Corrigé du Partiel Ma 211 - 16/11/2019

Question de cours :

* $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ de classe C^2 sur $\mathbb{R}^2$.

$$(x,y,z) \longmapsto \varphi(x,y,z)$$

$\overrightarrow{rot}\,\overrightarrow{grad}\,\varphi = \vec{0}$??

$$\overrightarrow{grad}(\varphi) = \left(\frac{\partial \varphi}{\partial x} ; \frac{\partial \varphi}{\partial y} ; \frac{\partial \varphi}{\partial z}\right).$$

Puisque φ est de classe C^2 sur $\mathbb{R}^2$ alors d'après le thm de Schwarz ; $\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$; $\frac{\partial^2 \varphi}{\partial x \partial z} = \frac{\partial^2 \varphi}{\partial z \partial x}$; $\frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi}{\partial z \partial y}$.

Alors
$$\overrightarrow{rot}\,\overrightarrow{grad}\,\varphi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \\ \frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \\ \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$$

* $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ de classe C^2.

$$(x,y,z) \longmapsto (F_1(x,y,z) ; F_2(x,y,z) ; F_3(x,y,z))$$

$div\,\overrightarrow{rot}\,F = 0$??

$$\overrightarrow{rot}\,F = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

donc
$$div\,\overrightarrow{rot}\,F = \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}}$$
$$= 0 \quad \text{(Théorème de schwarz)}$$

* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^2 et $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^2

$\overrightarrow{grad}(fg) = f\,\overrightarrow{grad}\,g + g\,\overrightarrow{grad}\,f$??

On a
$$\overrightarrow{grad}(fg) = \left(\frac{\partial}{\partial x}(fg) ; \frac{\partial}{\partial y}(fg) ; \frac{\partial}{\partial z}(fg)\right)$$

alors

$\overrightarrow{grad}(fg) = \left(g \times \frac{\partial f}{\partial x} + f \frac{\partial g}{\partial x} ; g \times \frac{\partial f}{\partial y} + f \frac{\partial g}{\partial y} ; g \frac{\partial f}{\partial z} + f \frac{\partial g}{\partial z} \right)$

$= \left(g \frac{\partial f}{\partial x}, g \frac{\partial f}{\partial y}, g \frac{\partial f}{\partial z} \right) + \left(f \frac{\partial g}{\partial x}, f \frac{\partial g}{\partial y}, f \frac{\partial g}{\partial z} \right)$

$= g \, \overrightarrow{grad} f + f \, \overrightarrow{grad} g.$

Exercice 1 :

$\boxed{a}$ $f : \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbb{R}$

$\vec{x} = (x_1, x_2, x_3) \longmapsto f(\vec{x}) = \frac{1}{\|\vec{x}\|} = \left(\sqrt{x_1^2 + x_2^2 + x_3^2} \right)^{-1}$

$= (x_1^2 + x_2^2 + x_3^2)^{\frac{-1}{2}}$

$\overrightarrow{grad} f$??

$\overrightarrow{grad} f = \left(\frac{\partial f}{\partial x_1} ; \frac{\partial f}{\partial x_2} ; \frac{\partial f}{\partial x_3} \right)$

$= \left(-x_1 (x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{2}} ; -x_2 (x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{2}} ; -x_3 (x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{2}} \right)$

donc $\overrightarrow{grad} f = \frac{-\vec{x}}{\|\vec{x}\|^3}$.

$\boxed{b}$ $g : \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow \mathbb{R}$

$\vec{x} = (x_1, x_2, x_3) \longmapsto g(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|^2} = \left(\frac{x_1}{x_1^2 + x_2^2 + x_3^2} ; \frac{x_2}{x_1^2 + x_2^2 + x_3^2} ; \frac{x_3}{x_1^2 + x_2^2 + x_3^2} \right)$

$= (g_1 ; g_2 ; g_3)$

• $\operatorname{div} g$??

$\operatorname{div}(g) = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}$

$= \frac{x_1^2 + x_2^2 + x_3^2 - 2x_1^2}{(x_1^2 + x_2^2 + x_3^2)^2} + \frac{x_1^2 + x_2^2 + x_3^2 - 2x_2^2}{(x_1^2 + x_2^2 + x_3^2)^2} + \frac{x_1^2 + x_2^2 + x_3^2 - 2x_3^2}{(x_1^2 + x_2^2 + x_3^2)}$

$= \frac{-\cancel{x_1^2} + \cancel{x_2^2} + x_3^2 + \cancel{x_1^2} - \cancel{x_2^2} + \cancel{x_3^2} + x_1^2 + x_2^2 - \cancel{x_3^2}}{(x_1^2 + x_2^2 + x_3^2)^2}$

$= \frac{1}{x_1^2 + x_2^2 + x_3^2} = \frac{1}{\|\vec{x}\|^2}$

$$\bullet\ \overrightarrow{\mathrm{rot}}(g) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1} \\ \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2x_2x_3}{\|\vec{x}\|^4} + \frac{2x_3x_2}{\|\vec{x}\|^4} \\ \frac{-2x_1x_3}{\|\vec{x}\|^4} + \frac{2x_1x_3}{\|\vec{x}\|^4} \\ \frac{-2x_1x_2}{\|\vec{x}\|^4} + \frac{2x_1x_2}{\|\vec{x}\|^4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}.$$

Exercice 2 :

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f(x,y) \end{array}$$ de classe C^1.

[a] $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ alors $f = \text{cte}$??

On a $\frac{\partial f}{\partial x}(x,y) = 0$ alors $f(x,y) = g(y)$ or $\frac{\partial f}{\partial y} = 0$ donc $\frac{\partial g}{\partial y} = 0$ càd $g(y) = \alpha$ où $\alpha \in \mathbb{R}$. D'où, $\boxed{f(x,y) = \alpha}$.

[b] $\frac{\partial f}{\partial x} = a$; $\frac{\partial f}{\partial y} = b$ où $(a,b) \in \mathbb{R}^2$ alors $f(x,y) = \alpha + \beta x + \gamma y$??

On a $\frac{\partial f}{\partial x} = a$ donc $f(x,y) = \int a\,dx = ax + g(y)$

or $\frac{\partial f}{\partial y} = b$ càd $g'(y) = b$ donc $g(y) = by + c$ où $c \in \mathbb{R}$.

D'où, $\boxed{f(x,y) = ax + by + c}$.

Exercice 3 :

$\vec{F}(x,y,z) = (F_1 ; F_2 ; F_3) = (x^2+yz ; 3y^2+3xz ; 5z^2-2x-3y+xyz)$

① $\overrightarrow{grad}\,\vec{F}$ pas de sens car $\vec{F}$ est un vecteur.

② $div\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 6y + 10z + xy$

③ $\overrightarrow{rot}\,\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix} = \begin{pmatrix} -3+xz-3x \\ y+2-yz \\ 2z \end{pmatrix}$

④ $\overrightarrow{grad}\,div\,\vec{F} = \begin{pmatrix} 2+y \\ 6+x \\ 10 \end{pmatrix}$

⑤ $div\,\overrightarrow{grad}\,\vec{F}$ pas de sens.

⑥ $\overrightarrow{rot}\,\overrightarrow{grad}\,\vec{F}$ pas de sens.

⑦ $\overrightarrow{rot}\,div\,\vec{F}$ pas de sens.

⑧ $div\,div\,\vec{F}$ pas de sens

⑨ $div\,\overrightarrow{rot}\,\vec{F} = z - 3 + 1 - z + 2 = 0$

⑩ $\overrightarrow{rot}\,\overrightarrow{rot}\,\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} -3+xz-3x \\ y+2-yz \\ 2z \end{pmatrix} = \begin{pmatrix} 0+y \\ x-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$

⑪ $\overrightarrow{grad}\,div\,\overrightarrow{rot}\,\vec{F} = \vec{0}$.

⑫ $div\,\overrightarrow{grad}\,div\,\vec{F} = 0$.

⑬ $\overrightarrow{rot}\,\overrightarrow{grad}\,div\,\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} 2+y \\ 6+x \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1-1 \end{pmatrix} = \vec{0}$.

Exercice 4 :

$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ de classe C^1 ; $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ de classe C^1

$(x,y,z) \longmapsto (F_1, F_2, F_3)$; $(x,y,z) \longmapsto (G_1, G_2, G_3)$

a) $(\vec{F} \cdot \vec{\nabla})\vec{G} = F_1 \frac{\partial \vec{G}}{\partial x} + F_2 \frac{\partial \vec{G}}{\partial y} + F_3 \frac{\partial \vec{G}}{\partial z}$.

Ecriture facile pour le développement des vecteurs

b) $$(J_G)F = \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} & \frac{\partial G_1}{\partial z} \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & \frac{\partial G_2}{\partial z} \\ \frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} & \frac{\partial G_3}{\partial z} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= \begin{pmatrix} F_1 \frac{\partial G_1}{\partial x} + F_2 \frac{\partial G_1}{\partial y} + F_3 \frac{\partial G_1}{\partial z} \\ F_1 \frac{\partial G_2}{\partial x} + F_2 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_2}{\partial z} \\ F_1 \frac{\partial G_3}{\partial x} + F_2 \frac{\partial G_3}{\partial y} + F_3 \frac{\partial G_3}{\partial z} \end{pmatrix}$$

$$= F_1 \begin{pmatrix} \frac{\partial G_1}{\partial x} \\ \frac{\partial G_2}{\partial x} \\ \frac{\partial G_3}{\partial x} \end{pmatrix} + F_2 \begin{pmatrix} \frac{\partial G_1}{\partial y} \\ \frac{\partial G_2}{\partial y} \\ \frac{\partial G_3}{\partial y} \end{pmatrix} + F_3 \begin{pmatrix} \frac{\partial G_1}{\partial z} \\ \frac{\partial G_2}{\partial z} \\ \frac{\partial G_3}{\partial z} \end{pmatrix}$$

$$= F_1 \frac{\partial \vec{G}}{\partial x} + F_2 \frac{\partial \vec{G}}{\partial y} + F_3 \frac{\partial \vec{G}}{\partial z}.$$

$$= (\vec{F} \cdot \vec{\nabla})\vec{G}.$$

c) Coordonnées de $(\vec{F} \cdot \vec{\nabla})\vec{G}$ d'après la question (b), on obtient :

1ère composante $= F_1 \frac{\partial G_1}{\partial x} + F_2 \frac{\partial G_1}{\partial y} + F_3 \frac{\partial G_1}{\partial z}$

2ème composante $= F_1 \frac{\partial G_2}{\partial x} + F_2 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_2}{\partial z}$

3ème composante $= F_1 \frac{\partial G_3}{\partial x} + F_2 \frac{\partial G_3}{\partial y} + F_3 \frac{\partial G_3}{\partial z}$

d-> $\overrightarrow{grad}\,(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}\wedge\overrightarrow{rot}\,\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{G}\wedge\overrightarrow{rot}\,\vec{F}$

* $\vec{F}\cdot\vec{G} = (F_1, F_2, F_3)\cdot(G_1, G_2, G_3) = F_1G_1 + F_2G_2 + F_3G_3.$

alors $\overrightarrow{grad}\,(\vec{F}\cdot\vec{G}) = \begin{pmatrix} G_1\frac{\partial F_1}{\partial x} + F_1\frac{\partial G_1}{\partial x} + G_2\frac{\partial F_2}{\partial x} + F_2\frac{\partial G_2}{\partial x} + G_3\frac{\partial F_3}{\partial x} + F_3\frac{\partial G_3}{\partial x} \\ G_1\frac{\partial F_1}{\partial y} + F_1\frac{\partial G_1}{\partial y} + G_2\frac{\partial F_2}{\partial y} + F_2\frac{\partial G_2}{\partial y} + G_3\frac{\partial F_3}{\partial y} + F_3\frac{\partial G_3}{\partial y} \\ G_1\frac{\partial F_1}{\partial z} + F_1\frac{\partial G_1}{\partial z} + G_2\frac{\partial F_2}{\partial z} + F_2\frac{\partial G_2}{\partial z} + G_3\frac{\partial F_3}{\partial z} + F_3\frac{\partial G_3}{\partial z} \end{pmatrix}$

D'autre part;

$(\vec{F}\cdot\vec{\nabla})\vec{G} = F_1\frac{\partial\vec{G}}{\partial x} + F_2\frac{\partial\vec{G}}{\partial y} + F_3\frac{\partial\vec{G}}{\partial z}.$

$(\vec{G}\cdot\vec{\nabla})\vec{F} = G_1\frac{\partial\vec{F}}{\partial x} + G_2\frac{\partial\vec{F}}{\partial y} + G_3\frac{\partial\vec{F}}{\partial z}.$

$$\vec{F}\wedge\overrightarrow{rot}(\vec{G}) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}\wedge\begin{pmatrix} \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} \\ \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} \\ \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \end{pmatrix} = \begin{pmatrix} F_2\frac{\partial G_2}{\partial x} - F_2\frac{\partial G_1}{\partial y} - F_3\frac{\partial G_1}{\partial z} + F_3\frac{\partial G_3}{\partial x} \\ F_3\frac{\partial G_3}{\partial y} - F_3\frac{\partial G_2}{\partial z} - F_1\frac{\partial G_2}{\partial x} + F_1\frac{\partial G_1}{\partial y} \\ F_1\frac{\partial G_1}{\partial z} - F_1\frac{\partial G_3}{\partial x} - F_2\frac{\partial G_3}{\partial y} + F_2\frac{\partial G_2}{\partial z} \end{pmatrix}$$

$$\vec{G}\wedge\overrightarrow{rot}\,\vec{F} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}\wedge\begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix} = \begin{pmatrix} G_2\frac{\partial F_2}{\partial x} - G_2\frac{\partial F_1}{\partial y} - G_3\frac{\partial F_1}{\partial z} + G_3\frac{\partial F_3}{\partial x} \\ G_3\frac{\partial F_3}{\partial y} - G_3\frac{\partial F_2}{\partial z} - G_1\frac{\partial F_2}{\partial x} + G_1\frac{\partial F_1}{\partial y} \\ G_1\frac{\partial F_1}{\partial z} - G_1\frac{\partial F_3}{\partial x} - G_2\frac{\partial F_3}{\partial y} + G_2\frac{\partial F_2}{\partial z} \end{pmatrix}$$

Conclusion :

$$(\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{F}\wedge\overrightarrow{rot}(\vec{G}) + \vec{G}\wedge\overrightarrow{rot}(\vec{F}) = \overrightarrow{grad}\,(\vec{F}\cdot\vec{G}).$$

Exercice 5 :

$\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ de classe C^2 ; $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ de classe C^2

$t \longmapsto \varphi(t)$; $u \longmapsto \psi(u)$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y) = \varphi(\underbrace{x+y}_{t}) + \psi(\underbrace{x-y}_{u})$

$\begin{cases} t = x+y \\ u = x-y \end{cases}$

$\frac{dt}{dx} = 1$; $\frac{dt}{dy} = -1$

$\frac{du}{dx} = 1$; $\frac{du}{dy} = -1$

$\frac{\partial^2 f}{\partial x^2} \overset{??}{=} \frac{\partial^2 f}{\partial y^2}$

On a $\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial t} \times \frac{\partial t}{\partial x} + \frac{\partial \psi}{\partial u} \times \frac{\partial u}{\partial x} = \varphi'(x+y) + \psi'(x-y)$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \varphi'}{\partial t} \times \frac{\partial t}{\partial x} + \frac{\partial \psi'}{\partial u} \times \frac{\partial u}{\partial x} = \varphi''(x+y) + \psi''(x-y)$

et $\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial t} \times \frac{\partial t}{\partial y} + \frac{\partial \psi}{\partial u} \times \frac{\partial u}{\partial y} = \varphi'(x+y) - \psi'(x-y)$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial \varphi'}{\partial t} \times \frac{\partial t}{\partial y} - \frac{\partial \psi'}{\partial u} \times \frac{\partial u}{\partial y} = \varphi''(x+y) + \psi''(x-y)$

D'où, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}$

Exercice 6 :

$\omega = P\,dx + Q\,dy + R\,dz$

$= \left(\frac{1}{y} - \frac{z}{x^2} + 1\right) dx + \left(\frac{1}{z} - \frac{x}{y^2} + 1\right) dy + \left(\frac{1}{x} - \frac{y}{z^2} + 1\right) dz$

ω est fermée ssi $\overrightarrow{rot}(P,Q,R) = \vec{0}$

$\overrightarrow{rot}(P,Q,R) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \frac{-1}{z^2} + \frac{1}{z^2} \\ \frac{-1}{x^2} + \frac{1}{x^2} \\ \frac{-1}{y^2} + \frac{1}{y^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}.$

D'où, ω est fermée sur $(\mathbb{R}_+^*)^3$.

ω est exacte s'il existe f t.q. $df = \omega$.

$$df = \omega \Leftrightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = P dx + Q dy + R dz$$

càd $\frac{\partial f}{\partial x} = P$; $\frac{\partial f}{\partial y} = Q$ et $\frac{\partial f}{\partial z} = R$.

- $\frac{\partial f}{\partial x} = P \Rightarrow f(x,y,z) = \int P dx = \int \left(\frac{1}{y} - \frac{z}{x^2} + 1\right) dx$

donc $f(x,y,z) = \frac{x}{y} + \frac{z}{x} + x + g(y,z)$

- $\frac{\partial f}{\partial y} = Q \Leftrightarrow \frac{-x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2} + 1 \Leftrightarrow \frac{\partial g}{\partial y} = \frac{1}{z} + 1$

donc $g(y,z) = \int \left(\frac{1}{z} + 1\right) dy = \frac{y}{z} + y + h(z)$

Alors $f(x,y,z) = \frac{x}{y} + \frac{z}{x} + x + \frac{y}{z} + y + h(z)$.

- $\frac{\partial f}{\partial z} = R \Leftrightarrow \frac{1}{x} - \frac{y}{z^2} + h'(z) = \frac{1}{x} - \frac{y}{z^2} + 1$ alors $h'(z) = 1$

D'où, $h(z) = z + \alpha$ où $\alpha \in \mathbb{R}$.

On en déduit que $f(x,y,z) = \frac{x}{y} + \frac{z}{x} + x + \frac{y}{z} + y + z + \alpha$

et donc f est exacte sur $(\mathbb{R}_+^*)^3$.