Differentiability

Functions of single variable

Calculus - Lecture 25

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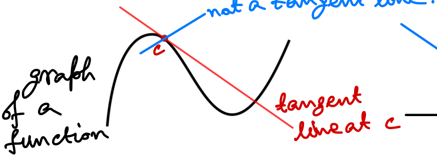
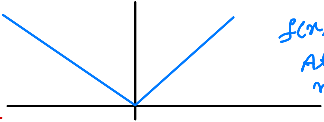
1. The derivative of a function at a point describes the rate of change of the function near that point.
2. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that a unique tangent line exists. A tangent line is a line that 'just touches' a particular point.





$f(x) = |x|$
At $x = 0$, f has many tangent lines.

3. The slope of a linear equation, written in the form $y = mx + b$, is m. This can be calculated by picking any two points, and dividing the change in y by the change in x, i.e., the slope $m = \dfrac{\text{change in } y}{\text{change in } x}$.

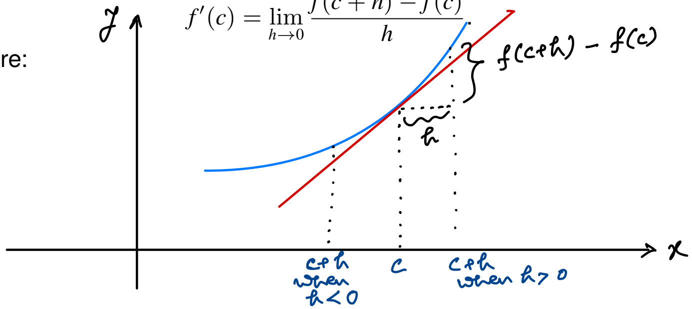
(x_2, y_2)

(x_1, y_1)

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

1. Consider two points $(c, f(c))$ and $(c+h, f(c+h))$ on the graph, where h is a small real number (positive or negative).
2. As before, the slope of the line passing through these two points can be calculated with the formula
$$\text{slope} = \frac{f(c+h)-f(c)}{(c+h)-c} = \frac{f(c+h)-f(c)}{h}.$$
3. As h approaches to 0, the slope of the line (passing through $(c, f(c))$ and $(c+h, f(c+h))$) gets closer and closer to the slope of the tangent line to the function at c if it exists. This is formally written and denoted by
$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}.$$

Picture:



A rigorous definition of continuity

Definition

Let $c \in (a,b)$. A function $f:(a,b)\to\mathbb{R}$ is called **differentiable at** c if

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad \text{exists.}$$

In this case, the limit is called the **derivative of** f **at** c. It is denoted by $f'(c)$ or $\frac{df}{dx}(c)$ or $\frac{dy}{dx}(c)$, where $y=f(x)$.

Remark

If derivative of f exists at c, then $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \underbrace{\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}$.

another interpretation of derivative of f at c.

Remark

The existence of $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ is **equivalent** to that

1. both $\lim_{h\to 0-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0+}\frac{f(c+h)-f(c)}{h}$ exist, and
2. $\lim_{h\to 0-}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0+}\frac{f(c+h)-f(c)}{h}$.

Examples: Differentiability of a function

Example (1)

Consider a constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = k$ for all $x \in \mathbb{R}$, where $k \in \mathbb{R}$ is some constant. Then $f'(c) = 0$ for every $c \in \mathbb{R}$.

Proof.

Let $c \in \mathbb{R}$. Then $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \dfrac{k-k}{h} = \lim_{h \to 0} 0 = 0.$ □

Example (2)

Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^m$ for all $x \in \mathbb{R}$, where $m \geqslant 1$ is some positive integer. Then $f'(c) = mc^{m-1}$ for every $c \in \mathbb{R}$.

Proof.

Let $c \in \mathbb{R}$. Then $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \dfrac{(c+h)^m - c^m}{h}$

$$= \lim_{h \to 0} \frac{\left(c^m + mc^{m-1}h + \binom{m}{2}c^{m-2}h^2 + \cdots + h^m\right) - c^m}{h} = mc^{m-1}.$$

□

Example (3)

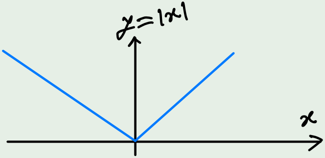
Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ for all $x \in \mathbb{R}$, i.e.,

$$f(x) = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

1. If $c > 0$, then $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{(c+h) - (c)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$
2. If $c < 0$, $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{-(c+h) + (c)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$
3. If $c = 0$, then
   1. $\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{h}{h} = 1$, while
   2. $\lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{-h}{h} = -1.$

   Picture:

$f'(c) = 1$ if $c > 0$
$f'(c) = -1$ if $c < 0$



Hence if $c \neq 0$, then f is differentiable at c, i.e., $f'(c)$ exists, while f is not differentiable at 0, i.e., $f'(0)$ does not exist.

Differentiable functions are continuous

Theorem

Let $c \in (a, b)$. If $f : (a, b) \to \mathbb{R}$ is differentiable at c, then f is continuous at c.

Proof.

Since $f : (a, b) \to \mathbb{R}$ is differentiable at c, the limit $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{(x - c)}$ exists.

Therefore

$$\lim_{x \to c}[f(x) - f(c)] = \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{(x - c)} (x - c) \right\}$$
$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c) = f'(c) \cdot 0 = 0.$$

Thus $\lim\limits_{x \to c} f(x) = f(c)$, hence the function f is continuous at c. $\square$

Remark

The converse to the above theorem is not necessarily true, e.g., $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ for all $x \in \mathbb{R}$ is continuous, but it is not differentiable at 0.

$f(x) = |x|$ is continuous at every point of $\mathbb{R}$.

Theorem

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions at a point $c \in (a, b)$. Then

1. $f + g$ is differentiable at c, and $(f + g)'(c) = f'(c) + g'(c)$.
2. $f - g$ is differentiable at c, and $(f - g)'(c) = f'(c) - g'(c)$.
3. For any $r \in \mathbb{R}$, the function (rf) is differentiable at c, and $(rf)'(c) = rf'(c)$.
4. fg is differentiable at c, and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
5. If $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at c, and
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

Proof of (4). Since $f, g : (a, b) \to \mathbb{R}$ are differentiable functions at c, the limits

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x\to c} \frac{g(x) - g(c)}{x - c} \quad \text{exist.}$$

Note that $\dfrac{(fg)(x) - (fg)(c)}{x - c} = \dfrac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} =$
$g(x) \cdot \dfrac{f(x) - f(c)}{x - c} + f(c) \cdot \dfrac{g(x) - g(c)}{x - c} \longrightarrow f'(c)g(c) + f(c)g'(c)$ as $x \to c$.

For every integer $m \geqslant 1$, we have proved that $\frac{d}{dx}(x^m) = mx^{m-1}$.
Using this result and the previous theorem, one obtains the following results.

Example (Polynomial function)

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial over $\mathbb{R}$. Then

$$f'(x) = n\, a_n\, x^{n-1} + (n-1)\, a_{n-1}\, x^{n-2} + \cdots + 2\, a_2 x + a_1 \quad \text{for all } x \in \mathbb{R}.$$

Example (Rational function)

Consider two polynomials over $\mathbb{R}$:

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \quad \text{and}$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0.$$

The function $f(x) = \frac{p(x)}{q(x)}$ is differentiable at every point $c \in \mathbb{R}$, where $q(c) \neq 0$.

$$f'(c) = \frac{p'(c)q(c) - p(c)q'(c)}{[q(c)]^2} \quad \text{for every point } c \in \mathbb{R} \text{ such that } q(c) \neq 0.$$

Theorem (Chain Rule)

Let I and J be two open intervals. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(I) \subseteq J$.
Let $c \in I$ and f is differentiable at c, and g is differentiable at $f(c)$. Then the composite function $g \circ f$ is differentiable at c, and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Example

We know that $\frac{d}{dx}(\sin(x)) = \cos(x)$ for all $x \in \mathbb{R}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ for all $x \neq 0$.
Using these, one obtains that $\frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$ for all $x \neq 0$.

Proof of the example.

Consider $f(x) = \frac{1}{x}$ for all $x \neq 0$, and $g(x) = \sin(x)$ for all $x \in \mathbb{R}$. Then the result follows from the above theorem. □

Let $c \in (a,b)$. Let $f : (a,b) \to \mathbb{R}$ be a function.

Definition

The function f is said to be **increasing at** c if there exists $\delta > 0$ such that

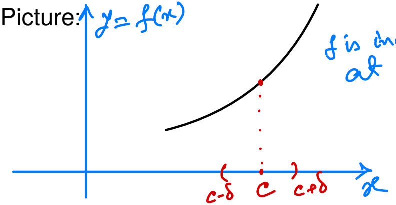
1. $f(x) < f(c)$ for all $x \in (a,b)$ satisfying $c - \delta < x < c$ and
2. $f(c) < f(x)$ for all $x \in (a,b)$ satisfying $c < x < c + \delta$.

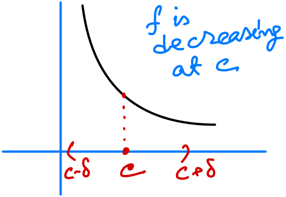
Definition

The function f is said to be **decreasing at** c if there exists $\delta > 0$ such that

1. $f(x) > f(c)$ for all $x \in (a,b)$ satisfying $c - \delta < x < c$ and
2. $f(c) > f(x)$ for all $x \in (a,b)$ satisfying $c < x < c + \delta$.

Picture:



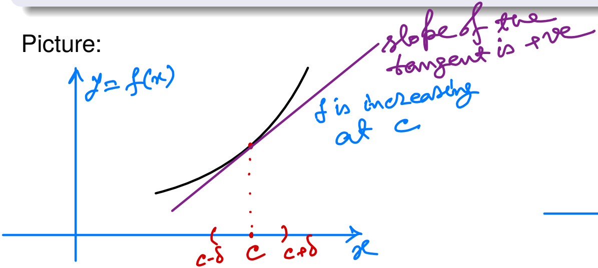


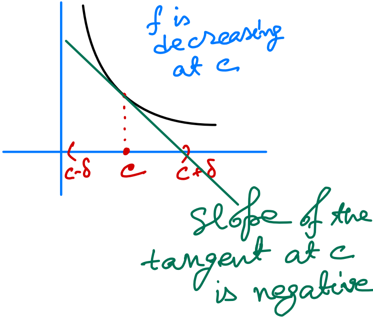
Theorem

Let $c \in (a,b)$. Let $f : (a,b) \to \mathbb{R}$ be differentiable at c.

1. If $f'(c) > 0$, then f is increasing at c.
2. If $f'(c) < 0$, then f is decreasing at c.
3. If $f'(c) = 0$, then we cannot conclude anything. E.g., $f(x) = x^3$ or $-x^3$.

Picture:





Local maxima and minima

Let $c \in (a,b)$. Let $f : (a,b) \to \mathbb{R}$ be a function.

Definition

f is said to have a **local maximum at** c if there exists $\delta > 0$ such that

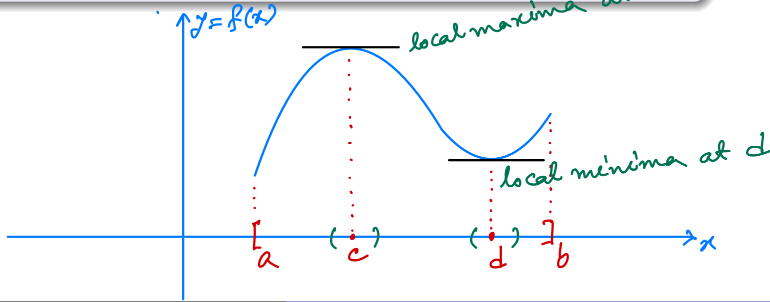
$$f(x) \leqslant f(c) \text{ for all } x \in (a,b) \text{ satisfying } c - \delta < x < c + \delta.$$

Definition

f is said to have a **local minimum at** d if there exists $\delta > 0$ such that

$$f(x) \geqslant f(d) \text{ for all } x \in (a,b) \text{ satisfying } d - \delta < x < d + \delta.$$

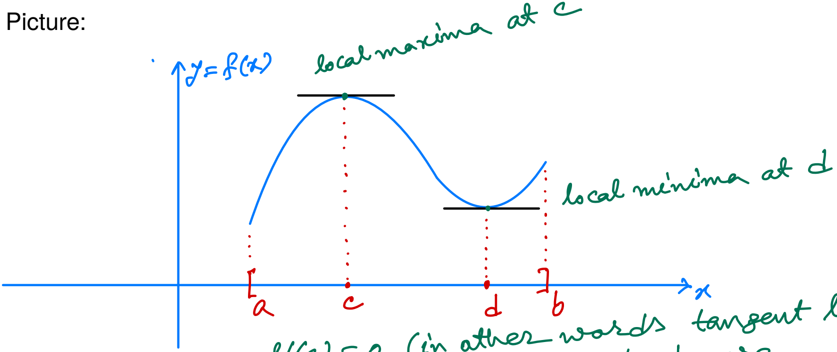
Picture:



Theorem

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Let f has a local maximum or a local minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Picture:



$f'(c) = 0$ (in other words tangent lines at c and d are parallel to the x-axis.

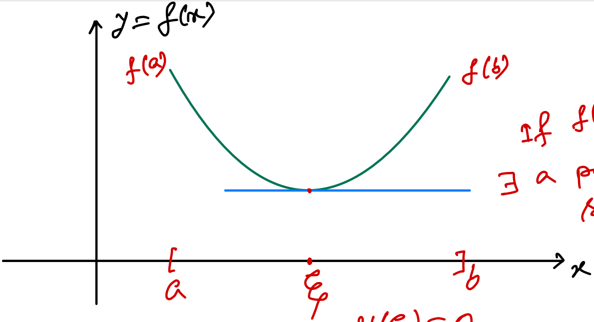
Theorem (Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

1. (i) $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$,
2. (ii) $f : (a, b) \to \mathbb{R}$ is differentiable at every point of (a, b) and
3. (iii) $f(a) = f(b)$.

Then there exists at least a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Picture:



$f'(\xi) = 0$

If $f(a) = f(b)$ $\exists$ a point $\xi \in (a, b)$ such that the tangent at ξ is parallel to the x-axis.

Corollary

If $P(x)$ is a polynomial of degree n with n real roots, then all the roots of $P'(x)$ are also real.

Using Rolle's Theorem, try to prove this result.

<u>proof</u>

$p(x)$ has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ (say).

All $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct.

$p(x)$ is continuous on $[\alpha_i, \alpha_{i+1}]$ for $1 \le i \le n-1$
differentiable at every point of (α_i, α_{i+1})

$p(\alpha_i) = 0 = p(\alpha_{i+1})$

Then by Rolle's Thm, $\exists$ $\beta_i \in (\alpha_i, \alpha_{i+1})$
s.t. $p'(\beta_i) = 0$
for $1 \le i \le n-1$.

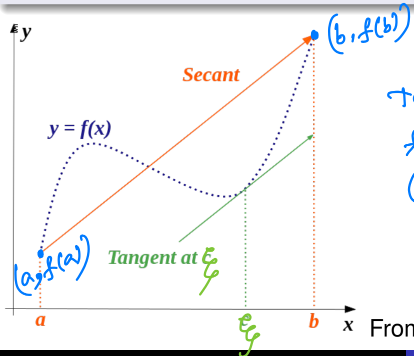
Theorem (Mean Value Theorem)

Let $f : [a,b] \to \mathbb{R}$ be a function such that

1. $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$,
2. $f : (a,b) \to \mathbb{R}$ is differentiable at every point of (a,b).

Then there exists at least a point $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



From Wikipedia

Tangent at ξ is parallel to the line joining $(a, f(a))$ and $(b, f(b))$.

Note : Rolle's Thm is a special case of MVT.

Theorem (Application 1)

Let $f : [a,b] \to \mathbb{R}$ be a function such that

1. $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$,
2. $f : (a,b) \to \mathbb{R}$ is differentiable at every point of (a,b), and
3. $f'(c) = 0$ for all $c \in (a,b)$.

Then f is a constant function on $[a,b]$, i.e., there exists a constant $k \in \mathbb{R}$ such that $f(x) = k$ for all $x \in [a,b]$.

Proof.

Consider any two points $s < t$ in $[a,b]$. By MVT, there exists $\xi \in (s,t)$ such that $\dfrac{f(t) - f(s)}{t - s} = f'(\xi)$. Since $f'(\xi) = 0$, one obtains that $f(t) = f(s)$. $\square$

Theorem (Application 2)

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function such that $f'(c) = 0$ for all $c \in (a, b)$. Then:
f is a constant function on (a, b), i.e., there exists a constant $k \in \mathbb{R}$ such that $f(x) = k$ for all $x \in (a, b)$.

Proof.

Consider any two points $s < t$ in (a, b). By MVT, there exists $\xi \in (s, t)$ such that $\dfrac{f(t) - f(s)}{t - s} = f'(\xi)$. Since $f'(\xi) = 0$, one obtains that $f(t) = f(s)$. □

Theorem (Application 3)

Let $g, h : (a, b) \to \mathbb{R}$ be two differentiable functions such that $g'(c) = h'(c)$ for all $c \in (a, b)$. Then:
there exists a constant $k \in \mathbb{R}$ such that $g(x) = h(x) + k$ for all $x \in (a, b)$.

Proof.

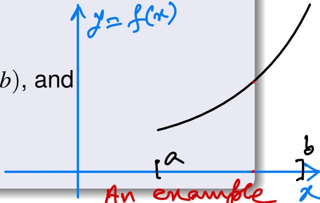
Consider the function $f = g - h$, and use Application 2. □

Theorem (Application 4)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

1. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$,
2. $f : (a, b) \to \mathbb{R}$ is differentiable at every point of (a, b), and
3. $f'(c) \geqslant 0$ for all $c \in (a, b)$.

Then f is a monotone increasing function on $[a, b]$.



An example of strictly increasing function

Theorem (Application 5)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

1. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$,
2. $f : (a, b) \to \mathbb{R}$ is differentiable at every point of (a, b), and
3. $f'(c) > 0$ for all $c \in (a, b)$.

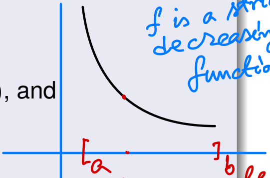
Then f is a strictly increasing function on $[a, b]$.

Theorem (Application 6)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

1. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$,
2. $f : (a, b) \to \mathbb{R}$ is differentiable at every point of (a, b), and
3. $f'(c) \leqslant 0$ for all $c \in (a, b)$.

Then f is a monotone decreasing function on $[a, b]$.



f is a strictly decreasing function

An example

Theorem (Application 7)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

1. $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$,
2. $f : (a, b) \to \mathbb{R}$ is differentiable at every point of (a, b), and
3. $f'(c) < 0$ for all $c \in (a, b)$.

Then f is a strictly decreasing function on $[a, b]$.

Thank you!