all the requirements outlined by the question.

(b). To find the dual, let's define two new random variables, y_v for each flow conservation constraint for each of the vertices $v \in U \cup V$ and z_{yz} for each capacity constraint on the edge $y \to z$ in the graph.

$$\min \sum_{(y,z)\in E} c_{yz} z_{yz}$$

subject to:

$$-y_y + y_z + z_{yz} \geq 0, \quad \forall (y,z) \in E$$

$$y_s = 1$$

$$y_t = 0$$

$$z_{yz} \geq 0, \quad \forall (y,z) \in E$$

□

Problem 2. *JE Book, Chapter H, Problem 7b*
Suppose you are given an arbitrary directed graph $G = (V, E)$ with arbitrary edge weights $l : E \to R$, and two special vertices. Each edge in G is colored either red, white, or blue to indicate how you are permitted to modify its weight:

- *You may increase, but not decrease, the length of any red edge.*
- *You may decrease, but not increase, the length of any blue edge.*
- *You may not change the length of any **white** edge.(changed typo in the book)*

Your task is to modify the edge weights—subject to the color constraints—so that every path from s to t has exactly the same length. Both the given weights and the new weights of the edges can be positive, negative, or zero. To keep the following problems simple, assume every edge in G lies on at least one path from s to t, and that G has no isolated vertices.

(a). Describe a linear program that is feasible if and only if it is possible to make every path from s to t the same length.

(b). Construct the dual of the linear program from part (a).

Solution.

(a). Solution added explicitly as the primal is needed to compute the dual and also because I was unable to complete Assignment - 5 in time.

let's define a variable called $dist(v)$ for each vertex v that determines it's distance from the source. Hence $dist(s) = 0$. we will also include the following variables $r_{(u,v)} \geq 0$, $b_{(u,v)} \geq 0$ that can be added or reduced for red and blue edges respectively between vertices u and v. The main objective as for the standard SSSP LP is

$$maximize \;\; dist(t)$$

The constraints are defined as follows

For each red edge we have length that can be increased

$$dist(v) - dist(u) \leq l(u,v) + r_{(u,v)}$$

Similarly for blue edges we have

$$dist(v) - dist(u) \leq l(u,v) - b_{(u,v)}$$

For white edges we have

$$dist(v) - dist(u) \leq l(u,v)$$

To enforce uniqueness of the solutions, ie, every path from the source to sink must have the same weight and since every edge is part of at least one of the paths we modify the above inequalities to equalities

In the end the final LP problem is defined as

$$maximize \;\; dist(t)$$

$$dist(v) - dist(u) = l(u,v) + r_{(u,v)}$$

$$dist(v) - dist(u) = l(u,v) - b_{(u,v)}$$

$$dist(v) - dist(u) = l(u,v)$$

$$r_{(u,v)} \geq 0 \mid b_{(u,v)} \geq 0 \mid dist(s) = 0$$

since dist() can be negative as edge weights can be negative we leave it unrestricted.

(b). To construct the dual we introduce variables for each constraint as follows, $\alpha_{u,v}$ for each red edge constraint, $\beta_{u,v}$ for each blue edge constraint,$\gamma_{u,v}$ for each white edge constraint.These dual variables are unrestricted as the original constraints in the primal are equalities. we also introduce the variable δ_s for the constraint that $dist(s) = 0$

Objective function of the dual is

$$minimize \sum_{(u,v)\in red} \alpha_{(u,v)} l(u,v) + \sum_{(u,v)\in blue} \beta_{(u,v)} l(u,v) + \sum_{(u,v)\in white} \gamma_{(u,v)} l(u,v)$$

For primal variable $r_{(u,v)} \geq 0$, The coefficient of $r_{(u,v)}$ in the primal constraint is -1. The corresponding dual constraint is

$$(-1) \cdot \alpha_{(u,v)} \leq 0 \implies \alpha_{(u,v)} \geq 0$$

For primal variable $b_{(u,v)} \geq 0$, The coefficient of $b_{(u,v)}$ in the primal constraint is 1. The corresponding dual constraint is

$$(+1) \cdot \beta_{(u,v)} \leq 0 \implies \beta_{(u,v)} \leq 0$$

for dual constraints of $dist(v)$, we need to check it's coefficients for all the vertices in the primal objective function, and in all the constraints of the primal. we get a series of conditions for each vertex $v \in V$ as follows,

$$\sum_{(u,v)\in E} y_{(u,v)} - \sum_{(v,w)\in E} y_{(v,w)} + \delta_v = c_v$$

where,

$y_{(u,v)}$ *can be any one of* $\alpha_{(u,v)},\ \beta_{(u,v)},\ \gamma_{(u,v)}$ *depending on the color of the edge* (u, v)

$$\delta_v = \delta_s \ if \ v = s, 0 \ otherwise.$$

$$c_v = 1 \ if \ v = t, 0 \ otherwise.$$

Usually the constraints on all of alpha, gamma, delta should be unrestricted because they correspond to equality constraints in the primal. but since we have obtained inequalities above for alpha and beta, I omit them here and mention the rest.

$$\gamma_{(u,v)}, \delta_s \ \text{are unrestricted}$$

Hence we can conclude the Dual conversion of the primal for the above problem.

□

Problem 3. *Prove that the dual of the dual of any linear program is the original primal linear program.*

Solution. From the standard proof we can show that any linear program can be converted into canonical form. If we can prove that dual of dual of a canonical linear program is itself, we have proved that dual of dual of any linear program is the original primal linear program itself.

	Primal	Dual	Dual's canonical	Dual's Dual	Restated
Objective	max cx	min yb	max -b$^T y^T$	min z$^T(-c^T)$	max cz
Constraints	Ax $\leq b$	yA $\geq c$	-A$^T y^T \leq -c^T$	-z$^T A^T \geq -b^T$	Az $\leq b$
limits	x ≥ 0	y ≥ 0	y$^T \geq 0$	z$^T \geq 0$	z ≥ 0

First let's start with the Primal of canonical and it's dual in standard form. Then we convert this dual into a standard canonical form. This conversion is accurate because $yb = b^T y^T$.

Minimization problem can be converted to maximization problem when a negative is applied, similar modifications can be made to constraints as well. Now lets find the dual's dual using the original table.

z is introduced as it's transpose to maintain consistency that x,y,z are all column vectors. Now we can apply similar transformations like before to convert say min problem to max problem and so on..

In the last column if z is replaced with x, it looks exactly like the original primal. Hence we have shown that Dual of a Dual is same as the original canonical Linear program and by extension Dual of the Dual of any Linear program is the original Primal Linear program itself $\square$

Problem 4. *Recall that in class we saw the LP relaxation for the max s-t flow where the objective function is to maximize the flow leaving s (we are assuming that there are no edges entering s and no edges leaving t). Write an LP relaxation for the max s-t flow where the objective function is to maximize the flow entering t. Write the dual of this LP and show that it is also equal to the min s-t cut.*

Solution. As we have already discussed in question 1, for the Max Flow linear program characterized as maximizing flow leaving the source we have,

let the variable f_{xy} be defined as the flow from vertex x to y & c_{xy} be defined as the capacity of the edge from vertex x to y. let V be the set of vertices in the graph and E be the set of edges in the graph.

$$maximize \ \sum_{w} f_{sw}$$

here s is the source vertex and w is any vertex where there is an edge $s \to w$ in the graph. We can convert this to a LP that maximizes the flow entering sink. we can simply update the maximization condition. All other conditions will remain the same as it is still the same problem.

$$maximize \ \sum_{w} f_{wt}$$

t is the sink vertex and w is any vertex where there is an edge $w \to t$ in the graph. This problem is subjected to the constraints as

$$\sum_{w} f_{vw} - \sum_{x} f_{xv} = 0 \ \ \forall \ v \in (V - \{s,t\}), \ x, w \in V$$

$$f_{yz} < c_{yz} \ \forall \ edges \ y \to z \ \in \ E$$

$$f_{yz} \geq 0 \ \ \forall \ edges \ y \to z \ \in \ E$$

The dual of this linear program can be derived as follows, let's define two new random variables, y_v for each flow conservation constraint of each of the vertices $v \in V$ and z_{yz} for each capacity constraint on the edge $f_{yz} \in E$.

$$\min \sum_{(y \to z) \in E} c_{yz} z_{yz} + (0) y_v = c_{yz} z_{yz}$$

Since the equality constraint equates to zero the coefficient in the objective is nullified. The coefficients (Denoted as f^c_{yz}) of f_{yz} in the primal objective function are derived as $f^c_{yz} = 1$ if $y \to z$ is an edge to sink ie, $w \to t$. $f^c_{yz} = 0$ otherwise. Constraints can be derived as follows

For each f_{yz}, we can write a dual constraint as

$$-y_y + y_z + z_{yz} \geq f^c_{yz}, \quad \forall (y, z) \in E$$

to explain,

- Coefficient of y_y set to -1 as it's coefficient from the flow conservation at node y.
- Coefficient of y_z set to 1 as it's coefficient from the flow conservation at node z.
- Coefficient of z_{yz} set to 1 as it's coefficient from the capacity constraint for that edge .
- f^c_{yz} will be set based on its value in the primal objective function.

Since y_t does not exist as the flow is not conserved at the sink, The constraint equation is undefined for any edge where sink is present and since for all the edges that don't have sink as one of their vertices has the coefficient $f^c_{yz} = 0$, we have

$$-y_y + y_z + z_{yz} \geq 0$$

In the primal the flow conservation constraints are not defined at source s and sink t as they define the net flow. We set them as below because the net flow from s to t in the primal corresponds to the difference $y_s - y_t$ in the dual.

$$y_s = 1 \mid y_t = 0$$

$$z_{yz} \geq 0, \quad \forall (y, z) \in E \mid y_v \in V \textit{ is unrestricted}$$

Hence by converting the primal into the Dual, we can verify that the dual is indeed the LP for the min-cut problem.

□