Arbitrage Opportunity, Impossible Frontier, and Logical Circularity in CAPM Equilibrium

Abstract: The capital market for CAPM is incomplete and is a Hilbert space, we find out the analytic expression for the SDF mimicking payoff in this market. The CAPM formula holds under the partial equilibrium of purely risky assets, which is equivalent to the condition that the market portfolio is the tangency portfolio. Since the general solution to asset prices in CAPM has only one dimension, given all individual investor's endowments and mean-variance preferences, the condition of CAPM equilibrium turns out to be an equation of only one variable. With the closed-form solution to CAPM equilibrium, we see more clearly that the risk-return characteristics is a false impression from the partial equilibrium of purely risky assets. Thus it is illusory, for the assets are priced as a whole and the prices are endogenous, the return of market portfolio is not exogenous but endogenous. By way of numerical examples, we show that CAPM equilibrium may coexist with arbitrage opportunities, and that the crisis of impossible frontier is due to market disequilibrium. We point out the incorrect practice of beta pricing by presenting negative prices of European call options.

Keywords: CAPM, Mean-Variance Rule, Beta, Arbitrage Opportunity, Risk Premium

JEL Classification: G12, G11, D52

The CAPM is considered the backbone of modern price theory for financial markets. It is widely accepted that the expected return of any primitive security is determined by its systematic risk (also known as market risk or non-diversifiable risk). Systematic risk is measured by the beta value, which is defined to be the covariance of the stock's return and market return divided by the variance of market return. If the expected return of a stock is determined by its beta, we will face the chicken-and-egg problem: a stock's return is determined from its beta, but beta itself depends on the return directly. On the other hand, to calculate the beta, we need to know the market return first, and to calculate the market return, we need to know the returns of each stock. Johnstone (2017, p.503) refers to this situation as "the logical circularity built into the CAPM equilibrium mechanism". Does this logical circularity really exist?

Given the wealth and preference of each investor, we obtain the analytical solution of the asset price for mean-variance market equilibrium (CAPM equilibrium). Based on the analytical solution, we find that

1. CAPM equilibrium allows arbitrage opportunities. If the CAPM market is free of arbitrage, the CAPM formula is essentially a risk-neutral pricing formula
2. Impossible frontier (Brennan and Lo, 2010, 2015) is attributed to the market not reaching CAPM equilibrium
3. The negative prices of plain vanilla European options under beta pricing (CAPM formula) are due to the incompleteness of CAPM market

The CAPM equilibrium is essentially the equilibrium of fund allocation. As long as we do not look at any individual security in isolation, break the mindset of risk determining returns, and rely on the overall pricing facts of portfolio diversification, we will find that CAPM's logical circularity is nothing more than an imaginary trap. For readability and the sake of completeness, we place all omitted proofs, technical details and reverse-engineering tips for numerical examples in the Appendix.

1 Diversification

Given the mean vector and variance matrix for the returns on securities, Markowitz (1952) suggests the mean-variance portfolio theory to calculate the optimal portfolio. When discussing the diversification of assets, returns are usually used. Obviously, if payoffs are given, it is necessary to assume that the equilibrium stock prices are positive for the calculation of the stock returns.

1.1 Market Settings

In a single-step model, there are two time points, 0 and 1. Given N primitive securities (stocks) and I investors, $N \geqslant 2$, and $I \geqslant 2$. Let X_i be the payoff of stock i at time 1, then the payoff vector of N stocks is $\mathbf{x} = [X_1, X_2, \cdots, X_N]'$. Let $\boldsymbol{\eta} = \mathrm{E}(\mathbf{x})$, and $\boldsymbol{\Omega} = \mathrm{var}(\mathbf{x})$, we assume that the first two moments $\boldsymbol{\eta}$ and $\boldsymbol{\Omega}$ are known, and the variance matrix is positive definite, $\boldsymbol{\Omega} > 0$, such that there is no redundant security. Defining

$$a = \mathbf{p}'\boldsymbol{\Omega}^{-1}\mathbf{p} \qquad b = \mathbf{p}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} \qquad c = \boldsymbol{\eta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}$$

where $\mathbf{p} = [P_1, P_2, \cdots, P_N]' > 0$ is the price vector of stocks (P_i is the market value of stock i, not the market price per share of stock). Apparently, $a > 0$, and c is a constant.

If $\mathbf{1}$ is a conforming vector of ones, then the total market payoff is $X_M = \mathbf{1}'\mathbf{x}$, with mean $E = \mathrm{E}(X_M) = \mathbf{1}'\boldsymbol{\eta}$, variance $Q = \mathrm{var}(X_M) = \mathbf{1}'\boldsymbol{\Omega}\mathbf{1} > 0$, and total market value $P_M = \mathbf{1}'\mathbf{p} > 0$. Let the return[1] of N stocks be $\mathbf{r} = [R_1, R_2, \cdots, R_N]'$, then $R_i = X_i/P_i$, and

$$\mathbf{r} = \mathbf{P}^{-1}\mathbf{x}$$

where $\mathbf{P} = \mathrm{diag}(\mathbf{p})$, the diagonal matrix of price vector $\mathbf{p}$. Let

$$\boldsymbol{\mu} = \mathrm{E}(\mathbf{r}) = \mathbf{P}^{-1}\boldsymbol{\eta} \qquad \mathbf{V} = \mathrm{var}(\mathbf{r}) = \mathbf{P}^{-1}\boldsymbol{\Omega}\mathbf{P}^{-1}$$

from the perspective of return, we see that

$$a = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \qquad b = \mathbf{1}'\mathbf{V}^{-1}\boldsymbol{\mu} \qquad c = \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu}$$

Since the first two moments of payoffs are known, the following three variables, price vector $\mathbf{p}$, expected return vector $\boldsymbol{\mu}$ and variance matrix $\mathbf{V}$ are equivalent, in the sense that any one of them is known, the other two are determined.

We assume that there is a risk-free bond, with payoff $X_0 > 0$ and return $R_0 > 1$ (risk-free interest simple rate is $R_0 - 1$). Since we have risky assets, say, $P_M > 0$, it's required that $R_0 < \frac{b}{a}$. Define

$$h(x) \equiv (\boldsymbol{\eta} - x\mathbf{p})'\boldsymbol{\Omega}^{-1}(\boldsymbol{\eta} - x\mathbf{p}) = (\boldsymbol{\mu} - x\mathbf{1})'\mathbf{V}^{-1}(\boldsymbol{\mu} - x\mathbf{1}) = ax^2 - 2bx + c \geqslant 0$$

[1] In finance, the word return may refer to profit (gain or loss in absolute terms) or rate of return (holding period return measured in percentage). In financial economics and this text, gross rate of return (final value over inital value), is often called return for short.

there is $ac - b^2 \geqslant 0$ by the discriminant of $h(x)$. If no otherwise stated, we assume that the expected returns are not identical, symbolically, $\boldsymbol{\mu} \neq x\mathbf{1}$. Then the constant $c > 0$, and for any real number x, $h(x) > 0$, thus $ac - b^2 > 0$.

1.1.1 Payoff Space

Let M be the market of N stocks and the risk-free bond, then its payoff space is

$$\mathsf{X} = \{h_0 X_0 + \mathbf{h}'\mathbf{x} : h_0 \in \mathbb{R}, \mathbf{h} \in \mathbb{R}^N\}$$

where $\mathbf{h} = [h_1, h_2, \cdots, h_N]'$, the ith element h_i is the percentage of shares held, that is, the number of shares held over the number of shares issued by stock i. Generally, market M is an incomplete market. However, the payoff space X is a subset of Hilbert space $\mathsf{L}^2(\Omega, \mathcal{F}, \mathrm{P})$. When the risk-free bond is excluded, the market for purely risky assets is denoted by M_*, with payoff space $\mathsf{X}_* = \{\mathbf{h}'\mathbf{x} : \mathbf{h} \in \mathbb{R}^N\} \subset \mathsf{X}$.

When discussing the payoff of portfolio, the number of shares is usually used, for example $X = \mathbf{h}'\mathbf{x}$. When the value of the portfolio is positive, we can use value weights to represent the portfolio, and use the return for analysis. For example, $R = \mathbf{z}'\mathbf{r}$ where $\mathbf{z} = [z_1, z_2, \cdots, z_N]'$, the ith element $z_i = \frac{h_i P_i}{\mathbf{h}'\mathbf{p}}$ is the value proportion of asset i in the portfolio. If a stock is sold short, the corresponding component is negative. Since the weights are normalized, $\mathbf{z}'\mathbf{1} = \sum_{i=1}^N z_i = 1$.

1.1.2 Postulates on Asset Prices

In addition to the perfect market assumptions, we also apply the following postulates on asset prices in the financial market (Abad, 2019)

1. (Law of One Price) Each primitive security has a unique price. Which states that there exists a pricing function $\wp$, such that $\mathbf{p} = \wp(\mathbf{x})$.
2. (Law of Linear Combination) The payoff of a portfolio equals the linear combination of payoffs of its constituent assets. The price of a portfolio equals the linear combination of prices of its constituent assets. The law of linear combination asserts that the pricing function is linear:

$$\wp(aX + bY) = a\,\wp(X) + b\,\wp(Y) \qquad \forall a, b \in \mathbb{R}, X, Y \in \mathsf{X}$$

3. (Limited Liability, or Positivity) For any limited liability ($X \geqslant 0$, and $\mathrm{P}(X > 0) > 0$, written by $X \gneq 0$) portfolio, the price is positive.

Thus, for the risk-free bond, $\wp(X_0) = X_0/R_0$, and $\wp(1) = 1/R_0$. A price vector satisfying the law of one price and law of linear combination is *semi-feasible*; And a price vector satisfying the above three postulates is *feasible*.

Primitive securities (stocks) are limited liabilities, which is not to say that the payoff of a company will not go negative such as insolvency, but that investors will receive an option due to securitization. The payoff of a risky security is $X = \max(Y - R_0 D_0, 0) \gneq 0$, where Y is the real payoff of the firm, and D_0 is the debt principal. Considering this institutional arrangement, the price of any primitive security must be positive. For convenience, we will call the positive semi-feasible price the *presentable* price.

1.1.3 Return Hyperplane

For any payoff $X \in \mathsf{X}$, if $\wp(X) > 0$, then we can define the return by $R = \frac{X}{\wp(X)}$, thus $\wp(R) = 1$. The return is the standardization of a payoff, a payoff with unit price. For this reason, the return hyperplane is

$$\mathsf{X}_1 = \{X \in \mathsf{X} : \wp(X) = 1\}$$

Similarly, the return hyperplane in payoff space X_* is denoted as

$$\mathsf{X}_{*1} = \{X \in \mathsf{X}_* : \wp(X) = 1\} \subset \mathsf{X}_1$$

1.2 Linear Pricing

Hilbert space theory provides great convenience for studying pricing functions (Chamberlain and Rothschild, 1983; Chamberlain, 1983). When the number of assets is finite, given the prices of current primitive securities, we are very lucky that the pricing function will be uniquely determined even if the market is incomplete.

1.2.1 Mimicking Payoff

By Riesz representation theorem, for any $X \in \mathsf{X}$, a linear pricing function is uniquely represented by

$$P = \wp(X) = \mathrm{E}(X_{:} X) \qquad X_{:} \in \mathsf{X} \tag{1.1}$$

where $X_{:}$ is the *mimicking payoff* of SDF (stochastic discount factor). We have the following proposition:

Proposition 1 (mimicking payoff): In Market M, for any presentable price vector

$$X_{:} = \frac{1+c-bR_0}{R_0} - \frac{1}{R_0}\mathbf{r}'\mathbf{V}^{-1}\left(\boldsymbol{\mu} - R_0\mathbf{1}\right) \tag{1.2}$$

is the unique mimicking payoff.

A more general form of Eq (1.2) is (zero or negative price allowed)

$$X_{:} = \frac{1+c-\boldsymbol{\eta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{R_0} + (\mathbf{x}-\boldsymbol{\eta})'\boldsymbol{\Omega}^{-1}\mathbf{p} \tag{1.3}$$

which states that the mimicking payoff is a function of price vector, where the price $\mathbf{p}$ is semi-feasible, not necessary in equilibrium. Eq (1.3) reveals the key to relative pricing: the prices of stocks are exogenous data, they are given beforehand. Eq (1.1) gives $\mathbf{p} = \mathrm{E}(X_{:}\mathbf{x})$, which is the identity about the prices of stocks. We know that the prices of stocks are determined by capital market equilibrium (absolute pricing). In a CAPM equilibrium, even if the payoffs of primitive securities have not changed, as long as any investor's preference and/or endowment is changed, the equilibrium price will be adjusted (see Proposition 4), and the mimicking payoff will be changed accordingly.

From Eq (1.3), we have

$$\mathrm{E}(X_{:}) = \frac{1}{R_0} = \wp(1) = \mathrm{E}(X_{:}\cdot 1)$$

The price of mimicking portfolio (with payoff $X_{:}$) is $\wp(X_{:}) = \frac{1+h(R_0)}{R_0^2} > 0$. Hence, the return of mimicking portfolio is

$$R_{:} = \frac{X_{:}}{\wp(X_{:})} = \frac{R_0}{1+h(R_0)}\left(1+c-bR_0 - \mathbf{r}'\mathbf{V}^{-1}\left(\boldsymbol{\mu}-R_0\mathbf{1}\right)\right) \tag{1.4}$$

with mean $\mu_{:} = \mathrm{E}(R_{:}) = \frac{R_0}{1+h(R_0)} = \mathrm{E}(R_{:}^2)/R_0$, and

$$\mathrm{E}\left(R_{:}R\right) = \mathrm{E}\left(R_{:}^2\right) = \frac{1}{\wp(X_{:})} = R_0\,\mathrm{E}(R_{:}) = \frac{R_0^2}{1+h(R_0)} \qquad \forall R \in \mathsf{X}_1 \tag{1.5}$$

There is no risk-free bond in market M_*, let the mimicking payoff of risk-free payoff 1 on payoff space X_* be

$$X_{!} = \mathrm{Pj}\left(1 \,|\, \mathsf{X}_*\right) = \frac{1}{c+1}\mathbf{r}'\mathbf{V}^{-1}\boldsymbol{\mu} \tag{1.6}$$

whose price is $\wp(X_{!}) = \frac{b}{c+1}$, in addition

$$\mathrm{E}\left(X_{!}X\right) = \mathrm{E}(X) \qquad \forall X \in \mathsf{X}_*$$

The payoff space X_* is a subspace of X, similarly, there is mimicking payoff $X_{*:} \in \mathsf{X}_*$, such that

$$P = \wp(X) = \mathrm{E}(X_{*:}X) \qquad \forall X \in \mathsf{X}_*$$

Note that $X_{*:}$ is equal to the projection of $X_{:}$ on X_*

$$X_{*:} = \mathrm{Pj}\left(X_{:} \,|\, \mathsf{X}_*\right) = \mathbf{r}'\mathbf{V}^{-1}\left(\mathbf{1} - \frac{b}{c+1}\boldsymbol{\mu}\right) \tag{1.7}$$

with price $\wp(X_{*:}) = \frac{ac-b^2+a}{c+1} > 0$, and return

$$R_{*:} = \frac{X_{*:}}{\wp(X_{*:})} = \frac{1}{ac-b^2+a}\mathbf{r}'\mathbf{V}^{-1}\left((c+1)\mathbf{1} - b\boldsymbol{\mu}\right) \tag{1.8}$$

We have $\mu_{*:} = \mathrm{E}(R_{*:}) = \frac{b}{ac-b^2+a} = \frac{b}{c+1}\,\mathrm{E}(R_{*:}^2)$, and

$$\mathrm{E}\left(R_{*:}R\right) = \mathrm{E}(R_{*:}^2) = \frac{1}{\wp(X_{*:})} = \frac{c+1}{ac-b^2+a} \qquad \forall R \in \mathsf{X}_{*1} \tag{1.9}$$

1.2.2 Excess Return Space

If a portfolio's payoff is equal to an excess return, then the portfolio has price zero. Thus, the set of excess return is

$$\mathsf{X}_0 = \{X \in \mathsf{X} : \wp(X) = 0\}$$

Obviously, X_0 is a linear space, the kernel space of pricing function $\wp$. For any excess return $R \in \mathsf{X}_0$, there is

$$\mathrm{E}\left(R_{:}R\right) = 0 \qquad \forall R \in \mathsf{X}_0 \tag{1.10}$$

Let R_ι be the projection of risk-free payoff 1 on excess return space X_0

$$R_\iota = \mathrm{Pj}\left(1 \,|\, \mathsf{X}_0\right) = 1 - \frac{R_{:}}{R_0} \tag{1.11}$$

there is $\mu_\iota = \mathrm{E}(R_\iota) = \frac{h(R_0)}{1+h(R_0)}$. We see that for excess return $R \in \mathsf{X}_0$

$$\mathrm{E}(R_\iota R) = \mathrm{E}(R) \qquad \forall R \in \mathsf{X}_0 \tag{1.12}$$

Furthermore, for any return $R \in \mathsf{X}_1$, given $\mu = \mathrm{E}(R)$, we have

$$\mathrm{E}(RR_\iota) = \mu - \mu_{:} = \mu - \frac{R_0}{h(R_0)}\mu_\iota \qquad \forall R \in \mathsf{X}_1 \tag{1.13}$$

Similarly, in market M_*, the space of excess return is

$$\mathsf{X}_{*0} = \{X \in \mathsf{X}_* : \wp(X) = 0\}$$

For any excess return $R \in \mathsf{X}_{*0}$, there is

$$\mathrm{E}\left(R_{*:}R\right) = 0 \qquad \forall R \in \mathsf{X}_{*0} \tag{1.14}$$

The projection of risk-free payoff 1 on excess return space X_{*0} is

$$R_{*\iota} = \mathrm{Pj}\left(1 \,|\, \mathsf{X}_{*0}\right) = X_! - \frac{b}{c+1}R_{*:} = \frac{1}{ac-b^2+a}\mathbf{r}'\mathbf{V}^{-1}\left(a\boldsymbol{\mu} - b\mathbf{1}\right) \tag{1.15}$$

with mean $\mu_{*\iota} = \mathrm{E}(R_{*\iota}) = \frac{ac-b^2}{ac-b^2+a}$. Note that if $b = 0$, there is $X_! = R_{*\iota}$. Furthermore, for any excess return $R \in \mathsf{X}_{*0}$, there is

$$\mathrm{E}(R_{*\iota}R) = \mathrm{E}(R) \qquad \forall R \in \mathsf{X}_{*0} \tag{1.16}$$

and for any return $R \in \mathsf{X}_{*1}$, we have

$$\mathrm{E}(R_{*\iota}R) = \mu - \mu_{*:} = \mu - \frac{b}{ac-b^2}\mu_{*\iota} \qquad \forall R \in \mathsf{X}_{*1} \tag{1.17}$$

1.3 Orthogonal Decomposition

In market M, for any portfolio with return R, and mean $\mu = \mathrm{E}(R)$, we have the following decomposition (Eq 3.15 of Hansen and Richard, 1987)

$$R = R_{:} + \theta(\mu)R_{\iota} + R_e \qquad \forall R \in \mathsf{X}_1 \tag{1.18}$$

where

$$\theta(\mu) \equiv \frac{\mathrm{E}(R_{\iota}R)}{\mathrm{E}(R_{\iota}^2)} = \frac{\mu - \mu_{:}}{\mu_{\iota}} = \mu + \frac{\mu - R_0}{h(R_0)} \tag{1.19}$$

with $R_{:}$, R_{ι} and R_e are pairwise orthogonal, say

$$\mathrm{E}(R_{:}R_{\iota}) = \mathrm{E}(R_{\iota}R_e) = \mathrm{E}(R_eR_{:}) = 0$$

Noting that the mean of an excess return is usually not zero, but for R_e, there is $\mathrm{E}(R_e) = 0$. In market M_*, there is a starred version in the following corresponding form

$$R = R_{*:} + \theta_*(\mu)R_{*\iota} + R_{*e} \qquad \forall R \in \mathsf{X}_{*1} \tag{1.20}$$

where

$$\theta_*(\mu) \equiv \frac{\mathrm{E}(R_{*\iota}R)}{\mathrm{E}(R_{*\iota}^2)} = \frac{\mu - \mu_{*:}}{\mu_{*\iota}} = \mu + \frac{a\mu - b}{ac - b^2} \tag{1.21}$$

with $R_{*:}$, $R_{*\iota}$ and R_{*e} are pairwise orthogonal

$$\mathrm{E}(R_{*:}R_{*\iota}) = \mathrm{E}(R_{*\iota}R_{*e}) = \mathrm{E}(R_{*e}R_{*:}) = 0$$

and $\mathrm{E}(R_{*e}) = 0$.

1.4 Efficient Frontier

The set of the minimum variance portfolio on the mean-variance plane is called the portfolio frontier. The analytical solution of mean-variance frontier is first presented by Merton (1972). More details can be found in Szegö (1980), and widely spread textbooks such as Ingersoll (1987) and Huang and Litzenberger (1988), and newly textbook Danthine and Donaldson (2014). Lagrangian approach is mostly used in finding the closed form solution to the mean-variance frontier. However, Chamberlain and Rothschild (1983) and Hansen and Richard (1987) using methods of Hilbert space, which is better in revealing the relationship between the mean-variance frontier portfolio and the mimicking portfolio of SDF.

1.4.1 Mean-variance Frontier

Let's start from the market M_*. Given portfolio $R = \mathbf{z}'\mathbf{r}$ with $\mu = \mathrm{E}(R) = \mathbf{z}'\boldsymbol{\mu}$, for the mean-variance frontier F_* in M_*, the following statements are equivalent:

1. Portfolio R is on frontier F_*, or $R \in \mathsf{F}_*$
2. Eq (1.20), the orthogonal decomposition takes $R_{*e} = 0$, say
$$R = R_{*:} + \theta_*(\mu) R_{*\iota} \tag{1.22}$$
3. The weight vector of portfolio equals to ($\mathbf{z}'\mathbf{1} = 1$)
$$\mathbf{z}(\mu) = \frac{a\mu - b}{ac - b^2}\mathbf{V}^{-1}\boldsymbol{\mu} + \frac{c - b\mu}{ac - b^2}\mathbf{V}^{-1}\mathbf{1} = \frac{c\mathbf{V}^{-1}\mathbf{1} - b\mathbf{V}^{-1}\boldsymbol{\mu}}{ac - b^2} + \frac{a\mathbf{V}^{-1}\boldsymbol{\mu} - b\mathbf{V}^{-1}\mathbf{1}}{ac - b^2}\mu \tag{1.23}$$
The weight of the frontier portfolio is a linear function of the expected return. The frontier portfolio is located on a straight line in $\mathbb{R}^N$
4. The variance of portfolio is
$$v(\mu) = \mathrm{var}(R) = \frac{h(\mu)}{ac - b^2} = \frac{a\mu^2 - 2b\mu + c}{ac - b^2} \tag{1.24}$$

In the mean-variance plane, the portfolio frontier F_* is a parabola. If $\boldsymbol{\mu} = x\mathbf{1}$, then $ac - b^2 = 0$, the expected return of portfolios is fixed at $\mu = x$, the mean-variance frontier has
$$\mathbf{z} = \frac{1}{a}\mathbf{V}^{-1}\mathbf{1} \qquad v = \mathrm{var}(R) = \frac{1}{a}$$
the frontier F_* degenerates to the singular point $(x,\, 1/a)$ on the mean-variance plane (variance on the horizontal axis).

The market M contains risk-free bond, the mean-variance frontier F is more concise than F_*. Let the portfolio be $R = z_0 R_0 + \mathbf{z}'\mathbf{r}$ with $\mu = \mathrm{E}(R) = z_0 R_0 + \mathbf{z}'\boldsymbol{\mu}$, then the following statements are equivalent:

1. Portfolio R is on frontier F, or $R \in \mathsf{F}$
2. Eq (1.18), the orthogonal decomposition takes $R_{*e} = 0$, say
$$R = R_{:} + \theta(\mu) R_{\iota} \tag{1.25}$$
3. The weight vector of portfolio equals to ($\mathbf{z}'\mathbf{1} = 1 - z_0$)
$$z_0(\mu) = \frac{1}{h(R_0)}\left((aR_0 - b)\,\mu + c - bR_0\right) \tag{1.26}$$
$$\mathbf{z}(\mu) = \frac{\mu - R_0}{h(R_0)}\mathbf{V}^{-1}\left(\boldsymbol{\mu} - R_0\mathbf{1}\right)$$
The weight of the frontier portfolio is a linear function of the expected return. The frontier portfolio is located on a straight line in $\mathbb{R}^{N+1}$
4. The variance of portfolio is
$$v(\mu) = \mathrm{var}(R) = \frac{(\mu - R_0)^2}{h(R_0)} = \frac{(\mu - R_0)^2}{aR_0^2 - 2bR_0 + c} \tag{1.27}$$

In the mean-variance plane, the portfolio frontier F is a parabola. If $\boldsymbol{\mu} = x\mathbf{1}$ and $x \neq R_0$, since $h(R_0) = (x - R_0)^2 a > 0$, the above conclusions still hold. If $\boldsymbol{\mu} = R_0\mathbf{1}$, then $h(R_0) = 0$, the frontier F degenerates to the singular point $(R_0,\,0)$ on the mean-variance plane. The portfolio weights is $z_0 = 1$ and $\mathbf{z} = 0$, all risky assets disappear.

By the uniqueness of portfolio on the frontier (F or F_*): Given any frontier portfolio's expected return, the weight (vector), variance and return will be determined uniquely. For frontier portfolio p and q

$$\mathbf{z}_p = \mathbf{z}_q \iff R_p = R_q \iff v_p = v_q \iff \mu_p = \mu_q$$

Therefore, we refer $\mathbf{z}_p$, R_p, v_p, or even p to the frontier portfolio that has expected return μ_p in the portfolio selection context.

1.4.2 Frontier Portfolio

There is a global minimum variance portfolio in frontier F_*, denoted by R_o, and

$$\mu_o = \frac{b}{a} \qquad v_o = \frac{1}{a} \qquad \mathbf{z}_o = \frac{1}{a}\mathbf{V}^{-1}\mathbf{1}$$

The upper branch of frontier F_*, $\mu \geqslant \mu_o$, where the largest expected return is obtained given the same risk (variance), is called the mean-variance *efficient frontier*. We see that portfolio $R_{*:}$ is a frontier portfolio (if $b > 0$, $\mu_{*:} = \frac{b}{ac-b^2+a} < \frac{b}{a} = \mu_o$, $R_{*:}$ is on the inefficient branch), and it gives the minimum second moment return, such that $\mathrm{E}(R_{*:}^2) = \min_{R\in\mathsf{X}_{*1}} \mathrm{E}(R^2)$. Noting that $R_{*:} + R_{*\iota}$ is on F_* too (maybe inefficient). In addition, since $\theta_*(\mu_o) = \mu_o$, there is $R_o = R_{*:} + \mu_o R_{*\iota}$.

The risk-free bond is the global minimum variance portfolio in frontier F. Hence, the upper branch, $\mu \geqslant R_0$, is the mean-variance efficient frontier. Analogously, $R_:$ and $R_: + R_\iota$ are frontier portfolios (inefficient), $R_:$ is the minimum second moment return such that $\mathrm{E}(R_:^2) = \min_{R\in\mathsf{X}_1} \mathrm{E}(R^2)$. In addition, since $\theta(R_0) = R_0$, there is $R_0 = R_: + R_0 R_\iota$.

In the mean-variance plane ($\mu - v$), frontier F is tangent to F_* at $R_\wr = \mathbf{z}_\wr'\mathbf{r}$. For the tangency portfolio $R_\wr$, we have (given $aR_0 - b \neq 0$ and $\boldsymbol{\mu} \neq x\mathbf{1}$)

$$\mu_\wr = \frac{bR_0 - c}{aR_0 - b} \tag{1.28}$$

$$v_\wr = \frac{(\mu_\wr - R_0)^2}{h(R_0)} = \frac{h(R_0)}{(aR_0 - b)^2} = \frac{\mu_\wr - R_0}{b - aR_0}$$

$$\mathbf{z}_\wr = \frac{1}{b - aR_0}\mathbf{V}^{-1}(\boldsymbol{\mu} - R_0\mathbf{1})$$

and

- SDF mimicking payoff and tangency portfolio $R_\wr$

$$X_: = \frac{1 + c - bR_0}{R_0} + \frac{aR_0 - b}{R_0} R_\wr \tag{1.29}$$

- Sharpe ratio: Let $\sigma = \sqrt{\mathrm{var}(R)}$, Sharpe ratio is defined by

$$\mathrm{S}(R) = \frac{\mu - R_0}{\sigma}$$

  tangency portfolio $R_\wr$ has the highest Sharpe ratio, and

$$\mathrm{S}(R) \leqslant \mathrm{S}(R_\wr) = \sqrt{h(R_0)} = \frac{\sigma_:}{\mu_:} = -\frac{\mu_: - R_0}{\sigma_:} \qquad \forall R \in \mathsf{X}_1 \tag{1.30}$$

  where $\sigma_: = \sqrt{\mathrm{var}(R_:)}$. In fact, any portfolio on the efficient branch of F obtains the highest Sharpe ratio, the so-called HJ lower bound (Hansen and Jagannathan, 1991).

- Tangency beta: For any portfolio $R = z_0 R_0 + \mathbf{z}'\mathbf{r}$, the tangency beta is defined as

$$\beta(R) \equiv \frac{\mathrm{cov}(R, R_\wr)}{\mathrm{var}(R_\wr)} = \frac{\mathbf{z}'\mathbf{V}\mathbf{z}_\wr}{v_\wr} = \frac{\mu - R_0}{\mu_\wr - R_0}$$

  Given the market settings, $\beta(R)$ depends only on expected return μ, thus, it is rewritten more explicitly as

$$\beta(\mu) = \beta(R) = \frac{\mu - R_0}{\mu_\wr - R_0} = \frac{b - aR_0}{h(R_0)}(\mu - R_0) = (b - aR_0)(\theta(\mu) - \mu)$$

  Evidently, $\beta(\mu_\wr) = 1$, and the tangency beta of any portfolio with the mean $\mu_\wr$ is 1

An important characteristic of the frontier F_* says that the covariance of any portfolio in the market M_* and the frontier portfolio is completely determined by the expectations of the two. Given frontier portfolio $R_p \in \mathsf{F}_*$, for any portfolio with return $R \in \mathsf{X}_{*1}$

$$\mathrm{cov}\,(R, R_p) = \frac{\mu_p \mu a - (\mu_p + \mu)\, b + c}{ac - b^2} = \frac{a(\mu_p - \mu_o)}{ac - b^2}(\mu - \mu_o) + \frac{1}{a} \tag{1.31}$$

In particular

$$\mathrm{cov}\,(R, R_o) = \frac{1}{a} \qquad \forall R \in \mathsf{X}_{*1}$$

Given $R_p \in \mathsf{F}_*$, for any $R_n, R_m \in \mathsf{X}_{*1}$

$$\mathrm{cov}(R_n, R_p) - \mathrm{cov}(R_m, R_p) = \frac{a(\mu_p - \mu_o)}{ac - b^2}(\mu_n - \mu_m) \qquad \forall R_n, R_m \in \mathsf{X}_{*1},\ R_p \in \mathsf{F}_* \tag{1.32}$$

When $\mu_n = \mu_m$, there is $\mathrm{cov}(R_n, R_p) = \mathrm{cov}(R_m, R_p)$, the covariance of an F_* frontier portfolio and any portfolio with the same mean is the same. Furthermore, the covariance of any zero-mean excess return and the F_* frontier portfolio is zero (and orthogonal)

$$\mathrm{E}(R) = 0 \implies \mathrm{cov}(R, R_p) = \mathrm{E}(RR_p) = 0 \qquad \forall R \in \mathsf{X}_{*0},\ R_p \in \mathsf{F}_* \tag{1.33}$$

On the frontier F, for any $R_p \in \mathsf{F}$, and any portfolio $R \in \mathsf{X}_1$

$$\mathrm{cov}\,(R, R_p) = \frac{\mu_p - R_0}{h(R_0)}(\mu - R_0) = \frac{\mu_p - R_0}{b - aR_0}\beta(\mu) \tag{1.34}$$

Which states that the covariance depends only on the mean of portfolio R. Thus, the covariance of an F frontier portfolio and any portfolio with the same mean takes the same value. Furthermore, the covariance of any zero-mean excess return and the F frontier portfolio is zero (and orthogonal)

$$\mathrm{E}(R) = 0 \implies \mathrm{cov}(R, R_p) = \mathrm{E}(RR_p) = 0 \qquad \forall R \in \mathsf{X}_0,\ R_p \in \mathsf{F} \tag{1.35}$$

1.5 Two Fund Separation

Whether it's frontier F_* or F, given two distinct frontier portfolios R_p and R_q ($R_p \neq R_q$), then portfolio R is on the frontier if and only if portfolio R is the affine combination of frontier portfolios R_p and R_q

$$R = \tau R_p + (1-\tau)R_q = R_q + \tau \cdot (R_p - R_q) \tag{1.36}$$

where $\tau = \tau_{pq}(\mu)$ is a function of portfolios R_p, R_q and R

$$\tau_{pq}(\mu) \equiv \frac{\mu - \mu_q}{\mu_p - \mu_q} = \frac{\beta(\mu) - \beta(\mu_q)}{\beta(\mu_p) - \beta(\mu_q)} = \frac{\mathrm{E}(RR_p) - \mathrm{E}(R_pR_q)}{\mathrm{E}(R_p^2) - \mathrm{E}(R_pR_q)} = \frac{\mathrm{cov}(R, R_p) - \mathrm{cov}(R_p, R_q)}{\mathrm{var}(R_p) - \mathrm{cov}(R_p, R_q)} \tag{1.37}$$

Furthermore, on frontier F_*

$$\tau_{pq}(\mu) = \frac{\theta_*(\mu) - \theta_*(\mu_q)}{\theta_*(\mu_p) - \theta_*(\mu_q)}$$

and if on frontier F

$$\tau_{pq}(\mu) = \frac{\theta(\mu) - \theta(\mu_q)}{\theta(\mu_p) - \theta(\mu_q)} = \frac{\sigma - \sigma_q}{\sigma_p - \sigma_q}$$

Eq (1.36) shows that the portfolio frontier is spanned by two funds (portfolios).

On frontier F, let $R_p = R_\wr$, and $R_q = R_0$, then $\tau_{pq}(\mu) = \beta(\mu)$, Eq (1.36) becomes

$$R = R_0 + \beta(\mu)(R_\wr - R_0) \qquad \forall R \in \mathsf{F}$$

written in form of weight vector, Eq (1.26) takes the following form

$$\begin{bmatrix} z_0(\mu) \\ \mathbf{z}(\mu) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} + \beta(\mu) \cdot \left(\begin{bmatrix} 0 \\ \mathbf{z}_\wr \end{bmatrix} - \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \right) = \begin{bmatrix} 1 - \beta(\mu) \\ \beta(\mu) \cdot \mathbf{z}_\wr \end{bmatrix} \tag{1.38}$$

It shows that the portfolio on F consists of risk-free bond and tangency portfolio. Investors with mean-variance preference only choose the portfolio on the efficient frontier. Therefore, the decision-making[2] of investment only needs to determine the proportion of risk-free bond, and the rest should simply be put into the tangency portfolio.

1.6 Zero-covariance Decomposition

Given frontier portfolio p on F_*, if $\mu_p \neq \mu_o$, then there exists a unique frontier portfolio q such that $\mathrm{cov}(R_p, R_q) = 0$. portfolio q is called the zero-covariance portfolio relative to p, with

$$\mu_q = \frac{c - b\mu_p}{b - a\mu_p} \qquad v_q = \frac{v_p}{av_p - 1} \qquad \mathbf{z}_q = \frac{1}{b - a\mu_p}\mathbf{V}^{-1}(\boldsymbol{\mu} - \mu_p\mathbf{1}) \tag{1.39}$$

[2] Tobin (1958) finds that the weight vector of risky assets is fixed, as if there were a single composite risk asset (p.84). He first proposes that asset allocation can be divided into two stages, first between the two types of assets of risk-free bond and risky securities, and then within the risky securities (p.85)

p and q are zero-covariance portfolio mutually relative to each other, and

$$(a\mu_p - b)(a\mu_q - b) = b^2 - ac < 0$$

which shows that p and q are on different sides of the global minimum variance portfolio o.

In particular, let R_Z be the zero-covariance portfolio relative to tangency portfolio $R_\wr$, we have

$$\mu_Z = R_0 \qquad v_Z = \frac{h(R_0)}{ac - b^2} \qquad \mathbf{z}_Z = \frac{aR_0 - b}{ac - b^2}\mathbf{V}^{-1}\boldsymbol{\mu} + \frac{c - bR_0}{ac - b^2}\mathbf{V}^{-1}\mathbf{1}$$

In Eq (1.36), let $R_p = R_\wr$, and $R_q = R_Z$, then for any frontier portfolio R_l on F_*, there is $\tau_{\wr Z}(\mu_l) = \beta(\mu_l)$, and thus

$$R_l = \beta(\mu_l)R_\wr + (1 - \beta(\mu_l))R_Z \qquad \forall R_l \in \mathsf{F}_*$$

For any portfolio $R_n \in \mathsf{X}_{*1}$ in market M_*, let frontier portfolio $R_l \in \mathsf{F}_*$ with $\mu_l = \mu_n = \mathrm{E}(R_n)$. Denote $R_{*e} = R_n - R_l$, for $\beta(\mu_n) = \beta(\mu_l)$, we have

$$R_n = R_l + R_{*e} = \beta(\mu_n)R_\wr + (1 - \beta(\mu_n))R_Z + R_{*e} = R_Z + \beta(\mu_n)(R_\wr - R_Z) + R_{*e} \quad (1.40)$$

Since

$$\mathrm{cov}(R_\wr, R_Z) = \mathrm{cov}(R_Z, R_{*e}) = \mathrm{cov}(R_\wr, R_{*e}) = 0$$

Eq (1.40) is the zero-covariance decomposition on tangency portfolio.

On frontier F, the risk-free bond R_0 is the zero-covariance portfolio relative to any portfolio. For any portfolio $R_n \in \mathsf{X}_1$ in market M, let $R_e = R_n - R_l$, where $R_l \in \mathsf{F}$ and $\mu_l = \mu_n = \mathrm{E}(R_n)$. Similar to Eq (1.40), let's take $R_p = R_\wr$ and $R_q = R_0$ in Eq (1.36), we have the following zero-covariance decomposition

$$R_n = R_l + R_e = \beta(\mu_n)R_\wr + (1 - \beta(\mu_n))R_0 + R_e = R_0 + \beta(\mu_n)(R_\wr - R_0) + R_e$$

where $\beta(\mu_n) = \tau_{\wr 0}(\mu_n) = \frac{\mu_n - R_0}{\mu_\wr - R_0}$. For any primitive security (stock) i with return R_i, comparing Eq (1.40), we have

$$R_{e_i} - R_{*e_i} = (1 - \beta(\mu_i))(R_Z - R_0) \quad (1.41)$$

Which is the excess return between the frontier portfolios on F and F_* that have the same mean as stock i.

2 CAPM Formula

Investors using mean-variance analysis only hold efficient portfolios. According to the portfolio separation, investors only need to properly allocate funds between risk-free bond and tangency portfolio (Eq 1.38). The optimal portfolio held by any mean-variance investor is a combination of R_0 and $R_\wr$. Thus, when the market is in equilibrium, the market portfolio R_M must be equal to the tangency portfolio $R_\wr$. Accordingly, the analysis of market equilibrium can be divided into two parts.

1. Risky securities: Given the total value of risky securities (how much capital is allocated to overall risky securities), the price vector of risky securities is calculated. Specifically, M_* equilibrium is to calculate the price vector $\mathbf{p}$ (or $\boldsymbol{\mu}$) of risky securities through the equilibrium condition $R_M = R_\wr$, and we set $R_M = X_M/P_M$ where P_M is given. The equilibrium condition of market M_* is that the market portfolio is equal to the tangency portfolio, and all risky securities are treated as a whole to achieve equilibrium. From the perspective of return, the M_* equilibrium is to find return vector $\mathbf{r}$ given R_M
2. Risk-free bond: Given the price vector of risky securities, find the net holding value of risk-free bond. Specifically, given the price vector $\mathbf{p}$ of risky securities, each investor calculates the optimal holding value of purely risky securities (equivalent to solving the optimal value of risk-free bond held) according to its endowment and preference, and then sum them up to be the total value of risky securities, P_M. From the perspective of return, the equilibrium of this part is to find R_M given return vector $\mathbf{r}$

These two parts are interwoven. When the market M is in equilibrium, the price of the risky securities obtained by the M_* equilibrium is exactly equal to the input price for the risk-free bond allocation, and the total value of the risky securities obtained from the risk-free bond allocation is equal to the value required by M_* equilibrium. M_* equilibrium is a partial equilibrium of all risky securities in market M. In this section, we first analyze the partial equilibrium of the overall risky securities. The equilibrium of market M needs to add the risk-free bond, and consider all individual investor's endowments and preferences, which will be discussed in the next section.

2.1 Partial Equilibrium of Risky Securities

Let's define market beta (also beta value, or beta coefficient) of stock i to be

$$\beta_i \equiv \frac{\mathrm{cov}(R_i, R_M)}{\mathrm{var}(R_M)}$$

If market M_* is in equilibrium, $R_M = R_l$, for any stock i, market beta β_i equals to tangency beta $\beta(\mu_i)$

$$\beta_i = \frac{\mathrm{cov}(R_i, R_l)}{\mathrm{var}(R_l)} = \beta(\mu_i) = \frac{\mu_i - R_0}{\mu_l - R_0} = \frac{\mu_i - R_0}{\mu_M - R_0} \tag{2.1}$$

Which indicates

$$\mu_i - R_0 = \beta_i(\mu_M - R_0) \qquad i = 1, 2, \cdots, N \tag{2.2}$$

This happens to be the classic CAPM formula. Besides, by zero-covariance decomposition (1.40), the excess return of stock i becomes

$$R_i - R_0 = \beta_i(R_M - R_0) + (1 - \beta_i)(R_Z - R_0) + R_{*e_i} \tag{2.3}$$

We arrive the CAPM formula by taking expectation. The CAPM formula describes the zero-covariance structure of expected return within partial equilibrium of risky securities. The excess return R_{*e_i} is regarded as the error term in the context of regression. In fact, it is the horizontal distance between return R_i and frontier F_*: let $R_l \in \mathsf{F}_*$ and $\mu_l = \mu_i$, then $R_{*e_i} = R_i - R_l$. The economic meaning of R_{*e_i} is the portion diversifiable without risk-free bond. Substituting Eq (1.41) into Eq (2.3) gives

$$R_i - R_0 = \beta_i(R_M - R_0) + R_{e_i}$$

Similarly, the excess return R_{e_i} is the horizontal distance between return R_i and frontier F.

2.2 Price of Risky Securities

Since the price vector of the primitive securities is assumed to be positive when using return, given the mean and variance of future payoff, the security price and the expected return are informationally equivalent, that is, if we know either, we know both. Thus, the CAPM formula (2.2) in form of expected return has a form in price, the certainty equivalent pricing formula

$$P_i = \frac{1}{R_0}\left(\mathrm{E}(X_i) - \frac{\mathrm{E}(R_M) - R_0}{\sigma_M^2}\mathrm{cov}(X_i, R_M)\right) \qquad i = 1, 2, \cdots, N \tag{2.4}$$

which can be rewritten equivalently in form of payoff[3]

$$P_i = \frac{1}{R_0}\left(\eta_i - \frac{E - P_M R_0}{Q} Q_i\right) \qquad i = 1, 2, \cdots, N \tag{2.5}$$

where $Q_i = \mathrm{cov}(X_i, X_M)$. In our market setting, stock i's price P_i is known once the total value of risky securities P_M is given. Please note that formula (2.5) is just the payoff expression for partial equilibrium condition of risky securities. The left and right sides of the formula, P_i

[3] See Eq (1) in Jensen and Long (1972, p.153), or Eq (7.23) in Fama and Miller (1972, p.296).

and P_M, are interdependent and cannot be interpreted as a linear pricing formula. There is no doubt that formula (2.5) is a necessary constraint of equilibrium price. Diversification makes risky securities closely linked through CAPM equilibrium, so it cannot be priced in isolation, but must be priced as a whole (Rubinstein, 2002, p.1043).

Eq (2.5) is a system of linear equations on price $\mathbf{p}$, we find that its solution space is one-dimensional:

Proposition 2 (general solution to asset prices in CAPM): In market M, when all investors follow the mean-variance criterion, the price vector of risky securities must be

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\boldsymbol{\Omega}\mathbf{1} \qquad x \in \mathbb{R} \tag{2.6}$$

The price vector of risky securities is determined by the equilibrium of the market M, and x in Eq (2.6) is a semi-feasible equilibrium solution (see Proposition 4), which is affected by the endowment and/or preference of any individual investor. If $x = 0$, the formula (2.6) is the discounted expected pricing, and the market is risk neutral. We see that the total value of risky securities is a linear function of x

$$P_M = \mathbf{1}'\mathbf{p} = \frac{\mathbf{1}'\boldsymbol{\eta}}{R_0} + x\mathbf{1}'\boldsymbol{\Omega}\mathbf{1} = \frac{QR_0x + E}{R_0} \tag{2.7}$$

The price formula (2.6) is equivalent to CAPM formula, and the total value of risky securities is given in Eq (2.7). At this time, we see more clearly that the CAPM formula is only a partial equilibrium relationship of the purely risky securities. The total market value of the purely risky securities needs to be given in order to determine the specific equilibrium price. Otherwise, we can only obtain the general solution as in Eq (2.6). Since asset prices in CAPM must satisfy Eq (2.6), we have

$$a = \mathbf{p}'\boldsymbol{\Omega}^{-1}\mathbf{p} = \frac{QR_0^2x^2 + 2ER_0x + c}{R_0^2}, \qquad b = \mathbf{p}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} = \frac{ER_0x + c}{R_0} \tag{2.8}$$

Thus

$$\begin{aligned} aR_0 - b &= QR_0x^2 + Ex = P_M R_0 x \\ bR_0 - c &= ER_0x \end{aligned}$$

and

$$h(R_0) = aR_0^2 - 2bR_0 + c = QR_0^2x^2$$

2.3 Expected Return and Beta

By definition, the expected return of stock i is

$$\mu_i = \mathrm{E}(R_i) = \frac{\mathrm{E}(X_i)}{P_i} = \frac{\eta_i}{P_i} \qquad i = 1, 2, \cdots, N \tag{2.9}$$

and the beta value is

$$\beta_i = \frac{\mathrm{cov}(R_i, R_M)}{\mathrm{var}(R_M)} = \frac{Q_i}{Q} \cdot \frac{P_M}{P_i} \qquad i = 1, 2, \cdots, N \tag{2.10}$$

Let $D_i = \frac{\eta_i}{Q_i}$, there is

$$\frac{\mu_i}{\beta_i} = \frac{Q}{P_M} D_i \qquad i = 1, 2, \cdots, N \tag{2.11}$$

Noting that Eq (2.11) comes directly from the definition of μ_i and β_i, it holds for any presentable price vector, even the market is not in equilibrium. However, CAPM formula comes from the M_* equilibrium price. Substituting M_* equilibrium price Eq (2.5) into Eq (2.9) and (2.10), we have

$$\mu_i = \frac{Q R_0 D_i}{P_M R_0 - E + Q D_i} \qquad i = 1, 2, \cdots, N \tag{2.12}$$

and (Johnstone, 2017, p.502)

$$\beta_i = \frac{P_M R_0}{P_M R_0 - E + Q D_i} \qquad i = 1, 2, \cdots, N \tag{2.13}$$

Joining Eq (2.12) and (2.13), there will be the security market line (SML)

$$\frac{\mu_i - R_0}{\beta_i} = \mu_M - R_0 = \frac{\mu_j - R_0}{\beta_j} \qquad i, j = 1, 2, \cdots, N \tag{2.14}$$

which verifies CAPM formula (2.2).

Proposition 3 (CAPM): If all investors use the mean-variance criterion, the following statements are equivalent in market M

1. CAPM formula: risk premium (excess return) is proportional to market beta

   $$\boldsymbol{\mu} - R_0\mathbf{1} = (\mu_M - R_0)\boldsymbol{\beta}$$

   where μ_M is the expected return of market portfolio R_M, and $\boldsymbol{\beta}$ is the market beta

   $$\boldsymbol{\beta} = \frac{\mathrm{cov}(\mathbf{r}, R_M)}{\mathrm{var}(R_M)} = \frac{P_M}{Q}\mathbf{P}^{-1}\boldsymbol{\Omega}\mathbf{1}$$

   CAPM formula is often written in the form of individual assets, such as Eq (2.2) or Eq (2.14)

2. Equilibrium of purely risky securities (M_* equilibrium): the market portfolio is equal to tangency portfolio, $R_M = R_\wr$

3. The market portfolio is on frontier F: $R_M \in \mathsf{F}$

4. Capital market line (CML): for any efficient portfolio $R \in \mathsf{F}$, there is

$$\frac{\mu - R_0}{\sigma} = \frac{\mu_M - R_0}{\sigma_M} \tag{2.15}$$

where μ and σ are the mean and standard deviation of return R respectively

5. Price vector of risky securities follows Eq (2.6), with total value of risky securities given in Eq (2.7)

From Eq (2.13) and Proposition 3 we see that: Beta value is not a physical attribute of primitive securities[4]. CAPM formula is equivalent to $R_M = R_\wr$ in market M_*, a partial equilibrium of the entire market M, where the market return R_M is exogenous. However, in the mean-variance equilibrium of market M, the market return $R_M = X_M/P_M$ is endogenous. Even if there is nothing change in the payoff of the security itself, any variables that can affect the total market value of risky securities, P_M, is possible to change the beta of the security by Eq (2.13). For instances

1. Modifying R_0, the risk-free return (change P_M by Eq 2.7)
2. Changes in any individual investor's preference and/or endowment will change the equilibrium total market value P_M, which we will further analyze in §3 (equilibrium solution x in Eq 2.7)
3. When the payoff of some other stock j is changed or a new stock is added to the market, due to the integrity of the pricing, the beta of stock i is usually modified. Please see the analysis in §5.2.2

In addition, there is a possibility of multiple equilibria, such as μ_i and β_i in Table 1 of Example 4.1 taking multiple values consistent with the CAPM formula.

In short, the CAPM formula (2.2) as a result of partial equilibrium of purely risky securities cannot be interpreted as μ_i being determined by β_i. In fact, given future payoffs and risk-free interest rate, the μ_i and β_i of stock i are jointly determined by the equilibrium total market value P_M (Eq 2.12 and 2.13) of all securities, not β_i determines μ_i or μ_i determines β_i (Example 5.2 shows a situation where μ_i remains unchanged while β_i is changed).

Logically, from the definition $\beta_i = \frac{\text{cov}(R_i, R_M)}{\text{var}(R_M)}$, beta is derived from the return, how can we reversely think that the return (knowing μ_i is equivalent to knowing R_i) is determined by

[4] Beta has been entrenched as a characteristic of primitive assets, even derivatives. Black and Scholes (1973) use "option's beta" as an alternative method to find out the Black-Scholes partial differential equation (PDE).

beta? We are misled by the risk dogma that the expected return is determined by its risk. The so-called systematic risk β_i here is specious. It's time to get rid of this misinterpretation of circular argument to cater to the risk dogma.

2.4 Comments

Markowitz (1952) divides the portfolio selection process into two stages. The first stage forms the beliefs about the future performances of securities, the second stage chooses the proportionate composition of portfolio. Markowitz (1952) considers the second stage in the process of selecting a portfolio. Which starts with the given mean vector and variance matrix of return about the securities involved and ends with the selection of a mean-variance efficient portfolio (minimum variance for given expected return or more and maximum expected return for given variance or less). Given the first two moments of the returns of risky securities, Tobin (1958) adds risk-free bond, and shows that the mean-variance investors hold the same risky portfolio, and the selection of fixed proportions within risky portfolio is separate from the decision on the allocation proportion of risk-free bond (portfolio separation). Based on these findings, Sharpe (1964) starts with the capital market line (Fig 1, p.426), and ends with the discovery of security market line (Fig 9, p.440): The expected return on an asset is proportional to its beta[5]. This linear relationship is the so-called CAPM formula. Because the CAPM formula is in line with the intuition for risk premium (higher risk requires higher expected return), it quickly attracted the attention of a lot of researchers and the financial professionals. For examples, Jensen (1968) defines α to measure fund performance, and financial industry rush for β after the agitation by Welles (1971). Jensen (1972) summarizes the empirical studies on CAPM, he mentions that Douglas (1969) and Jacob (1971) find that market data go against the CAPM formula. Jensen (1972) concludes that the expected return of a security has a positive correlation with its beta but deviates from the CAPM formula.

Now we know that the CAPM formula is the result of the partial equilibrium of risky securities, which is equivalent to the market portfolio being a frontier portfolio. Analytical solution to the portfolio frontier was not made public until Merton (1972), which was eight years after the CAPM formula was discovered. Before that, researchers could only describe

[5]Beta is defined as the covariance of the asset return with the market return, divided by the variance of market return. Fama (1968) firstly uses symbol β for modern form of CAPM.

portfolio frontier by drawing what they feel. Continuing this hazy understanding, the perception that expected return is determined by its systematic risk (beta) becomes a belief. Fama and French (1992) declares that CAPM is dead; Black (1993) refutes that the announcements of the death of beta seem premature; CAPM is dead or alive, Fama and French (1996, 2004), Perold (2004), Levy and Roll (2012) and Lai (2015) are still debating. A special supplement in *Abacus* (Vol 49#S1) collects debates over support and opposition to beta around Dempsey (2013). And a special section in *Review of Financial Studies* (Vol 29#1) consists of three studies on the cross-section of expected returns prompted by Karolyi (2016).

We need to look back and see if the basis of these arguments is based on solid rocks or seemingly stable quicksand. Are there any important details that have been overlooked? Lintner (1965, p.25) reaches the CAPM formula by assuming that current security prices are exogenous data. Hansen and Richard (1987, p.587) points out that security prices are endogenous, Coles and Loewenstein (1988, p.285) reminds that returns depend on the endogenously determined equilibrium asset prices. Despite these facts, the partial equilibrium truth of CAPM are not well recognized. Johnstone (2017, p.503) spots the logical circularity built into the CAPM formula. Only when we realize that CAPM is only a partial equilibrium given risk-free return and market return, can we be relieved of the illusory market beta.

3 CAPM Equilibrium: Security Prices

Although the way of market clearing in quantity (quantity supplied is equal to quantity demanded) can still be used in capital market equilibrium, however, market clearing in quantity obscures the important characteristics of CAPM equilibrium in capital market, *the clearing in value*. For market clearing in value, the total value of risky assets is equal to the sum of all individual investor's optimal holding value of risky assets, and the total wealth of all investors is equal to the sum of the total market value of risky securities and the net holding value of risk-free bond. In particular, in the market of purely risky securities, the clearing of value derives from diversification. Where the value of each security is equal to the total funds allocated by investors to that security, the market portfolio becomes the tangency portfolio. In this sense, the essence of CAPM equilibrium is the allocation of funds.

3.1 Mean-Variance Decision

Assuming that investor k has initial wealth W_k, whose portfolio payoff being $X = h_0 X_0 + \mathbf{h}'\mathbf{x}$, with mean η, and variance q. Investor k is a mean-variance investor, whose merit function $m_k(X) = m_k(\eta, q)$ satisfies

$$\partial m_k / \partial \eta > 0 \qquad \partial m_k / \partial q < 0$$

In other words: Fixed the initial input $W_k > 0$, given a risk level (variance of payoff), the higher the expected payoff the better, and given an expected payoff, the lower the risk the better. As

$$m_k(\eta, q) = m_k \left(\mathbf{h}'\boldsymbol{\eta} + (W_k - \mathbf{h}'\mathbf{p}) R_0, \mathbf{h}'\boldsymbol{\Omega}\mathbf{h} \right)$$

the optimal portfolio of risky assets should be

$$\mathbf{h}_k = g_k \boldsymbol{\Omega}^{-1}(\boldsymbol{\eta} - R_0 \mathbf{p}) = g_k \mathbf{P}^{-1}\mathbf{V}^{-1}(\boldsymbol{\mu} - R_0 \mathbf{1}) \tag{3.1}$$

with *trade-off coefficient*

$$g_k = -\frac{1}{2} \cdot \frac{\partial m_k / \partial \eta}{\partial m_k / \partial q} > 0 \tag{3.2}$$

Thus, the optimal holding value of risky assets for investor k is

$$w_k = \mathbf{p}'\mathbf{h}_k = \mathbf{p}' g_k \boldsymbol{\Omega}^{-1}(\boldsymbol{\eta} - R_0 \mathbf{p}) = (b - aR_0) g_k > 0 \tag{3.3}$$

It happens that

$$\frac{1}{w_k} \mathbf{P}\mathbf{h}_k = \mathbf{z}_\wr$$

The portfolio of purely risky securities held by investors is exactly the tangency portfolio, which is independent of the investor's trade-off coefficient g_k. Therefore, the market portfolio must be equal to the tangency portfolio, $\mathbf{z}_M = \mathbf{z}_\wr$. The expectation and variance of payoff for investor's optimal portfolio are

$$\eta = \mathbf{h}_k'\boldsymbol{\eta} + (W_k - \mathbf{h}_k'\mathbf{p}) R_0 = W_k R_0 + g_k h(R_0) \tag{3.4}$$

$$q = \mathbf{h}_k' \boldsymbol{\Omega} \mathbf{h}_k = g_k^2 h(R_0)$$

and the mean-variance frontier of payoff is

$$q = \frac{(\eta - W_k R_0)^2}{h(R_0)}$$

Comparing Eq (1.27) we find that the payoff frontier and the return frontier are equivalent.

For investor k, the proportion of risk-free bond is

$$z_{k,0} = \frac{W_k - w_k}{W_k} = 1 + \frac{g_k}{W_k}(aR_0 - b)$$

Since market portfolio is equal to the tangency portfolio, by Eq (2.8) we have

$$z_{k,0} = 1 + \frac{g_k}{W_k}(Q R_0 x^2 + E x) = 1 + \frac{g_k}{W_k} P_M R_0 x \tag{3.5}$$

3.2 Trade-off Coefficient

Let the merit function for investor k be

$$m_k(X) = m_k(\eta, q) = \eta - \frac{G_k}{2W_k} q \tag{3.6}$$

where $W_k > 0$ is the wealth of investor k, $G_k > 0$ is the preference parameter. We see that

$$\frac{\partial m_k}{\partial \eta} = 1 > 0 \qquad \frac{\partial m_k}{\partial q} = -\frac{G_k}{2W_k} < 0$$

By Eq (3.2)

$$g_k = \frac{W_k}{G_k} \tag{3.7}$$

the trade-off coefficient is only related to the endowment and preference of investor, and is independent of the payoff of securities. We call this type of trade-off coefficient a *simple trade-off coefficient*.

Quadratic utility functions on the return are often employed in a mean-variance decision. Let the utility function for investor k be $U_k(R) = R - \frac{1}{2} G_k R^2$, then her merit function is

$$m_k(X) = m_k(\eta, q) = \mathrm{E}(W_k U_k(R)) = \eta - \frac{G_k}{2W_k}\eta^2 - \frac{G_k}{2W_k} q \tag{3.8}$$

with trade-off coefficient being $g_k = \frac{W_k}{G_k} - \eta$. Joining Eq (3.4) gives

$$g_k = \frac{W_k}{1 + h(R_0)} \left(\frac{1}{G_k} - R_0 \right) \tag{3.9}$$

If $0 < G_k < \frac{1}{R_0}$, then $g_k > 0$, investors will not violate the mean-variance rule. The trade-off coefficient g_k in Eq (3.9) is called *quadratic trade-off coefficient*, it relates not only to investor's endowment and preference, but also to equilibrium prices ($h(R_0) = QR_0^2 x^2$, a quadratic function of semi-feasible equilibrium solution x).

3.3 Equilibrium Price

Let the total wealth of all investors be $W = \sum_{k=1}^{I} W_k$, and the net holding amount of risk-free bond is

$$B = \sum_{k=1}^{I} (W_k - w_k) = W - \sum_{k=1}^{I} w_k = W - (b - aR_0) \sum_{k=1}^{I} g_k$$

If price vector $\mathbf{p}$ in CAPM general solution (2.6) satisfies $W = P_M + B$, it is called mean-variance semi-feasible equilibrium price. Evidently, semi-feasible equilibrium price must meet the following equation on x

$$xR_0 \sum_{k=1}^{I} g_k = -1 \tag{3.10}$$

A positive semi-feasible equilibrium price is called presentable CAPM price. Whenever there exists presentable CAPM price, the market is in mean-variance equilibrium, or CAPM equilibrium.

Proposition 4 (semi-feasible equilibrium price): In market M, the semi-feasible equilibrium prices follow Eq (2.6)

1. When all investors employ simple trade-off coefficients, the semi-feasible equilibrium solution will be
$$x = -\frac{1}{GR_0} \tag{3.11}$$
where $G = \sum_{k=1}^{I} g_k = \sum_{k=1}^{I} \frac{W_k}{G_k}$
2. When all investors employ quadratic trade-off coefficients, let $G = \sum_{k=1}^{I} \frac{W_k}{G_k} \neq \sum_{k=1}^{I} g_k$, if $(WR_0 - G)^2 < 4Q$, market equilibrium does not exist. If $(WR_0 - G)^2 \geqslant 4Q$, the semi-feasible equilibrium solution will be
$$x = \frac{WR_0 - G \pm \sqrt{(WR_0 - G)^2 - 4Q}}{2QR_0} \tag{3.12}$$

When all investors employ simple (or quadratic) trade-off coefficients, substituting x in Eq (3.11) (or 3.12) into Eq (2.6) we get $\mathbf{p}$. If $\mathbf{p} > 0$ then the CAPM equilibrium exists. Of course, investors can mix simple trade-off coefficients and quadratic trade-off coefficients, or even other forms of trade-off coefficients, we can still solve the equilibrium price by Eq (3.10). For any CAPM equilibrium price

$$P_i = \frac{\eta_i}{R_0} + xQ_i > 0 \qquad\qquad i = 1, 2, \cdots, N \tag{3.13}$$

We see that the price of stock i, P_i, is determined by the following factors: the risk-free rate R_0, the expectation of payoff η_i, the covariance of payoff $Q_i = \mathrm{cov}(X_i, X_M)$, and the equilibrium solution x that involves the endowments and preferences of all participants in the market (Eq 3.12 using the quadratic trade-off coefficient also involves the variance of the total market payoff). Obviously, the price of security cannot be priced in isolation, otherwise it violates the joined-up thinking contained in diversification. As long as the mean-variance diversification method is considered, risky securities are held in the form of a portfolio, the risky securities must be priced as a whole. If the payoff of firm i is independent of other firms' payoffs (or weakened to be uncorrelated), then $P_i = \frac{\mathrm{E}(X_i)}{R_0} + x\mathrm{var}(X_i)$. When all investors employ simple trade-off coefficients, the price P_i can be increased by raising the expectation of its payoff or reducing the variance of its payoff. In this case, the pricing of asset i has a high degree of autonomy, but the asset price is still affected by the equilibrium solution x, and the pricing of individual asset cannot be handled in isolation.

If the traditional market clearing in quantity method is used, the market clearing condition is $\sum_{k=1}^{I} \mathbf{h}_k = \mathbf{1}$ (the supply of each purely risky security is standardized to 1). By Eq (3.1) we have (Nielsen, 1992, p.805; Bossaerts et al., 2007, p.999)

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} - \frac{1}{R_0 \sum_{k=1}^{I} g_k} \boldsymbol{\Omega} \mathbf{1}$$

Which sets $x = -\frac{1}{R_0 \sum_{k=1}^{I} g_k}$ in Eq (2.6) as Eq (3.10) does. However, the benefit of clearing in value is to highlight the allocation equilibrium, where the pricing of individual asset is automatically realized in the process of allocation of funds.

3.4 Equilibrium Return

In equilibrium, the market portfolio is the tangency portfolio

$$\mathbf{z}_M = \mathbf{z}_\wr = \frac{1}{P_M} \mathbf{p} = \frac{\boldsymbol{\eta} + R_0 x \boldsymbol{\Omega} \mathbf{1}}{Q R_0 x + E} \tag{3.14}$$

The market return is $R_M = \frac{X_M}{P_M}$. Let $D = E/Q$, then the market return has expectation

$$\mu_M = \frac{E}{P_M} = \frac{R_0 D}{R_0 x + D}$$

and variance

$$v_M = \frac{Q}{P_M^2} = \left(\frac{R_0}{Q R_0 x + E} \right)^2 Q = \frac{1}{Q} \left(\frac{R_0}{R_0 x + D} \right)^2$$

Noting that Q is a constant, the greater the total market value, the less the volatility of market return (market payoff X_M is exogenous, but the market return R_M is endogenous).

In equilibrium, the price of stock i is P_i in Eq (3.13), thus the return on stock i is

$$R_i = \frac{X_i}{P_i} = \frac{R_0 X_i}{Q_i R_0 x + \eta_i}$$

with its mean being

$$\mu_i = \frac{R_0 D_i}{R_0 x + D_i} \tag{3.15}$$

and its beta being

$$\beta_i = \frac{R_0 x + D}{R_0 x + D_i} \tag{3.16}$$

When all investors employ simple trade-off coefficients, by Eq (3.11), there will be

$$\mu_i = \frac{G D_i}{G D_i - 1} R_0 \qquad \beta_i = \frac{G D - 1}{G D_i - 1}$$

In this scenario, β_i is not affected by R_0, because R_0 is inversely proportional to equilibrium solution x (Eq 3.11) and is finally cancelled out.

Since the asset return depends on the equilibrium solution x, the expectation and variance of the return cannot be determined by the asset itself. It is affected by each asset's payoff and each investor's endowment and preference. Hence, *the mean and variance of a return are not the identities of an asset*. Noting that $c = \boldsymbol{\eta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} = \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu}$ is a constant, it is not affected by the equilibrium price. There is always a constraint between the mean and variance of the return on risky securities. Risky securities form a whole for pricing, and any individual security cannot be isolated. As a result of equilibrium, beta is endogenous. The practice of taking β_i as the systematic risk of stock i to determine the expected return μ_i, is plainly a mistake of separating an individual security from the whole portfolio.

4 CAPM Equilibrium: Numerical Examples

Numerical examples are more intuitive and helpful to understand the CAPM equilibrium analysis. We first analyze the case of cash endowment, and then we use a numerical example to show that the CAPM equilibrium may violate the principle of no arbitrage. Finally, impossible frontier designated by Brennan and Lo (2010) seems to put the CAPM formula to death, and we use numerical examples to reveal the source of its error.

4.1 Cash Endowment

Assuming that the endowments are all in cash (risk-free bond), investors buy stocks from the secondary market.

Example 4.1 (equilibrium price): Given $R_0 = \frac{21}{20}$ for risk-free bond, and $N = 3$ stocks with

$$\boldsymbol{\eta} = \frac{52}{79}\begin{bmatrix} 218 \\ 92 \\ 1547 \end{bmatrix} \qquad \boldsymbol{\Omega} = \frac{540\,800}{6241}\begin{bmatrix} 2 & -1 & 13 \\ -1 & 2 & -13 \\ 13 & -13 & 338 \end{bmatrix}$$

Assuming that there are $I = 4$ investors, whose cash endowments are

$$[W_1; W_2; W_3; W_4] = [100; 200; 300; 600]$$

In these settings, $c = \boldsymbol{\eta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\eta} = \frac{19\,311}{160}$, the expectation of total payoff of stocks is $E = \mathbf{1}'\boldsymbol{\eta} = \frac{96\,564}{79}$, the variance of total payoff of stocks is $Q = \mathbf{1}'\boldsymbol{\Omega}\mathbf{1} = \frac{214\,156\,800}{6241}$, and investors' total wealth is $W = \sum_{k=1}^{I} W_k = 1200$.

4.1.1 Simple Trade-off Coefficient

Investors all use merit function (3.6), with preference coefficients being

$$[G_1; G_2; G_3; G_4] = \left[\frac{79}{18}; \frac{79}{26}; \frac{79}{30}; \frac{79}{8}\right] = [4.389; 3.038; 2.633; 9.875]$$

then $G = \sum_{k=1}^{I} \frac{W_k}{G_k} = \frac{20\,800}{79}$. By Eq (3.11) we have $x = -\frac{1}{GR_0} = -\frac{79}{21\,840}$, and equilibrium prices (more on left part of Table 1)

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\boldsymbol{\Omega}\mathbf{1} = \frac{1040}{79}\begin{bmatrix}10\\4\\65\end{bmatrix} = \begin{bmatrix}131.65\\52.66\\855.70\end{bmatrix}$$

Thus, the total market value of risky assets is $P_M = \mathbf{1}'\mathbf{p} = 1040$, and betas are

$$\boldsymbol{\beta} = \frac{P_M}{Q}\mathbf{P}^{-1}\boldsymbol{\Omega}\mathbf{1} = \frac{79}{495}[2; 5; 7] = [0.3192; 0.7980; 1.117]$$

From Eq (3.5), the weights of risk-free bond are $[z_{1,0}; z_{2,0}; z_{3,0}; z_{4,0}] = [0.1; -0.3; -0.5; 0.6]$. The net holding value of risk-free bond is $B = W - P_M = 160$.

Some textbooks claim that the risk-free bond is market cleaning in CAPM formula, that is, $B = 0$. Which implies that the risk-free rate R_0 is also determined endogenously

$$R_0 = \frac{GE - Q}{GW} = \frac{91}{100} < 1$$

which is not in line with usual financial market practice.

4.1.2 Quadratic Trade-off Coefficient

Investors all use merit function (3.8), with preference coefficients being

$$[G_1; G_2; G_3; G_4] = \left[\frac{3950}{5493}; \frac{3950}{6091}; \frac{395}{639}; \frac{7900}{9491}\right] = [0.719; 0.648; 0.618; 0.832]$$

then $G = \sum_{k=1}^{I} \frac{W_k}{G_k} = \frac{130\,636}{79}$. By Eq (3.12) we have $x_1 = -\frac{79}{21\,840}$, and $x_2 = -\frac{395}{54\,054}$. The two sets of equilibrium solutions are listed in Table 1: Comparing to the first solution (same as simple trade-off coefficient case), the second solution leads to a lower total market value ($\frac{\mathrm{d}P_M}{\mathrm{d}x} = Q > 0$)

- Volatility of market portfolio increases, by $\sigma_M = \sqrt{Q}/P_M$
- The price of each security goes down, which is consistent with Eq (3.13): $\frac{\mathrm{d}P_i}{\mathrm{d}x} = Q_i > 0$, for any i, there is $Q_i > 0$
- The expected return of any security increases. Since $D_i > 0$ for any i, Eq (3.15) shows that $\frac{\mathrm{d}\mu_i}{\mathrm{d}x} = -\frac{R_0^2 D_i}{(D_i + xR_0)^2} < 0$. The expected return changes inversely to the total market value

Table 1: Equilibrium under Quadratic Trade-off Coefficient

	$x_1 = -\frac{79}{21\,840}$	$x_2 = -\frac{395}{54\,054} < x_1$
P_M	1040	$\frac{1\,515\,280}{1659} = 913.37$
μ_M	$\frac{1857}{1580} = 1.175$	$\frac{38\,997}{29\,140} = 1.338$
σ_M	$\frac{3}{79}\sqrt{22} = 0.178\,1$	$\frac{63}{1457}\sqrt{22} = 0.202\,8$
$\mathbf{p}$	$\frac{1040}{79}\begin{bmatrix}10\\4\\65\end{bmatrix} = \begin{bmatrix}131.65\\52.66\\855.70\end{bmatrix}$	$\frac{1040}{164\,241}\begin{bmatrix}19\,982\\7508\\116\,753\end{bmatrix} = \begin{bmatrix}126.53\\47.54\\739.30\end{bmatrix}$
$\mathbf{z}_M$	$\frac{1}{79}\begin{bmatrix}10\\4\\65\end{bmatrix} = \begin{bmatrix}0.1266\\0.0506\\0.8228\end{bmatrix}$	$\frac{1}{144\,243}\begin{bmatrix}19\,982\\7508\\116\,753\end{bmatrix} = \begin{bmatrix}0.1385\\0.0521\\0.8094\end{bmatrix}$
$\boldsymbol{\beta}$	$\frac{79}{495}\begin{bmatrix}2\\5\\7\end{bmatrix} = \begin{bmatrix}0.319\,2\\0.798\,0\\1.117\,2\end{bmatrix}$	$\begin{bmatrix}\frac{2914}{9991}\\\frac{1457}{1877}\\\frac{1457}{1283}\end{bmatrix} = \begin{bmatrix}0.291\,6\\0.776\,2\\1.135\,6\end{bmatrix}$
$\boldsymbol{\mu}$	$\frac{1}{100}\begin{bmatrix}109\\115\\119\end{bmatrix} = \begin{bmatrix}1.09\\1.15\\1.19\end{bmatrix}$	$\begin{bmatrix}\frac{226\,611}{199\,820}\\\frac{47\,817}{37\,540}\\\frac{35\,343}{25\,660}\end{bmatrix} = \begin{bmatrix}1.134\,1\\1.273\,8\\1.377\,4\end{bmatrix}$
$\boldsymbol{\sigma}$	$\begin{bmatrix}\frac{1}{10}\\\frac{1}{4}\\\frac{1}{5}\end{bmatrix} = \begin{bmatrix}0.1\\0.25\\0.2\end{bmatrix}$	$\begin{bmatrix}\frac{2079}{19\,982}\\\frac{2079}{7508}\\\frac{297}{1283}\end{bmatrix} = \begin{bmatrix}0.104\,04\\0.276\,9\\0.231\,49\end{bmatrix}$
$\boldsymbol{z}_0$	$\frac{1}{10}\begin{bmatrix}1\\-3\\-5\\6\end{bmatrix} = \begin{bmatrix}0.1\\-0.3\\-0.5\\0.6\end{bmatrix}$	$\frac{1}{16590}\begin{bmatrix}3477\\-2351\\-5265\\10\,762\end{bmatrix} = \begin{bmatrix}0.2096\\-0.1417\\-0.3174\\0.6487\end{bmatrix}$
B	160	$\frac{475\,520}{1659} = 286.63$

The weight vector of risk-free bond for all investors is $\boldsymbol{z}_0 = [z_{1,0}; z_{2,0}; z_{3,0}; z_{4,0}]$. Comparing the two sets of solutions, the mean and volatility of individual stock's return both increase, but some beta rise and some fall. The market value of the entire market declines, but the mean and volatility of market return increase.

- Some beta increase and some decrease. By Eq (3.16), there is $\frac{\mathrm{d}\beta_i}{\mathrm{d}x} = \frac{D_i - D}{(D_i + xR_0)^2} R_0$, whose sign is determined by the sign of $D_i - D$. Since $D_1 - D = \frac{186\,361}{2745\,600}$, $D_2 - D = \frac{553}{68\,640}$, and $D_3 - D = -\frac{2291}{686\,400}$, thus β_1 and β_2 go down and β_3 goes up

- The weight of individual stock in market portfolio may increase or decrease. Since $z_{Mi} = \frac{\eta_i + R_0 x Q_i}{Q R_0 x + E}$, $\frac{\mathrm{d} z_{Mi}}{\mathrm{d} x} = \frac{EQ_i - Q\eta_i}{(E + QxR_0)^2} R_0$, whose sign is determined by the sign of $EQ_i - Q\eta_i$

In both sets of equilibrium solutions, the net holding values of risk-free bond are positive. If the risk-free rate R_0 is also determined endogenously, $B = 0$, and thus

$$R_0 = \frac{E}{W} + \frac{Q}{(E-G)W} = \frac{405\,587}{425\,900} = 0.952\,3$$

Noting that $R_0 < 1$, it is not reasonable here for the market clearing of risk-free bond.

4.2 Allow Arbitrage Opportunity

Dybvig and Ingersoll (1982, p.237) points out that when the market is complete, there may be arbitrage opportunities in CAPM equilibrium. The coming Example 4.2 shows that when the market is incomplete, there may also have arbitrage opportunities in CAPM equilibrium.

Example 4.2 (mean-variance rule and arbitrage possibility): Given two risky assets with

$$\boldsymbol{\eta} = \begin{bmatrix} \frac{23}{20} \\ \frac{185}{164} \end{bmatrix} = \begin{bmatrix} 1.150 \\ 1.128 \end{bmatrix} \qquad \boldsymbol{\Omega} = \begin{bmatrix} \frac{1}{16} & \frac{1}{25} \\ \frac{1}{25} & \frac{1}{25} \end{bmatrix}$$

and risk-free rate $R_0 = \frac{21}{20} = 1.05$. There are two investors with $W_1 = 1$, and $W_2 = 2$. If the simple trade-off coefficients are used, let $G_1 = \frac{100}{41}$, and $G_2 = \frac{400}{123}$, then $G = \frac{41}{40}$, $x = -\frac{1}{GR_0} = -\frac{800}{861}$, and the equilibrium prices are

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\boldsymbol{\Omega}\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, In equilibrium, there is $\mathbf{r} = \mathbf{x}$, say, $\boldsymbol{\mu} = \boldsymbol{\eta}$ and $\mathbf{V} = \boldsymbol{\Omega}$. Noting that $\mu_1 > \mu_2$ and $v_1 > v_2$, there may be arbitrage opportunities. Assuming that $R_1 - R_2 \sim \mathrm{LN}(\mu, \sigma^2)$, then the parameters μ and σ^2 of lognormal distribution are

$$\mu = 2\ln\left(\frac{9}{410}\right) - \frac{1}{2}\ln\left(\frac{15453}{672400}\right) = -5.751$$
$$\sigma^2 = \ln\left(\frac{15\,453}{672\,400}\right) - 2\ln\left(\frac{9}{410}\right) = 3.865$$

We see that $X_1 - X_2 = R_1 - R_2 > 0$, but $\wp(X_1 - X_2) = 0$. A zero-investment portfolio of one share asset 1 in long position and 1 share asset 2 in short position, becomes an arbitrage opportunity.

If quadratic trade-off coefficients are used, let $G_1 = \frac{2050}{3139}$, $G_2 = \frac{16\,400}{23\,139}$, then the two solutions are $x_1 = -\frac{800}{861}$, and $x_2 = -\frac{8200}{1533}$. The first set of prices is identical to the case of simple trade-off coefficients. The second set of prices is

$$\mathbf{p} = \begin{bmatrix} \frac{559}{1022} \\ \frac{13\,543}{20\,951} \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \frac{11\,753}{5590} \\ \frac{94\,535}{54\,172} \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \frac{261\,121}{1249\,924} & \frac{21\,411\,922}{189\,263\,425} \\ \frac{21\,411\,922}{189\,263\,425} & \frac{438\,944\,401}{4585\,321\,225} \end{bmatrix}$$

We still have $\mu_1 > \mu_2$ and $v_1 > v_2$. If $R_1 - R_2 \sim \text{LN}(\mu, \sigma^2)$, then

$$\mu = 2\ln(54116433) - \frac{1}{2}\ln(4725279613170837) - \ln(151410740) = -1.268$$

$$\sigma^2 = \ln(4725279613170837) - 2\ln(54116433) = 0.4784$$

Similarly, the zero-investment portfolio of buying asset 1 and selling short asset 2 is an arbitrage opportunity.

Jarrow and Madan (1997) finds that there may be a conflict between the mean-variance preference and the absence of arbitrage in the market. In CAPM equilibrium, there may exists $\mu_1 > \mu_2$ and $v_1 > v_2$. For merit function (3.6), if

$$G_k > 2\frac{\mu_1 - \mu_2}{v_1 - v_2} \tag{4.1}$$

then $m_k(W_k R_1) < m_k(W_k R_2)$. For merit function (3.8), if (Baron, 1977, p.1687)

$$G_k > 2\frac{\mu_1 - \mu_2}{\mu_1^2 - \mu_2^2 + v_1 - v_2} \tag{4.2}$$

then $m_k(W_k R_1) < m_k(W_k R_2)$. However, if $R_1 - R_2 \gneq 0$, the zero-investment portfolio of buying asset 1 and selling short asset 2 is an arbitrage opportunity. Markowitz (1991, 2014) emphasize that the mean-variance criterion is highly approximate to the logarithmic and exponential utility functions. However we must be careful because they have significant differences in excluding arbitrage opportunities. Therefore, we must first check whether there is an arbitrage opportunity in the CAPM equilibrium, and only when the CAPM equilibrium prices meet the condition of no arbitrage can the prices be feasible.

4.3 Possible Frontier

Brennan and Lo (2010) define the impossible frontier to be an efficient frontier for which every frontier portfolio has at least one negative weight or short position. They prove that the probability that a generically chosen frontier is impossible tends to one at a geometric rate as the number of assets grows. Brennan and Lo (2010, p.909) provide the following numerical example (their version is in simple return)

$$\boldsymbol{\mu} = \mathbf{1} + \begin{bmatrix} 0.096 \\ 0.12 \\ 0.15 \end{bmatrix} = \begin{bmatrix} \frac{137}{125} \\ \frac{28}{25} \\ \frac{23}{20} \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \frac{16}{625} & \frac{16}{625} & \frac{256}{15\,625} \\ \frac{16}{625} & \frac{1}{25} & \frac{1}{25} \\ \frac{256}{15\,625} & \frac{1}{25} & \frac{1}{16} \end{bmatrix}$$

then $a = \frac{32\,025}{656}$, $b = \frac{106\,079}{1968}$, $c = \frac{8792\,161}{147\,600}$, and

$$\mathbf{z}(0) = -\frac{1}{366} \begin{bmatrix} -8915 \\ 2017 \\ 6532 \end{bmatrix} \qquad \mathbf{z}(1) - \mathbf{z}(0) = \frac{25}{6} \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}$$

By Eq (1.23), there is $\mathbf{z}(\mu) = \mathbf{z}(0) + (\mathbf{z}(1) - \mathbf{z}(0))\mu$

$$\mathbf{z}(\mu) = -\frac{1}{366} \begin{bmatrix} -8915 \\ 2017 \\ 6532 \end{bmatrix} + \frac{25\mu}{6} \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}$$

It shows that[6]: When $\mu > \frac{1783}{1525} = 1.169$, the weight of the first asset is negative; When $\mu < \frac{2017}{1525} = 1.323$, the weight of the second asset is negative. Therefore, no matter what the expected return is, there is always at least one negative weight on the frontier. In fact, the reason for the impossible frontier is to set $(\boldsymbol{\mu}, \mathbf{V})$ at will. The return is endogenous, a given return is equivalent to a given equilibrium price. If the return can be arbitrarily set, then a set of positive prices chosen at liberty should be in equilibrium.

In the example, if we set $\mathbf{P}_0 = 1000\mathbf{I}$, in other words, investors guess that the initial asset prices are all at 1000, so the total value of the risky assets is $P_M = 3000$, and

$$\boldsymbol{\eta} = \mathbf{P}_0\boldsymbol{\mu} = \begin{bmatrix} 1096 \\ 1120 \\ 1150 \end{bmatrix} \qquad \boldsymbol{\Omega} = \mathbf{P}_0\mathbf{V}\mathbf{P}_0 = \begin{bmatrix} 25\,600 & 25\,600 & 16\,384 \\ 25\,600 & 40\,000 & 40\,000 \\ 16\,384 & 40\,000 & 62\,500 \end{bmatrix}$$

we have $Q = 292\,068$, $E = 3366$. For simplicity, let $R_0 = 1$, Eq (2.7) gives $x = -\frac{1}{798}$, hence, the real equilibrium price should be

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\boldsymbol{\Omega}\mathbf{1} = \frac{8}{133} \begin{bmatrix} 16\,813 \\ 16\,420 \\ 16\,642 \end{bmatrix} = \begin{bmatrix} 1011.31 \\ 987.67 \\ 1001.02 \end{bmatrix}$$

[6] Brennan and Lo (2010) use $\mathbf{z}(\mu) = \mathbf{z}_o + (\mu - \mu_o)(\mathbf{z}(1) - \mathbf{z}(0))$ to compute the weights

$$\mathbf{z}(\mu) = \frac{1}{11529} \begin{bmatrix} 15\,625 \\ -10\,496 \\ 6400 \end{bmatrix} + \frac{25}{6}\left(\mu - \frac{1739}{1575}\right) \begin{bmatrix} -5 \\ 1 \\ 4 \end{bmatrix}$$

their α is equal to $\frac{ab-a^2}{12(ac-b^2)}\left(\mu - \frac{b}{a}\right) = \frac{1281}{160}\left(\mu - \frac{1739}{1575}\right)$.

there is a slight adjustment to the initial price. In this equilibrium, the return is modified to be

$$\boldsymbol{\mu} = \mathbf{P}^{-1}\boldsymbol{\eta} = \begin{bmatrix} \frac{18\,221}{16\,813} \\ \frac{931}{821} \\ \frac{76\,475}{66\,568} \end{bmatrix} = \begin{bmatrix} 1.0837 \\ 1.1340 \\ 1.1488 \end{bmatrix}$$

We see that the impossible frontier appears because market prices are not the equilibrium prices.

In fact, given any initial set of positive prices, we can always find a possible frontier through Eq (2.7) for that example of Brennan and Lo (2010). As an illustration, let the initial price vector be $[3000; 5000; 2000]$, then

$$\boldsymbol{\eta} = \begin{bmatrix} 3288 \\ 5600 \\ 2300 \end{bmatrix} \qquad \boldsymbol{\Omega} = \begin{bmatrix} 230\,400 & 384\,000 & 98\,304 \\ 384\,000 & 1000\,000 & 400\,000 \\ 98\,304 & 400\,000 & 250\,000 \end{bmatrix}$$

$Q = 3245\,008$, $E = 11\,188$, and $P_M = 10000$. Thus $x = -\frac{297}{811\,252}$, and the equilibrium price and return are adjusted to be

$$\mathbf{p} = \frac{8}{202\,813} \begin{bmatrix} 76\,741\,359 \\ 125\,411\,350 \\ 51\,363\,541 \end{bmatrix} = \begin{bmatrix} 3027.08 \\ 4946.88 \\ 2026.05 \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \frac{27\,785\,381}{25\,580\,453} \\ \frac{2839\,382}{2508\,227} \\ \frac{116\,617\,475}{102\,727\,082} \end{bmatrix} = \begin{bmatrix} 1.0862 \\ 1.1320 \\ 1.1352 \end{bmatrix}$$

There is an adjustment of about 1% relative to the initial setting. We have to keep in mind that the returns are determined endogenously by the market equilibrium.

In view of the fatal blow to CAPM by the impossible frontier, academic journals such as *Critical Finance Review* have published confrontational articles: Levy and Roll (2015) use the endogenous return for discussion, and adjust $\boldsymbol{\mu}$ and $\mathbf{V}$ through the reverse engineering method (Levy and Roll, 2010), which is equivalent to directly set the equilibrium result by weights of the market index. Ingersoll (2015) implicitly assumes that the price is positive, he use payoff and argue by quantity cleaning that prices can always adjust so that demand will equal the positive supply. However, he is limited to the scope of partial equilibrium of risky securities, and does not take into account the role of all individual investor's endowments and preferences, and fails to see that the possible frontiers do come from the mean-variance allocation of social wealth. The futile defense in Brennan and Lo (2015) is based on endogenous return without entering the CAPM market equilibrium, and the impossible frontier is still groundless. The return of risky security is endogenous, using non-equilibrium returns will lead to impossible frontiers.

5 Beta Coefficient

Can beta be used to price new projects? Can beta be used to price options? If a new manufacturing technique will increase the performance of the company, will the expected return rise? Will the corporate value also rise? Contrary to popular belief, the answers to these questions are negative in the framework of mean-variance analysis. Limited by the partial equilibrium and our long established habit of seeing things in isolation from each other, beta has been dogmatic, this section attempts to uncover its misleading nature.

5.1 Beta Pricing

In the CAPM equilibrium, for any non-zero investment with a payoff of X in the market, that is, $P = \wp(X) \neq 0$, let the return be $R = X/P$ (strictly speaking, R is a return only when $P > 0$), we have the following two equivalent pricing method:

1. SDF mimicking payoff pricing: $1 = \mathrm{E}(X_{:}R)$, or if x is the mean-variance semi-feasible equilibrium solution
$$P = \mathrm{E}(X_{:}X) = \left(\frac{1}{R_0} - xE\right)\mathrm{E}(X) + x\,\mathrm{E}(XX_M) \tag{5.1}$$
2. Beta pricing: $\mathrm{E}(R) - R_0 = \beta(\mathrm{E}(R_M) - R_0)$, where $\beta = \frac{\mathrm{cov}(R,R_M)}{\mathrm{var}(R_M)}$

Using return to study pricing is only applicable when the portfolio price is non-zero. However, Eq (5.1) in payoff form is suitable for any portfolio in the market, including zero-investment portfolios. It is necessary to emphasize that beta pricing or equivalent formula (5.1) can only price portfolio in market M. If the pricing is for payoffs outside the market payoff space, it may lead to absurd prices.

Example 5.1 (negative option price): The setting of primitive securities comes from Dybvig and Ingersoll (1982, p.242–4) and Ingersoll (1987, p.200–5)
$$\boldsymbol{\eta} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{\Omega} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \qquad R_0 = 1$$
Assuming that there are two investors, with $W_1 = 1$, and $W_2 = 2$. Using simple trade-off coefficient, if the preference parameters are $G_1 = \frac{5}{8}$, and $G_2 = \frac{5}{6}$, then $G = 4$, and equilibrium solution $x = -\frac{1}{GR_0} = -\frac{1}{4}$. Thus, the equilibrium price is
$$\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\boldsymbol{\Omega}\mathbf{1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
Which shows that $\mathbf{r} = \mathbf{x}$, $\boldsymbol{\mu} = \boldsymbol{\eta}$, and $\mathbf{V} = \boldsymbol{\Omega}$.

Introducing a European option $X = (X_1 - K)1_{X_1 \geqslant K}$, where K is the strike price. If CAPM formula is used for pricing, the equivalent formula (5.1) is used to give

$$\begin{aligned} P &= \frac{3}{2}\,\mathrm{E}(X) - \frac{1}{4}\,\mathrm{E}(X_1 X) \\ &= -\frac{3}{2}K\,\mathrm{E}\left(1_{X_1 \geqslant K}\right) + \left(\frac{3}{2} + \frac{K}{4}\right)\mathrm{E}\left(X_1 1_{X_1 \geqslant K}\right) - \frac{1}{4}\,\mathrm{E}\left(X_1^2 1_{X_1 \geqslant K}\right) \end{aligned}$$

Let $N(x)$ be the standard normal distribution function, $\kappa = \sqrt{\frac{2}{\pi}}$, $\lambda = \exp(-\frac{1}{2})$ and $z = \frac{1}{2}\sqrt{\ln(2)}$. Assuming that X_1 and X_2 are both normally distributed or lognormally distributed, the option prices are as follows

	Normal distribution	Lognormal distribution
$K = 1$	$\kappa\lambda^{1/4} = 0.7041$	$\frac{7}{2}N(3z) - \frac{3}{2}N(z) - 2N(5z) = 0.1748$
$K = 2$	$\kappa - \frac{1}{2} = 0.2979$	$4N(z) - 3N(-z) - 2N(3z) = -0.1585$
$K = 4$	$\kappa\lambda - 3N(-1) = 0.007976$	$5N(-z) - 6N(-3z) - 2N(z) = -0.2650$
$K = 8$	$\kappa\lambda^9 - 7N(-3) = -0.0005856$	$7N(-3z) - 2N(-z) - 12N(-5z) = -0.1606$

The price of European call options should not be negative, the wrong pricing method leads to arbitrage opportunities.

When using the CAPM formula to price options, Jarrow and Madan (1997, p.23) observe that when the exercise price exceeded a certain level (out of the money by two standard deviations), the call option began to have a negative price. The reason for the negative option price is that we have used the wrong pricing kernel: mimicking payoff $X_:$ is only suitable for pricing the assets within the payoff space X, and not for pricing any asset $X \notin \mathsf{X}$. From the knowledge of functional analysis, we know that mimicking payoff $X_:$ can price the assets of a subset M_* of M, while M_* mimicking payoff $X_{*:}$ cannot price assets in M. In Example 5.1, If we try $X_{*:}$ to price risk-free payoff $1 \notin \mathsf{M}_*$, we have

$$\mathrm{E}(X_{*:} \cdot 1) = \mathrm{E}(X_{*:}) = \frac{b}{c+1} = \frac{32}{88+1} = \frac{32}{89}$$

However, $\wp(1) = \mathrm{E}(X_: \cdot 1) = \mathrm{E}(X_:) = \frac{1}{R_0} = 1$, here we can not use $X_{*:}$ as pricing kernel.

In short, when the market is incomplete, if the payoff of an option cannot be a linear combination of risk-free bond and N primitive securities, the option's payoff is not in the payoff space X of market M. Beta pricing is equivalent to use mimicking payoff $X_:$ of market M for pricing, it is beyond the scope of application under this circumstance, it will produce a wrong price of the option. If the market is complete, such as $\mathsf{X} = \mathbb{R}^2$, any linear combination or non-linear function of payoff, that is, any contingent claim, is still within the market's payoff space, there is no problem with beta pricing. Therefore, the CAPM method for the project's discount rate (expected return) and capital cost in corporate finance is suspicious if the market is not complete.

5.2 Changes in Assets

How will changes in the payoff of primitive security affect beta and security price? According to the formula (3.11), the equilibrium solution x is not affected by payoff ($\boldsymbol{\eta}$ and $\boldsymbol{\Omega}$) of any primitive asset using simple trade-off coefficients. Assuming that all individual investor's preferences and endowments are unchanged, then the equilibrium solution x remains unchanged, which facilitates us to investigate the impact of changes in asset payoff in the market.

5.2.1 Changes in Payoffs

If the CAPM equilibrium exists, the formula (3.13) can be rewritten as

$$\begin{bmatrix} P_1 \\ \mathbf{p}_{/1} \end{bmatrix} = \frac{1}{R_0}\begin{bmatrix} \eta_1 \\ \boldsymbol{\eta}_{/1} \end{bmatrix} + x\begin{bmatrix} q_1 & \mathbf{q}' \\ \mathbf{q} & \boldsymbol{\Omega}_{/1} \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}$$

where $\mathbf{p}_{/1}$ and $\boldsymbol{\eta}_{/1}$ are $\mathbf{p}$ and $\boldsymbol{\eta}$ excluding the first row respectively, $\boldsymbol{\Omega}_{/1}$ is $\boldsymbol{\Omega}$ deleting the first row and the first column, $\mathbf{q} = \operatorname{cov}(\mathbf{x}_{/1}, X_1)$ is the first column of $\boldsymbol{\Omega}$ without the first element. Given constant $K > 0$, if the payoff of the first asset increase from X_1 to $X_1 + K$, then the new equilibrium price will be

$$\begin{bmatrix} P_{1+} \\ \mathbf{p}_{/1+} \end{bmatrix} = \frac{1}{R_0}\begin{bmatrix} \eta_1 + K \\ \boldsymbol{\eta}_{/1} \end{bmatrix} + x\begin{bmatrix} q_1 & \mathbf{q}' \\ \mathbf{q} & \boldsymbol{\Omega}_{/1} \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} P_1 + \frac{K}{R_0} \\ \mathbf{p}_{/1} \end{bmatrix}$$

Since the uncertainty (payoff variance matrix) is not changed, the prices of those unchanged securities are not affected, and the price adjustment of the asset whose payoff is increased is equal to the risk-free discounting of the deterministic amount of change.

The teachings on current textbooks believe that beta will not change, however, the price of securities and market returns are endogenous. By Eq (2.10), there is

$$\beta_{1+} - \beta_1 = \frac{P_{M+}}{Q}\frac{Q_1}{P_{1+}} - \frac{P_M}{Q}\frac{Q_1}{P_1} = \frac{Q_1 K}{P_1 Q}\frac{P_1 - P_M}{K + P_1 R_0} \neq 0$$

The uncertainty of the future has not changed at all, but beta has changed. For assets with unchanged payoff

$$\beta_{i+} - \beta_i = \frac{P_{M+}}{Q}\frac{Q_i}{P_{i+}} - \frac{P_M}{Q}\frac{Q_i}{P_i} = \frac{K}{QR_0}\frac{Q_i}{P_i} \qquad i = 2, 3, \cdots, N$$

Beta has changed, although the expected return remains the same.

Example 5.2 (better performance, lower return): Following Example 4.2, if $K = \frac{1}{5}$, beta of security 1 goes down by $\beta_{1+} - \beta_1 = -\frac{164}{1825}$. And despite better future performance, expected return has fallen by

$$\mathrm{E}(R_{1+} - R_1) = \frac{\eta_1 + K}{P_1 + \frac{K}{R_0}} - \frac{\eta_1}{P_1} = \frac{P_1 R_0 - \eta_1}{P_1 R_0 + K} \frac{K}{P_1} = -\frac{2}{125}$$

Due to the wholeness of asset pricing, improving performance of a single company may not necessarily increase its expected return. For asset 2, with unchanged payoff, $\mathrm{E}(R_{2+} - R_2) = 0$, but $\beta_{2+} - \beta_2 = \frac{128}{1533}$, the expected return is unchanged while its beta is changed.

Another way to increase the payoff is to add a non-negative random variable, such as X_1 increases to $X_1 + Y$, assuming that the random variable $Y > 0$, and $\mathrm{cov}(Y, \mathbf{x}) = 0$. The new equilibrium price is

$$\begin{bmatrix} P_{1+} \\ \mathbf{p}_{/1+} \end{bmatrix} = \frac{1}{R_0} \begin{bmatrix} \eta_1 + \mathrm{E}(Y) \\ \boldsymbol{\eta}_{/1} \end{bmatrix} + x \begin{bmatrix} q_1 + \mathrm{var}(Y) & \mathbf{q}' \\ \mathbf{q} & \boldsymbol{\Omega}_{/1} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}$$

The price change of asset 1 is

$$P_{1+} - P_1 = \frac{\mathrm{E}(Y)}{R_0} + x\mathrm{var}(Y)$$

Example 5.3 (better performance, lower market value): Following Example 4.2, let $\mathrm{E}(Y) = \frac{1}{50}$, and $\mathrm{var}(Y) = \frac{1}{36}$, there is $P_{1+} - P_1 = -\frac{262}{38\,745} < 0$, performance improves but market value declines. This is contrary to common sense. However, noting that the parameter x is negative, for an individual firm, if the increase in the expected value of payoff is not enough to offset the discount brought by the increase in risk (variance), the overall effect is the decline in the firm's value. From the perspective of return, $\mathrm{E}(R_{1+} - R_1) = \frac{5381}{192\,415}$, and $\beta_{1+} - \beta_1 = \frac{255\,261\,294}{2126\,609\,063}$, both expected return and beta rise.

The increase in payoff may also be expansive. If the payoff X_1 expands to KX_1, where the constant $K > 1$. The equilibrium price after the expansion is

$$\begin{bmatrix} P_{1+} \\ \mathbf{p}_{/1+} \end{bmatrix} = \frac{1}{R_0} \begin{bmatrix} K\eta_1 \\ \boldsymbol{\eta}_{/1} \end{bmatrix} + x \begin{bmatrix} K^2 q_1 & K\mathbf{q}' \\ K\mathbf{q} & \boldsymbol{\Omega}_{/1} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}$$

Therefore, unless $\mathbf{q} = 0$, the prices of other assets will be changed

$$\mathbf{p}_{/1+} - \mathbf{p}_{/1} = x(K-1)\mathbf{q}$$

Obviously, if diversification is considered, the change of a single asset's future payoff will usually change the equilibrium prices of other assets. We see again that the price of security is endogenous, not determined by the performance on its own, but by all securities as a whole. From Eq (2.10), when $\mathbf{q} = 0$

$$\beta_{i+} - \beta_i = \frac{K-1}{QR_0} \cdot \frac{Q_i}{P_i} \cdot \frac{Q\eta_1 - (K+1)Eq_1}{q_1(K^2-1)+Q} \qquad i = 2, 3, \cdots, N$$

The prices of the unchanged risky securities are unchanged, and their returns are also unchanged. However, their betas cannot be maintained. If $q_1 = \frac{Q\eta_1}{E(K+1)}$ is further added, the betas for those risky securities with unchanged payoff can remain unchanged.

The payoff of security 1 is increased to K times, but the price is not K times of its original price

$$P_{1+} - KP_1 = xq_1 (K-1) K \neq 0 \tag{5.2}$$

It is important to emphasize that the pricing relationship of the CAPM equilibrium is not linear: equilibrium asset price is not a homogeneous function of its payoff. Because equilibrium pricing is a comprehensive examination of all assets. In addition,

$$\beta_{1+} - \beta_1 = \frac{P_{M+}Q_{1+}}{Q_+P_{1+}} - \frac{P_MQ_1}{QP_1}$$

is usually not zero.

Example 5.4 (non-linear pricing): Following Example 4.2, let $K = \frac{6}{5}$, then $P_{1+} - KP_1 = -\frac{4}{287}$, and $\beta_{1+} - \beta_1 = -\frac{9}{5402}$. Obviously, when the payoff increases to K times, the market value cannot increase to K times simultaneously, for the equilibrium pricing is non-linear in payoff. Beta is affected by payoffs and endogenous prices (Eq 2.10 or 3.16), and its movements are more complicated.

5.2.2 Adding an Additional Asset

Suppose that one additional security is added to the market, that is, the number of risky securities increases from N to $N+1$. In the original market, $\mathbf{p} = \frac{\boldsymbol{\eta}}{R_0} + x\mathbf{\Omega}\mathbf{1}$, while in the new market

$$\begin{bmatrix} \mathbf{p}_+ \\ P_{N+1} \end{bmatrix} = \frac{1}{R_0} \begin{bmatrix} \boldsymbol{\eta} \\ \eta_{N+1} \end{bmatrix} + x \begin{bmatrix} \mathbf{\Omega} & \mathbf{q} \\ \mathbf{q}' & q_{N+1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 1 \end{bmatrix}$$

where $\mathbf{q} = \mathrm{cov}(\mathbf{x}, X_{N+1})$. We have

$$\mathbf{p}_+ = \frac{\boldsymbol{\eta}}{R_0} + x(\mathbf{\Omega}\mathbf{1} + \mathbf{q}) \qquad P_{N+1} = \frac{\eta_{N+1}}{R_0} + x(\mathbf{q}'\mathbf{1} + q_{N+1})$$

Thus, $\mathbf{p}_+ - \mathbf{p} = x\mathbf{q}$, unless $\mathbf{q} = 0$ (the payoff of new asset is not correlated to the original assets'), the equilibrium price of the original assets in the new market will be adjusted. In addition, the increase of total market value is

$$P_{M+} - P_M = \mathbf{1}'(\mathbf{p}_+ - \mathbf{p}) + P_{N+1} = \frac{\eta_{N+1}}{R_0} + x\left(2\mathbf{q}'\mathbf{1} + q_{N+1}\right)$$

In the new market, the betas of original assets are

$$\boldsymbol{\beta}_+ = \left(x + \frac{E + \eta_{N+1}}{\left(Q + 2\mathbf{q}'\mathbf{1} + q_{N+1}\right) R_0}\right)\left(\mathbf{P}_+^{-1}\boldsymbol{\Omega}\mathbf{1} + \mathbf{P}_+^{-1}\mathbf{q}\right)$$

where $\mathbf{P}_+ = \mathrm{diag}(\mathbf{p}_+) = \mathbf{P} + x\mathrm{diag}(\mathbf{q})$, so usually $\boldsymbol{\beta}_+ - \boldsymbol{\beta} \neq 0$. But we find that beta remains the same in the following situations

1. $\mathbf{q} = 0$ and $\frac{\eta_{N+1}}{q_{N+1}} = \frac{E}{Q}$. For if $\mathbf{q} = 0$, we have $Q_{i+} = Q_i$, and $P_{i+} = P_i$, the pricing of original assets is isolated; While $\frac{\eta_{N+1}}{q_{N+1}} = \frac{E}{Q}$ gives $\frac{P_{M+}}{Q_+} = \frac{P_M}{Q}$, beta will not be changed by Eq (2.10)
2. $\mathbf{q} = \frac{Q\eta_{N+1} - Eq_{N+1}}{Q(E - \eta_{N+1})}\boldsymbol{\Omega}\mathbf{1}$ and $\boldsymbol{\eta} = \frac{E}{Q}\boldsymbol{\Omega}\mathbf{1}$. Under this condition, the prices of the original assets are proportional to the expected payoffs in both the old and new markets

The above situations are obtained on the premise that the equilibrium solution x has not changed. In fact, in the real market, the wealths of investors are constantly changing, and the equilibrium solution must be changed with time.

5.3 Risk Dogma

In most cases, investors are risk averse: At a given level of expected return, investors prefer investment options with the lowest risk. That is, if an investor chooses an asset with a higher risk, the expected return on that asset must be higher. This leads to the notion of risk premium: bearing risks requires some sort of compensation, and bearing more risks will require higher returns. Naturally, this concept is depicted by CML, and the expected return is a linear (or non-linear curve) increasing function of risk, which further evolves into the dogma of risk. The general idea of the risk dogma is that risk determines the expected return. However, the concept of risk in the risk dogma is vague, mixed with the feelings of uncertainty, correlation, and the possibility of loss. Sharpe (1964, p.436) points out that single risky assets will lie above the CML (Eq 2.15), but as long as the risk measure is changed from standard deviation to beta, and then SML (Eq 2.14, i.e., CAPM formula) is used, the linear relationship between expected return and risk is still valid. Under the guidance of risk dogma, systematic risk determines risk premium, so beta in CAPM represents systematic risk. Since then, the endogenous market return has been taken as a risk factor, and with the prevailing explanation[7] of CAPM, the risk dogma has been further strengthened.

After obtaining the analytical solution of the CAPM equilibrium, we find that the evolution of the concept of risk premium (compensation for risk taking) into the dogma of risk (risk determines return) was caused by a series of coincidences and misunderstandings:

1. Misunderstanding of the market portfolio: The portfolio separation of Tobin (1958) brings the market portfolio front and center. Although investors have different wealth and preferences, investors with mean-variance preferences all hold the same portfolio of risky assets. However, there is a precondition for this assertion in his paper, Tobin (1958) assumes that the mean vector and variance matrix of the returns of risky securities are given. Ignoring this premise, the endogenous return of market portfolio $R_M = X_M/P_M$ is misunderstood as unaffected by any investor's wealth and preference[8]. For examples, the weight vector of market portfolio in Eq (3.14) $\mathbf{z}_M = \frac{\boldsymbol{\eta}+R_0 x \boldsymbol{\Omega}\mathbf{1}}{QR_0 x+E}$ is not affected, or the total market value of risky assets in Eq (2.7) $P_M = xQ + E/R_0$ is not affected by the wealth distribution and individual preference of mean-variance investors. Therefore, the partial equilibrium of purely risky assets (M_* equilibrium) is indistinguishable from the equilibrium of the entire market (M equilibrium), and the CAPM formula as the result of the partial equilibrium is blindly regarded as an overall equilibrium pricing formula.
2. Coincidences: The mean-variance efficient frontier is one-dimensional, which shows that the portfolio frontier is spanned by two funds, and leads to the general solution of CAPM is one-dimensional. This makes the mean-variance analysis greatly simplified. Once the market return R_M (equivalent to P_M or μ_M) is given, then the price (equivalent to expected return) of each primitive security is fully determined. In SML (2.14), $\frac{\mu_i - R_0}{\beta_i}$ takes the same value for any primitive security and their combinations, which goes beyond the scope of CML. The CAPM formula is so simple and beautiful, it really deserves some explanation. The CAPM formula seems to be tailored for risk dogma, but actually it is a

[7]In finance textbooks, CAPM is described as the relationship between risk and return. For examples, Ross et al. (2015), Berk and DeMarzo (2017), and Brealey et al. (2017) in corporate finance, Bodie et al. (2017) in investments, and Fabozzi et al. (2012, Ch14) and Danthine and Donaldson (2014, Ch8) in financial economics.

[8]Merton (1990, p.44) claims that the market portfolio can be constructed without the knowledge of preferences, the distribution of wealth.

coincidence! If zeta of stock i is defined to be $\zeta_i = \text{cov}(R_i, R_M) = \beta_i\sigma_M^2$, then

$$\frac{\mu_i - R_0}{\zeta_i} = \frac{\mu_M - R_0}{\sigma_M^2} = \frac{\mu_j - R_0}{\zeta_j} \qquad i, j = 1, 2, \cdots, N$$

And if delta of stock i is defined to be $\delta_i = \rho_i\sigma_i = \beta_i\sigma_M$, then

$$\frac{\mu_i - R_0}{\delta_i} = \frac{\mu_M - R_0}{\sigma_M} = \frac{\mu_j - R_0}{\delta_j} \qquad i, j = 1, 2, \cdots, N$$

Given the market return, beta, zeta and delta are all proportional to the excess return. If the discoverers of CAPM formula first would find the delta form, in which the slope of risk compensation is equal to the Sharpe ratio of market portfolio, what is most likely to be called systematic risk should not be beta but delta. Because the CAPM formula is nearly perfect, the pricing function is a mapping of payoff to price (equivalent to expected return) is forgotten, and the proportional relationship between beta and expected return is read as the pricing function. Beta is defined by returns, for the information equivalence of the return, price, and expected return, beta is just a byname of price. Due to these coincidences, unknowingly, the independent variable of the pricing function was changed from payoff to price itself.

3. The mixed-blood of the CAPM formula, namely, the dual characteristics of the linear relationship and the partial equilibrium. The linear relationship of the CAPM formula in beta form is not a linear relationship between the payoff and price in the relative pricing function, but the linearity between expected return and systematic risk (beta). If it is only a linear relationship and there is not in partial equilibrium, it is unlikely to mistake linear relationship for pricing relationship. For example, the formula (2.11) is derived from the mathematical definition of μ_i and β_i, and does not require partial market equilibrium of the purely risky securities. Let's define psi of stock i to be $\psi_i = \beta_i D_i$, then

$$\frac{\mu_i}{\psi_i} = \frac{\mu_M}{D} = \frac{\mu_j}{\psi_j} \qquad i, j = 1, 2, \cdots, N$$

which shows that $\frac{\mu_i}{\psi_i}$ takes the identical value for any primitive security and their combinations. In the psi pricing, we need to know the price (μ_i or P_i) of the N primitive securities, which is the essential requirement of relative pricing. However, because the CAPM formula is the result of partial equilibrium, only P_M is needed to know, and the price vector of the primitive security is determined accordingly. Coupled with the aforementioned misunderstanding of the market portfolio, it seems that CAPM can directly price primitive securities, hence CAPM formula is unadvisedly considered to be an equilibrium pricing formula.

Therefore, CAPM is used as an equilibrium pricing formula[9] and reveals the relationship between risk and return. Because those coincidences are highly inductive, the related misunderstandings are so concealed that the obvious error of circular argument has long been ignored. CAPM has quickly become the mainstream, those who have doubts often doubt themselves first, lack confidence or dare not question openly. If the expected return of an individual security is determined by its beta, let's ask one question: The first moment (expectation) of a random variable (payoff or return) is determined by its second moment (variance and covariance), why? When explaining the CAPM formula, the return of the market portfolio is often considered to be directly observable, so isn't it easier to observe that of an individual stock? Calculating beta is more complicated than calculating expectations, so is it necessary to use the CAPM formula?

We know that absolute pricing determines the prices of primitive securities in the equilibrium process of fund allocation. In absolute pricing, price is a function of the payoff of primitive security and every investor's characteristics (endowment and preference). The pricing function is usually non-linear with respect to payoff. In contrast, relative pricing is the pricing of attainable payoffs based on the given prices of primitive securities (Eq 1.1 and 1.3) in the process of fair trading. In relative pricing, price is a function of payoff. Due to the law of linear combination, the pricing function must be linear and only applicable to the payoff space spanned by given primitive securities. The CAPM formula is not an equilibrium pricing formula (absolute pricing), because it is only a result of partial equilibrium, and it has a one-dimensional general solution. Only when P_M is given in advance[10], the CAPM formula of the form (2.5) (or Eq 5.1 given the equilibrium solution x first) is a relative pricing formula. In this case it can only be used to price the portfolio of primitive securities. ***As relative pricing, it does not make sense to discuss risk in the CAPM formula. Because the relative pricing is based on the equilibrium prices of primitive securities and is realized through an arbitrage mechanism (replication), whereas arbitrage is not affected by risk preference.*** When the CAPM equilibrium prices are free of arbitrage, the CAPM formula must be a risk-neutral pricing formula.

In the CAPM equilibrium, all securities are priced as a whole. It is meaningless to study

[9]In textbooks and academic journals, CAPM is taught as an equilibrium pricing formula. For example, the renowned doctoral textbook, Cochrane (2005, p.xiv) asserts that "the CAPM and its successor factor models are paradigms of the absolute approach."

[10]This premise is easily overlooked. For example, Jarrow (2018, Ch17) does not realize the exogeneity of market return when he derives CAPM by martingale approach.

the return or price determination mechanism of an individual security in isolation. It is likely that the pricing of an individual asset is the continuation of the inertial thinking of the pricing of consumer goods. Consumer goods are different from capital assets, for consumer goods are seldom involved in a diversification. In Example 4.1, if the correlation coefficients of payoffs are all changed to $-4/5$, we know from Eq (3.13) that

$$P_1 - \frac{\eta_1}{R_0} = P_2 - \frac{\eta_2}{R_0} = \frac{3536}{553} > 0$$

For the first two stocks, $P_i > \frac{\eta_i}{R_0}$, the price is higher than the discounted expected payoff, and the risk compensation is negative. Portfolio diversification makes risky assets into one composite asset. *A single asset exists as a component of the market portfolio, and does not exist in isolation.* The mistake of risk dogma is to treat individual security in isolation. Taking the market portfolio as a whole, for investors with simple trade-off coefficients, from Eq (2.7) and (3.11), there is

$$P_M - \frac{E}{R_0} = Qx = -\frac{Q}{GR_0} < 0$$

That is, $\mu_M > R_0$, which meets the requirements of risk compensation. We see that equilibrium pricing reflects the risk aversion of all investors as a whole.

From Equation (3.15), the expected return of an asset depends on the ratio of its own expected payoff to covariance of its payoff and market payoff. At the same time, the risk-free interest rate and equilibrium solution also have an impact on its expected return, where the equilibrium solution is determined by all individual investor's preferences and endowments. Obviously, the expected return is endogenous. It is the result of the equilibrium of the entire market according to the mean-variance criterion, and is not determined by the so-called risk (variance, beta, or covariance). The beta value is calculated from the equilibrium return, using beta value to explain the expected return is a circular argument. In addition, calculating the beta value and then solving the expected return by CAPM is not as easy as taking the expectation of return directly.

6 Conclusion

Because of the neglect of the partial equilibrium nature in CAPM and the ideological restraint of risk dogma, CAPM is considered to describe the relationship between risk and return. CAPM is taught as the cornerstone theory of finance. Beta is printed in textbooks as a fundamental concept, and engraved in the minds of financial theorists and practitioners. However, both theoretical analysis and numerical examples of the CAPM equilibrium show that beta pricing is only applicable to existing securities and their linear combinations in the market. Beta is not a systematic risk, and it cannot be extended to new projects or financial derivatives at will. It is extremely wrong to use beta to estimate the so-called cost of capital (Abad, 2020). CAPM is equivalent to the coincidence of market portfolio and tangency portfolio. There is a general solution to the security price in CAPM, and the general solution is one-dimensional. The logical circularity of CAPM does not exist. The logical circularity of CAPM is just a mindset trap of the risk dogma. The CAPM formula is not an equilibrium pricing formula (absolute pricing), but a relative pricing formula with known market return.

In the framework of CAPM equilibrium, risky securities are priced as a whole, and the security returns and the market return are endogenous. Examining the return of individual security in isolation leads to the wrong causal inference that beta determines the expected return. The crisis of impossible frontier is also a false alarm, because the payoff of an asset is exogenous, and the return is endogenous. The return is not the physical attribute of the asset, but the payoff is the only identifier of a financial asset. Returns set unrestrainedly often fail to meet the conditions of the CAPM equilibrium, which leads to the so-called impossible frontier. Last but not least, one caveat to remember is the coexistence of arbitrage opportunity and CAPM equilibrium. Once the CAPM equilibrium is absent of arbitrage opportunities, the CAPM formula is indeed a risk-neutral pricing formula.

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