

4.2 Applications of 2nd Derivative

Second Derivative and Concavity

The second derivative of a function provides information on how the first derivative is changing and leads to conclusions regarding the **concavity** of a graph.

Graphically, a function is **concave up** if its graph is curved with the opening upward (a in the figure). Similarly, a function is **concave down** if its graph opens downward (b) in the [Figure 4.10](#).

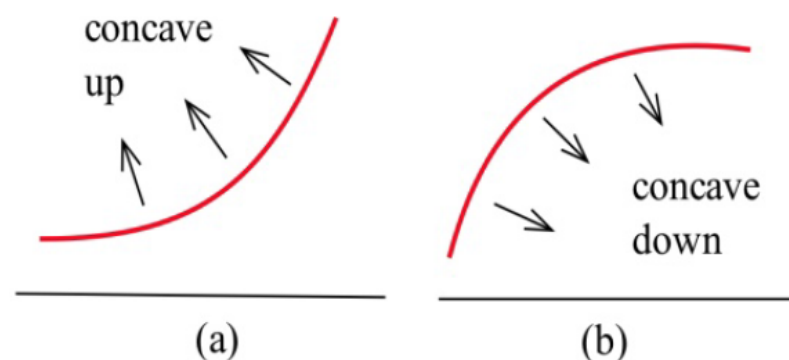


Figure 4.10

This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing. It's important to understand the difference between increasing/decreasing characteristics versus concavity of a graph.

As an example of this difference, notice for the left portion of Figure 3.4 which is labeled as “concave up”: this portion of the graph first decreases and then increases however this section of the graph is concave up.

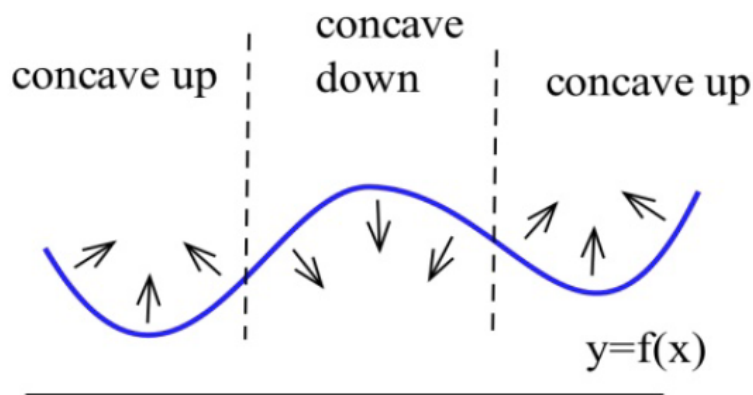


Figure 4.11

For example, **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In the figure below, $f(x)$ is the number of people who have the disease at time x , and two different situations are shown. In both (a) and (b), the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference in the two situations is the concavity of f , and that difference in concavity might have a big effect on your decision.

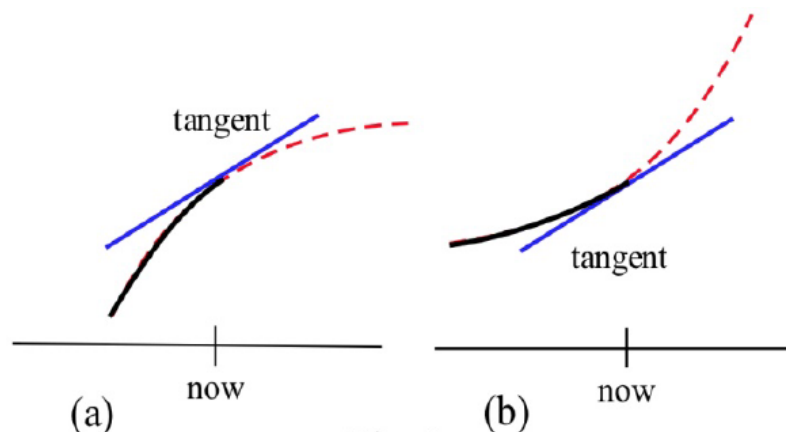


Figure 4.12

In (a), f is concave down at “now”, the slopes are decreasing, and it looks as if it’s tailing off. We can say “ f is increasing at a decreasing rate.” It appears that the current methods are starting to bring the epidemic under control.

In (b), f is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the derivative is increasing or decreasing. The derivative of a function f is a function that gives information about the slope of f . The derivative tells us if the original function is increasing or decreasing. Because f' is a function, we can take its derivative. This second derivative also gives us information about our original function f . The second derivative gives us a mathematical way to tell how the graph of a function is curved. The second derivative tells us if the original function is concave up or down.

Second Derivative

Let $y' = f'(x)$.

The **second derivative of f** is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or y'' . You can read this aloud as “y double prime.”

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as “the second derivative of f ”.

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval. We can say that f is increasing (or decreasing) **at an increasing rate**.

If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave down** on that interval.

We can say that f is increasing (or decreasing) **at a decreasing rate**.

Inflection Points

Definition

An inflection point is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

Example 1

Which of the labeled points in the graph below are inflection points?

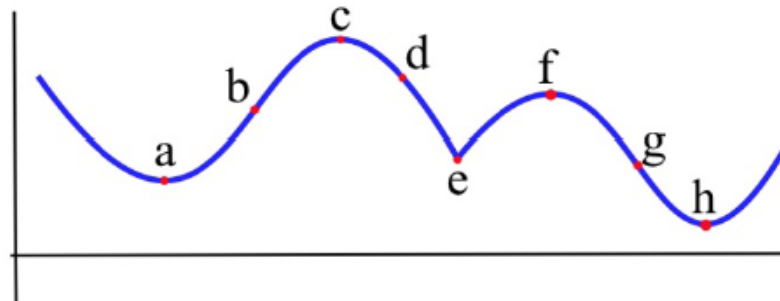


Figure 4.13

The concavity changes at points b and g. At points a and h, the graph is concave up on both sides, so the concavity does not change. At points c and f, the graph is concave down on both sides. At point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

Working Definition

An inflection point is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So

to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of f'' changes sign. The functions in the next example illustrate what can happen.

Example 2

Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$. For which of these functions is the point $(0, 0)$ an inflection point?

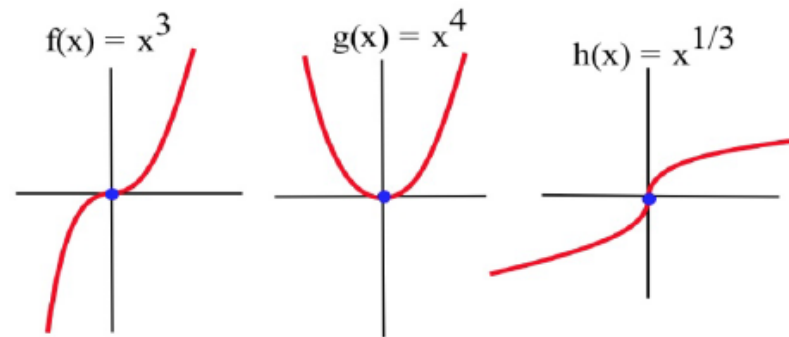


Figure 4.14

Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = x^{1/3}$ changes at $(0, 0)$, so $(0, 0)$ is an inflection point for f and h . The function $g(x) = x^4$ is concave up everywhere so $(0, 0)$ is not an inflection point of g .

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined (f' is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so f is concave down. If $x > 0$, then $f''(x) > 0$ so f is concave up. At $x = 0$ the concavity changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of x^3 .

If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which

$g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so g is concave up. If $x > 0$, then $g''(x) > 0$ so g is also concave up. At $x = 0$ the concavity **does not change** so the point $(0, g(0)) = (0, 0)$ is **not an inflection point of x^4** . Keep this example in mind!.

If $h(x) = x^{\frac{1}{3}}$, then $h'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and $h''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$. h'' is not defined if $x = 0$, but $h''(\text{negative number}) > 0$ and $h''(\text{positive number}) < 0$ so h changes concavity at $(0, 0)$ and $(0, 0)$ is an inflection point of h .