11.1 Multiple Integrals

- We can evaluate multiple integrals of continuous functions iteratively. Fubini’s theorem states that the order of integration does not matter:

$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

- So the method is to integrate along one dimension first then integrate the result along the other.

- An analogous method exists for higher dimensions.

- The technique for square domains is demonstrated in the next example.

EXAMPLE 1 The following function gives the temperature of a rectangular plate that is 8 m long and 6 m wide. What is the average temperature?

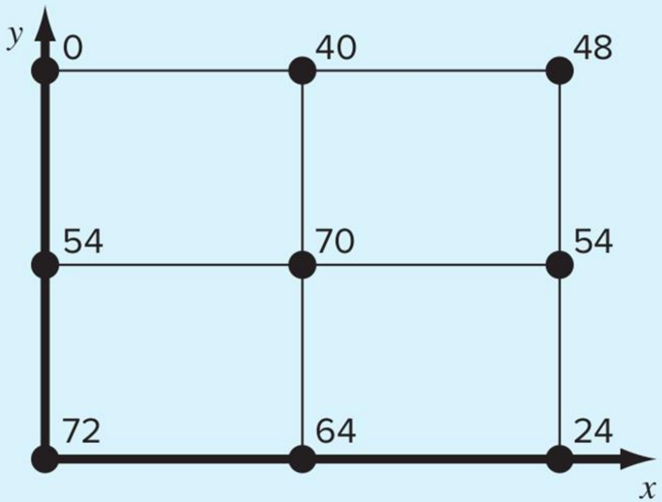
$$T(x, y) = 2xy + 2x - x^2 - 2y^2 + 72$$

Average value of function of 2 variables

$$\bar{f} = \frac{\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy}{(d - c)(b - a)}$$

Use composite trapezium rule

$$I = \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}$$



$$(8 - 0)\frac{0 + 2(40) + 48}{4} \rightarrow \mathbf{256}$$

$$(8 - 0)\frac{54 + 2(70) + 54}{4} \rightarrow \mathbf{496}$$

$$(8 - 0)\frac{72 + 2(64) + 24}{4} \rightarrow \mathbf{448}$$

$$(6 - 0)\frac{256 + 2(496) + 448}{4} = \mathbf{2544}$$

Average value

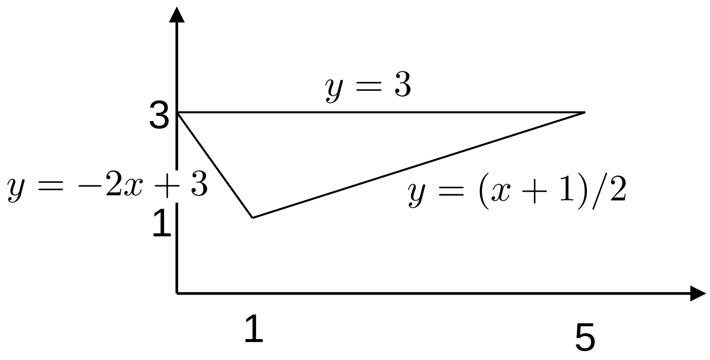
$$2544/(6 \times 8) = 53$$

- We could equally apply Simpson's rule in 2 dimensions in exactly the same way.

- In 3 dimensions we would partition the *z*-axis and have a diagram like the one above for each z_i. The result from each layer would then be used in the integral formula.

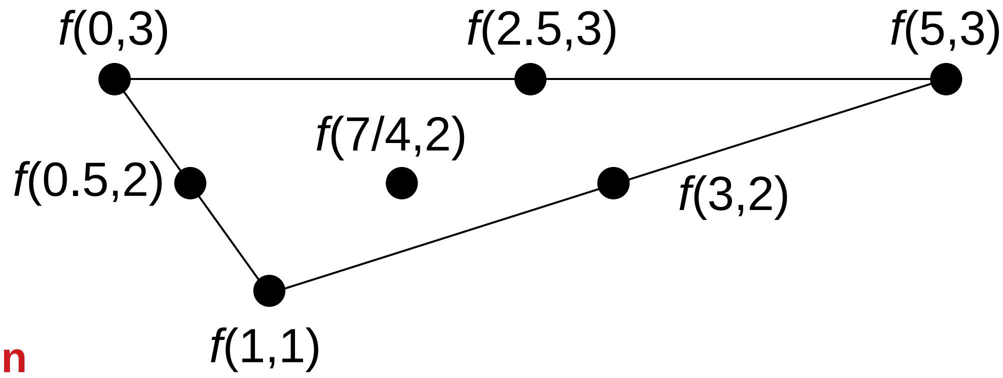
EXAMPLE 2 Calculate the following double integral over the non-rectangular region using the composite trapezium rule in 2 dimensions.

$$\int_1^3 \int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} \underbrace{(6x^2 - 40y)}_{f(x,y)} \, dx dy$$



From calculus the exact answer is –935/3 = –311.6667

- Partition the domain with n = 2 and apply the trapezium rule.



Integrate *x* direction

y = 3: $$\frac{5-0}{4}\left[f(0,3)+2f(2.5,3)+f(5,3)\right]=-365.6250$$

y = 2: $$\frac{3-1/2}{4}\left[f(0.5,2)+2f(7/4,2)+f(3,2)\right]=-142.32438$$

y = 1: $$0$$

Integrate *y* direction

$$\frac{3-1}{4}\left[-365.6250-2\times 143.3238\right]=-325.1562$$

- We can improve the approximation using a finer mesh.

EXAMPLE 3 Estimate the triple integral of $f(x,y,z) = xyz$ over the unit cube in the first octant with corner at the origin by using the trapezium rule in 3 dimensions with $n = 2$.

Matlab integral2 and integral3

EXAMPLE 4 Repeat **Example 1** using Matlab built-in functions.

```
>> T=@(x,y) 2*x.*y + 2*x - x.^2 - 2*y.^2 + 72;
>> format shortg
>> A=integral2(T,0,8,0,6)
A =

         2816

>> avg=A/((8-0)*(6-0))
avg =

      58.6667
```

Min *x* **Max *x*** **Min *y*** **Max *y***

EXAMPLE 5 Repeat **Example 2** using Matlab built-in functions and integrating along *dy* first then *dx*.

$$\int_0^1 \int_{3-2x}^{3} (6x^2 - 40y)\, dydx + \int_1^5 \int_{\frac{(x+1)}{2}}^{3} (6x^2 - 40y)\, dydx$$

```
>> f=@(x,y) 6*x.^2-40*y;
>> integral2(f,0,1,@(x)3-2*x,3)+integral2(f,1,5,@(x)x/2+1/2,3)
ans =

      -311.67
```

EXAMPLE 6 Repeat **Example 2** using Matlab built-in functions and integrating along *dx* first then *dy*.

```
>> f=@(y,x) 6*x.^2-40*y;
>> integral2(f,1,3,@(y)-y/2+3/2,@(y)2*y-1)
ans =

      -311.67
```

$$\int_1^3 \int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} (6x^2 - 40y)\, dxdy$$

Notes on non-rectangular domains

- The Matlab **integral2** function must have scalar inputs for the first 2 limits. Only the last 2 limits can be functions.

- Notice that **we had to redefine the function** we were integrating by **switching the position of *x* and *y*** so that we could integrate between functions in a different order.

- Trying to put the functions in the first 2 limit positions results in an error:

```
>> f=@(x,y) 6*x.^2-40*y;
>> integral2(f,@(y)-y/2+3/2,@(y)2*y-1,1,3)
Error using integral2 (line 71)
XMIN must be a floating point scalar.
```

- The same idea applies to triple integrals in non-rectangular domains.

- The first 2 limits must be scalars, the 3rd and 4th limits can be functions of 1 variable, the 5th and 6th limits can be functions of 2 variables.

EXAMPLE 7 Calculate the following triple integral over the non-rectangular domain.

$$\int_0^3 \int_0^{2-\frac{2x}{3}} \int_0^{6-2x-3y} 2x\, dz\, dy\, dx$$

```
>> f=@(x,y,z) 2*x;
>> integral3(f,0,3,0,@(x)2-2*x/3,0,@(x,y)6-2*x-3*y)
ans =

          9
```

Recursive Functions

- Recursive functions are functions that call themselves with different input argument values each time along with a stopping criterion.

- The classic example is to calculate a factorial:

```
function y = myfac(n)
if n > 0
    y = n*myfac(n-1);
else
    y = 1;
end
```

- It's possible to utilise recursive functions for the method in the next section and also for adaptive quadrature.

11.2 Richardson Extrapolation

- We now develop more efficient integration algorithms for single variables.

- Remember that we can use any method we like in multiple dimensions using the procedure outlined in the previous section, just with different integration formulas.

- This method begins by expressing an integral as the trapezium method approximation + some error for a given step size, *h*.

$$I = I\,(h) + E(h)$$

- Equating 2 different step sizes gives:

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

- The error for the composite trapezium rule is:

$$E \cong -\frac{b-a}{12} h^2 \bar{f}''$$

- Since $\bar{f}''$ is constant we write the ratio of the errors for different step sizes as:

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2} \longrightarrow E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

- Substituting into the last equation on the previous slide yields:

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2)$$

$$\longrightarrow E(h_2) = \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}$$

- Substituting that error into

$$I = I(h_2) + E(h_2)$$

gives:

$$I = I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}[I(h_2) - I(h_1)]$$

- The error for this formula is now $O(h^4)$ compared with the trapezium rule which had error $O(h^2)$. This can be proved using Taylor series expansions.

- If the 2nd interval is just half of the first then we have:

$$h_2 = h_1/2 \longrightarrow \boxed{I = \frac{4}{3}I(h_2) - \frac{1}{3}I(h_1)}$$

EXAMPLE 8 Use Richardson extrapolation with a halved step size to integrate the following function between 0 and 0.8.

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Trapezium rule for different step sizes

Segments	h	Integral	ε_t
1	0.8	0.1728	89.5%
2	0.4	1.0688	34.9%
4	0.2	1.4848	9.5%

True value is 1.640533

Estimate for 1 and 2 segments

$$I = \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.367467$$

$\varepsilon_t = 16.6\%$

Estimate for 2 and 4 segments

$$I = \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) = 1.623467$$

$\varepsilon_t = 1.04\%$

- Continuing this method we can obtain an even more accurate formula.

- From the previous Richardson extrapolation we have a **less accurate approximation, I_L**, using segments 1 & 2, and a **more accurate approximation, I_m**, using segments 2 & 4.

- Assuming that the error for I_L is approximately ch^4, where c is a constant, we have that the error for I_m, is approximately $c(h/2)^4$, since we halve the step size.

- It can be shown that the next level of Richardson extrapolation results in the $O(h^6)$ formula:

$$\boxed{I = \frac{16}{15} I_m - \frac{1}{15} I_l}$$

EXAMPLE 9 Combine the results of **Example 8** to produce a more accurate estimate of the integral using the $O(h^6)$ Richardson extrapolation.

$$I = \frac{16}{15}(1.623467) - \frac{1}{15}(1.367467) = 1.640533$$

ε_t = 0.00002%

- Another iteration of Richardson extrapolation yields the $O(h^8)$ formula:

$$\boxed{I = \frac{64}{63} I_m - \frac{1}{63} I_l}$$

- This process can be generalised giving:

$$\boxed{I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}}$$

- The repeated use of the integral formula on the previous slide is known as **Romberg integration**.

- The notation $I_{j,k}$ represents the improved integral.

- The notation $I_{j+1,k-1}$ and $I_{j,k-1}$ represent the more and less accurate integrals for the $(k-1)^{th}$ iteration of Richardson extrapolation using j segments.

- When $k = 1$ it represents the original trapezium rule.

- When $k = 2$ and $j = 1$ we have the $O(h^4)$ Richardson extrapolation formula:

$$I_{1,2} = \frac{4I_{2,1} - I_{1,1}}{3}$$

Romberg Integration Strategy

	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
	0.172800	1.367467		
(a)	1.068800			
	0.172800	1.367467	1.640533	
	1.068800	1.623467		
(b)	1.484800			
	0.172800	1.367467	1.640533	1.640533
	1.068800	1.623467	1.640533	
	1.484800	1.639467		
(c)	1.600800			

- (a) has j = 1 and 2 corresponding with 1 and 2 segments.
- (b) includes j = 3 corresponding with 4 segments.
- (c) includes j = 4 corresponding with 8 segments.

- We can define the relative approximate error to be the percent difference between the new estimate and the old one:

$$|\varepsilon_a| = \left|\frac{I_{1,k} - I_{2,k-1}}{I_{1,k}}\right| \times 100\%$$

- This can be used as stopping criterion in a loop.

- We evaluate the approximate error at the final point of each iteration. So for example after the first iteration we would have:

$$|\varepsilon_a| = \left|\frac{1.367467 - 1.068800}{1.367467}\right| \times 100\% = 21.8\%$$

- After the second iteration we would have:

$$|\varepsilon_a| = \left|\frac{1.640533 - 1.623467}{1.640533}\right| \times 100\% = 1.04\%$$

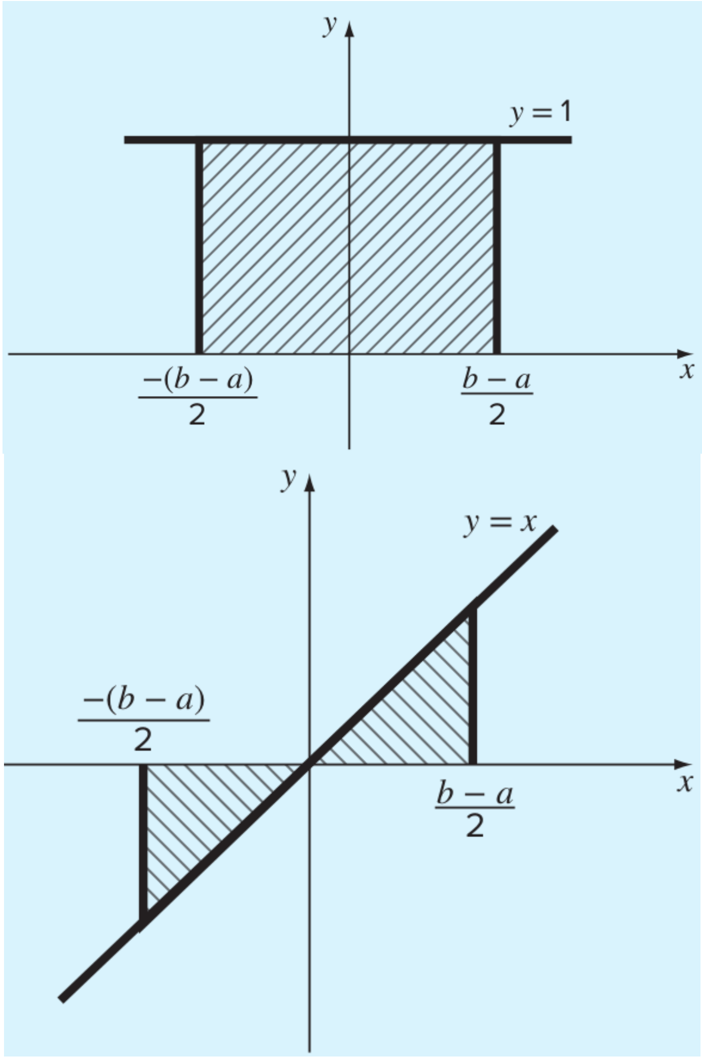
11.3 Gauss Quadrature

- This class of methods involves partitioning the domain into uneven spaces in order to take advantage of the curvature to cancel out some positive and negative errors.

- We will focus on the Gauss-Legendre formulas which we will derive using the method of undetermined coefficients.

- We develop the method of undetermined coefficients to derive the trapezium rule again. Once this method has been established we can use it to obtain the Gauss-Legendre formulas.

- The straight line functions on the right depict two functions that have 0 error when evaluated using the trapezium rule.



- We assume the form of the integral:

$$I \cong c_0 f(a) + c_1 f(b)$$

- The coefficients are to be determined.

- We have:

$$c_0 + c_1 = \int_{-(b-a)/2}^{(b-a)/2} 1\, dx$$

$$-c_0 \frac{b-a}{2} + c_1 \frac{b-a}{2} = \int_{-(b-a)/2}^{(b-a)/2} x\, dx$$

- Integrating both equations results in:

$$c_0 + c_1 = b - a \qquad\qquad -c_0\frac{b-a}{2} + c_1\frac{b-a}{2} = 0$$

- Solving simultaneously we obtain the coefficients:

$$c_0 = c_1 = \frac{b-a}{2}$$

- This gives us the trapezium rule:

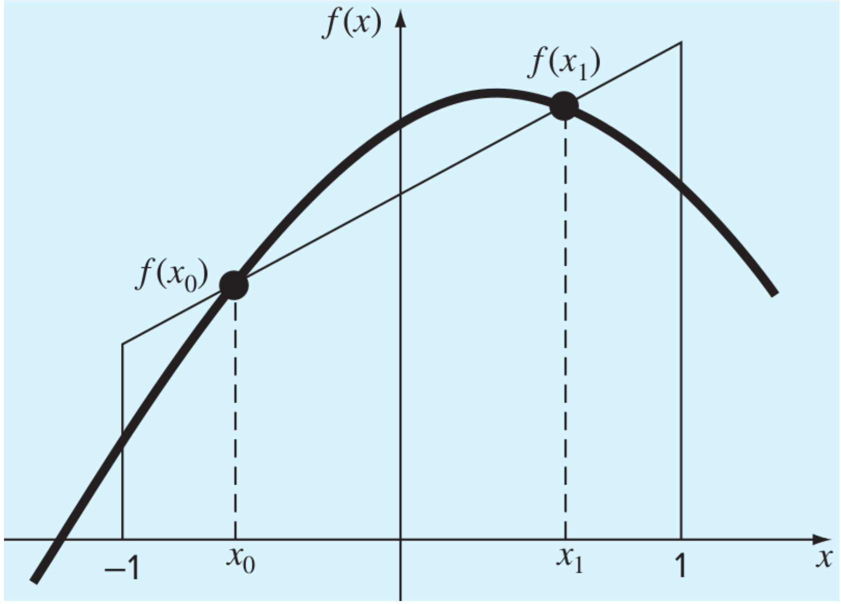
$$I = \frac{b-a}{2}f(a) + \frac{b-a}{2}f(b)$$

- Let us now use this technique to derive the Two-Point Gauss-Legendre formula.

Two-Point Gauss-Legendre

- Consider a general function evaluated at 2 points and assume the form of the integral:

$$I \cong c_0 f(x_0) + c_1 f(x_1)$$



- This time we have 4 unknowns, namely, c_0, c_1, x_0 and x_1.

- We assume our formula fits the integral of a constant function, linear function, parabolic function and cubic function.

- These 4 constraints determine both the weights of the function evaluations and the location of the domain partitions.

- The 4 integral equations can then be expressed as:

$$c_0 + c_1 = \int_{-1}^{1} 1\, dx = 2 \qquad c_0 x_0 + c_1 x_1 = \int_{-1}^{1} x\, dx = 0$$

$$c_0 x_0^2 + c_1 x_1^2 = \int_{-1}^{1} x^2\, dx = \frac{2}{3} \qquad c_0 x_0^3 + c_1 x_1^3 = \int_{-1}^{1} x^3\, dx = 0$$

- Integrating and solving for the 4 unknowns yields:

$$c_0 = c_1 = 1 \qquad x_0 = -\frac{1}{\sqrt{3}} \qquad x_1 = \frac{1}{\sqrt{3}}$$

- Therefore the integral formula is:

$$\boxed{I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

- If we wish to integrate between limits that are not –1 and 1 we can make a substitution:

$$x = \frac{(b+a) + (b-a)x_d}{2}$$

where x_d is the original variable we used to derive the formula.

- Notice when $x_d = -1$ we have $x = a$ and when $x_d = 1$ we have $x = b$.

- Substituting the above along with the following differential allows us to adapt the Two-Point Gauss-Legendre formula.

$$dx = \frac{b-a}{2}\, dx_d$$

EXAMPLE 9 Use the Two-Point Gauss-Legendre formula to evaluate the following integral between 0 and 0.8.

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

$$a = 0 \text{ and } b = 0.8 \longrightarrow x = 0.4 + 0.4x_d \qquad dx = 0.4dx_d$$

Transform integral to get new integrand *f*(*x*$_d$)

$$\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\, dx$$

$$= \int_{-1}^{1} \underbrace{[0.2 + 25(0.4 + 0.4x_d) - 200(0.4 + 0.4x_d)^2 + 675(0.4 + 0.4x_d)^3 - 900(0.4 + 0.4x_d)^4 + 400(0.4 + 0.4x_d)^5]0.4}_{f(x_d)}dx_d$$

Use formula

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.516741 + 1.305837 = 1.822578$$

Higher Point Gauss-Legendre Formulas

- Continuing the technique developed in the previous section to more points and higher order polynomial constraints it's possible to obtain formulas with more points.

- The general form is given by:

$$I \cong c_0 f(x_0) + c_1 f(x_1) + \cdots + c_{n-1} f(x_{n-1})$$

- A summary of the weights and domain partition points are given in the following table.

Points	Weighting Factors	Function Arguments	Truncation Error
1	$c_0 = 2$	$x_0 = 0.0$	$\cong f^{(2)}(\xi)$
2	$c_0 = 1$	$x_0 = -1/\sqrt{3}$	$\cong f^{(4)}(\xi)$
	$c_1 = 1$	$x_1 = 1/\sqrt{3}$	
3	$c_0 = 5/9$	$x_0 = -\sqrt{3/5}$	$\cong f^{(6)}(\xi)$
	$c_1 = 8/9$	$x_1 = 0.0$	
	$c_2 = 5/9$	$x_2 = \sqrt{3/5}$	
4	$c_0 = (18 - \sqrt{30})/36$	$x_0 = -\sqrt{525 + 70\sqrt{30}}/35$	$\cong f^{(8)}(\xi)$
	$c_1 = (18 + \sqrt{30})/36$	$x_1 = -\sqrt{525 - 70\sqrt{30}}/35$	
	$c_2 = (18 + \sqrt{30})/36$	$x_2 = \sqrt{525 - 70\sqrt{30}}/35$	
	$c_3 = (18 - \sqrt{30})/36$	$x_3 = \sqrt{525 + 70\sqrt{30}}/35$	
5	$c_0 = (322 - 13\sqrt{70})/900$	$x_0 = -\sqrt{245 + 14\sqrt{70}}/21$	$\cong f^{(10)}(\xi)$
	$c_1 = (322 + 13\sqrt{70})/900$	$x_1 = -\sqrt{245 - 14\sqrt{70}}/21$	
	$c_2 = 128/225$	$x_2 = 0.0$	
	$c_3 = (322 + 13\sqrt{70})/900$	$x_3 = \sqrt{245 - 14\sqrt{70}}/21$	
	$c_4 = (322 - 13\sqrt{70})/900$	$x_4 = \sqrt{245 + 14\sqrt{70}}/21$	
6	$c_0 = 0.171324492379170$	$x_0 = -0.932469514203152$	$\cong f^{(12)}(\xi)$
	$c_1 = 0.360761573048139$	$x_1 = -0.661209386466265$	
	$c_2 = 0.467913934572691$	$x_2 = -0.238619186083197$	
	$c_3 = 0.467913934572691$	$x_3 = 0.238619186083197$	
	$c_4 = 0.360761573048131$	$x_4 = 0.661209386466265$	
	$c_5 = 0.171324492379170$	$x_5 = 0.932469514203152$	

EXAMPLE 10 Repeat **Example 9** using the Three-Point Gauss-Legendre formula.

$$c_0 = 5/9 \qquad x_0 = -\sqrt{3/5}$$
$$c_1 = 8/9 \qquad x_1 = 0.0$$
$$c_2 = 5/9 \qquad x_2 = \sqrt{3/5}$$

$$I = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= 0.2813013 + 0.8732444 + 0.4859876 = 1.640533$$

11.4 Adaptive Quadrature

- Some functions have fast rates of change over small intervals requiring a very small *h* in that region. However it may be inefficient to make *h* that small for the whole region of integration.
- We deal with this by applying Simpson’s 1/3 rule to subintervals of an initial width.
- We then refine the domain by taking another smaller width and repeating the calculation.
- The approximate error is calculated based on these calculations and if deemed unacceptable the domain is further refined for that particular subinterval.
- Since the method relies on Simpson’s 1/3 rule it requires that we do a more detailed error analysis first.

Simpson's 1/3 Rule Error Analysis

- We are trying to approximate a definite integral using Simpson's rule:

$$I = \int_a^b f(x)\,dx \approx S(a,b)$$

where,

***h* = (*b*–*a*)/2**

$$S(a,b) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

- If we Taylor expand the integrand *f*(*x*) about the point *c* = (*a*+*b*)/2 then integrate we obtain:

$$\int_a^b f(x)\,dx = 2f(c)h + \frac{2}{3!}f''(c)h^3 + \frac{2}{5!}f^{(4)}(c)h^5 + \cdots$$

- Taylor expanding *S*(*a*,*b*) about the point *c* we obtain:

$$S(a,b) = 2f(c)h + \tfrac{1}{3}f''(c)h^3 + \tfrac{1}{36}f^{(4)}(c)h^5 + \cdots$$

- Subtracting the above Taylor expansion from the exact integral Taylor expansion on the previous page we obtain the Simpson's rule error as:

$$-\frac{1}{90}h^5 f^{(4)}(c) + \cdots \approx -\frac{1}{90}h^5 f^{(4)}(c)$$

- Now we halve the interval, *H* = *h*/2, and use Simpson's rule for *n* = 4:

$$S(a,b) = \frac{H}{3}\left[f(a) + 4f(a+H) + 2f(c) + 4f(b-H) + f(b)\right]$$

- The Taylor expansion for the exact integral becomes:

$$\int_a^b f(x)\,dx = 4Hf(c) + \frac{8}{3}H^3 f''(c) + \frac{8}{15}H^5 f^{(4)}(c) + \cdots$$

- The Taylor expansion for *S*(*a*,*b*) with step size *H* is:

$$S(a,b) = 4Hf(c) + \frac{8}{3}H^3 f''(c) + \frac{5}{9}H^5 f^{(4)}(c) + \cdots$$

- So subtracting the two expansions we obtain the Simpson's rule error for the halved step size *H* as:

$$-\frac{1}{45}H^5 f^{(4)}(c) + \cdots \approx -\frac{1}{45}H^5 f^{(4)}(c) = -\frac{1}{1440}h^5 f^{(4)}(c)$$

- It will be helpful to write the above error as:

$$-\frac{1}{1440}h^5 f^{(4)}(c) = -\frac{1}{16(90)}h^5 f^{(4)}(c)$$

- We now proceed with using these errors in the method of adaptive quadrature.

Adaptive Quadrature Methodology

- Using Simpson's rule twice with regular and halved step size, *h* and *h*/2, gives:

$$S(a,b) - \frac{1}{90}h^5 f^{(4)}(c) \approx S(a,c) + S(c,b) - \frac{1}{16(90)}h^5 f^{(4)}(c)$$

- Solving for the error:

$$\frac{1}{90}h^5 f^{(4)}(c) \approx \frac{16}{15}\left[S(a,b) - S(a,c) - S(c,b)\right]$$

- So the approximate error between the true integral and Simpson's approximation with a halved step size is:

$$\left|\int_a^b f(x)\,dx - S(a,c) - S(c,b)\right| \approx \underbrace{\frac{1}{15}\left|\left[S(a,b) - S(a,c) - S(c,b)\right]\right|}_{|E_a|}$$

- We choose a tolerance, ε, such that if $|E_a| < \varepsilon$, then we are happy that the accuracy is enough but otherwise we halve the step size again over that subinterval.

- In other words we require:

$$|[S(a,b) - S(a,c) - S(c,b)]| < 15\varepsilon$$

- If

$$|[S(a,b) - S(a,c) - S(c,b)]| > 15\varepsilon$$

then we halve the step size.

EXAMPLE 11 Use adaptive quadrature to approximate the following integral with an error tolerance of $\varepsilon = 0.01$ (true value is 2.648783...).

$$\int_0^{1.5} \tan x \, dx$$

$15\varepsilon = 0.15$

$$\textbf{Step 1: } S(0, 1.5) = \tfrac{0.75}{3}\left[f(0) + 4f(0.75) + f(1.5)\right] \approx 4.456951447$$

$$S(0, 0.75) = \tfrac{0.75}{6}\left[f(0) + 4f(0.375) + f(0.75)\right] \approx 0.313262845$$

$$S(0.75, 1.5) = \tfrac{0.75}{6}\left[f(0.75) + 4f(1.125) + f(1.5)\right] \approx 2.925412689.$$

$$|S(0, 1.5) - S(0, 0.75) - S(0.75, 1.5)| \approx 1.218276$$

Not < 0.15

Halve step size

$$\textbf{Step 2a: } \text{(Left Half) } S(0, 0.375) = \tfrac{0.75}{12}\left[f(0) + 4f(0.1875) + f(0.375)\right] \approx 0.0720338137$$

$$S(0.375, 0.75) = \tfrac{0.75}{12}\left[f(0.375) + 4f(0.5625) + f(0.75)\right] \approx .2404358582$$

$$|S(0, 0.75) - S(0, 0.375) - S(0.375, 0.75)| \approx 0.000793173$$

< 0.15

No need to halve again (stop)

Step 2b: (Right Half) $S(0.75, 1.125) =$

$$\frac{0.75}{12}\left[f(0.75) + 4f(0.9375) + f(1.125)\right] \approx 0.5295284776$$

$$S(1.125, 1.50) = \frac{0.75}{12}\left[f(1.125) + 4f(1.3125) + f(1.50)\right] \approx 1.958383993$$

$$|S(0.75, 1.50) - S(0.75, 1.125) - S(1.125, 1.50)| \approx 0.4375\ldots$$

Not < 0.15

Halve step size

Step 3a: (Left Half) $S(0.75, 0.9375) =$

$$\frac{0.75}{24}\left[f(0.75) + 4f(0.84375) + f(0.9375)\right] \approx 0.21218778$$

$$S(0.9375, 1.125) = \frac{0.75}{24}\left[f(0.9375) + 4f(1.03125) + f(1.125)\right] \approx 0.316703867$$

$$|S(0.75, 1.125) - S(0.75, 0.9375) - S(0.9375, 1.125)| \approx 0.000636$$ $\boxed{< 0.15}$

No need to halve again (stop)

$\textbf{Step 3b:}$ (Right Half) $S(1.125, 1.3125) =$

$$\frac{0.75}{24}\left[f(1.125) + 4f(1.21875) + f(1.3125)\right] \approx 0.523950957$$

$$S(1.3125, 1.50) = \frac{0.75}{24}\left[f(1.3125) + 4f(1.40625) + f(1.50)\right] \approx 1.311747793$$

$$|S(1.125, 1.50) - S(1.125, 1.3125) - S(1.3125, 1.50)| \approx 0.122\ldots$$ $\boxed{\text{Not} < 0.15}$

Halve step size

etc.

- The last thing we need to do once no more halving is necessary is to add up all the approximations to get the final integral.

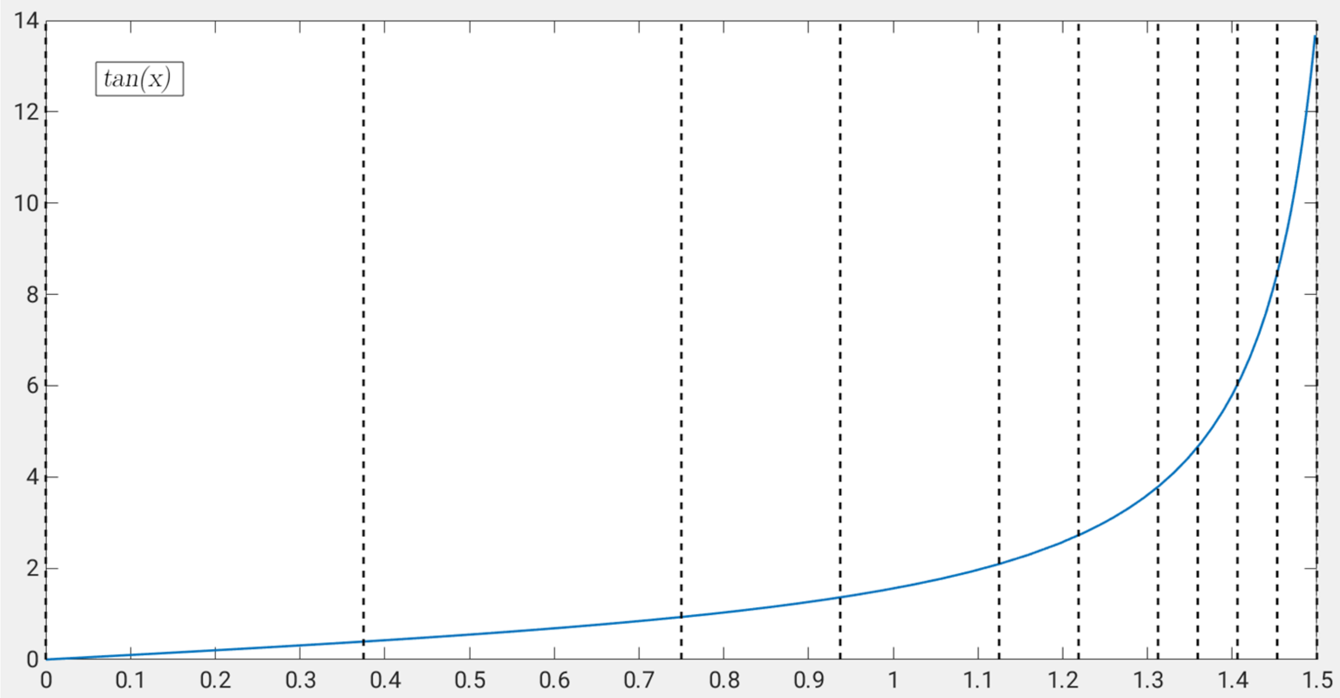
$$\int_0^{1.5} \tan x \, dx \approx S(0, 0.375) + S(0.375, 0.75) + S(0.75, 0.9375)$$

$$+ S(0.9375, 1.125) + S(1.125, 1.21875) + S(1.21875, 1.3125) +$$

$$S(1.3125, 1.359375) + S(1.359375, 1.40625) + S(1.40625, 1.453125)$$

$$+ S(1.453125, 1.5)$$

$$\boxed{= 2.649260871}$$

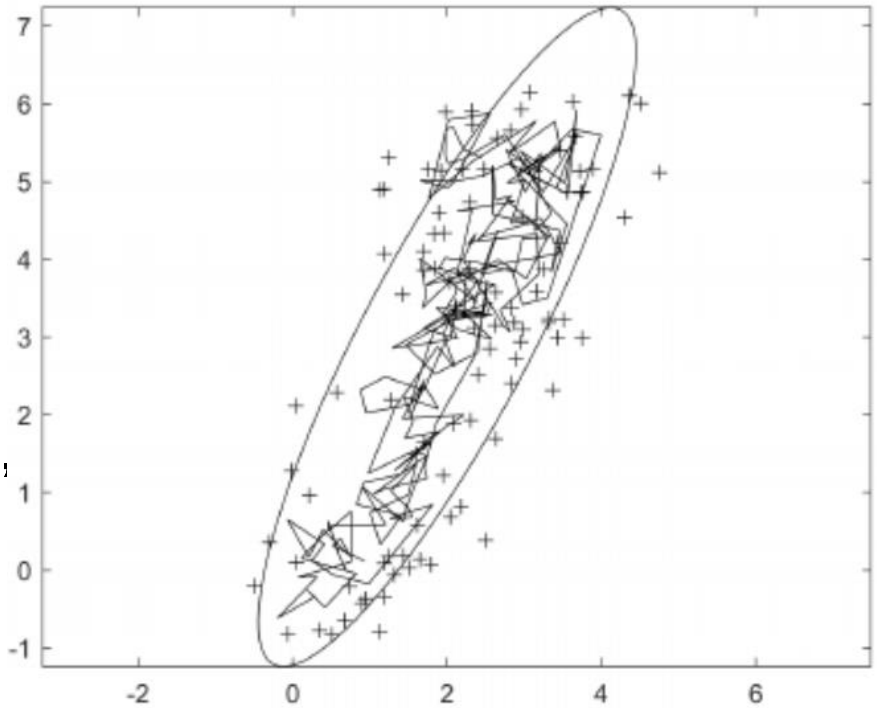


11.5 Monte Carlo Integration

- When doing numerical methods in higher dimensions the number of computations increases exponentially.

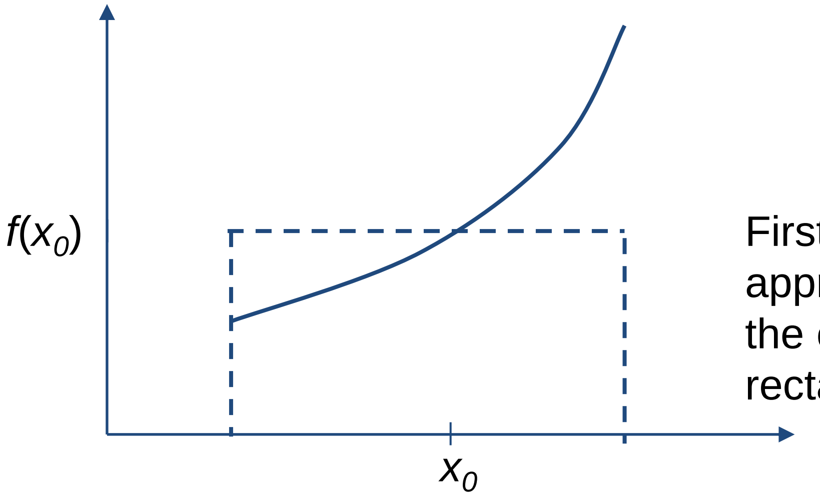
- It becomes computationally infeasible to subdivide the domain in all dimensions and work through them iteratively.

- Instead we can use random sampling to help us reduce the number of computations by selecting a representative set of points from the domain.



- We could apply a random walk, in the direction of greatest variance, for example, to sample a domain space.

- To apply the method to integration we can sample points in the domain and estimate the integral by calculating the hyperrectangle (just rectangle for single variable functions $f(x)$) at each point and then taking the mean of those values.



First sample point we approximate the area under the curve with a simple rectangle

EXAMPLE 12 Use Monte Carlo integration to approximate the following integral (true value is 4.236531...).

$$\int_0^2 e^{\sin(x)}\,dx$$

Random points, x_i	0.19	1.83	1.26	0.2	0.56	1.09	1.92
$\exp(\sin(x_i))$	1.2078	2.6289	2.5911	1.2197	1.7009	2.4269	2.5590
A_i	2.4157	5.2579	5.1822	2.4395	3.4018	4.8538	5.1181
Running average of *A*	2.4157	3.8368	4.2853	3.8238	3.7394	3.9252	4.0956

EXAMPLE 13 Use Monte Carlo integration to approximate the following integral (true value is 35.599174...).

$$\int_0^2 \int_0^3 x e^{x-y}\, dx\, dy$$

Random points, (x_i, y_i)	1.46 1.60	0.43 0.84	2.75 1.58	2.88 1.31	0.11 1.7	1.14 1.53	2.39 0.37
x_i*exp(x_i+y_i)	1.2692	0.2853	8.8604	13.8431	0.0224	0.7718	18.0165
A_i	7.6155	1.7122	53.1628	83.0588	0.1345	4.6310	108.0995
Running average of *A*	7.6155	4.6638	20.8302	36.3873	29.1368	25.0525	36.9163

Note here that A_i is calculated as the integrand value multiplied by 2 x 3 = 6.