Galerkin Finite Element Method Math

$U_t - U_{xx} = f(x,t)$ $\qquad\Omega \rightarrow (x,t) \in (0,1) \times (0,1)$

Strong form

BC: $U(x,0) = \sin(\pi x)$

$U(0,t) = U(1,t) = 0$

$f(x,t) = (\pi^2 - 1)e^{-t}\sin(\pi x)$

Weak form:

$$\int_\Omega u' v\,dx - \int_\Omega u'' v\,dx = \int_\Omega f(x,t)\,v\,dx$$

Integration by Part $\left(u'' v\big|_0^1 \rightarrow 0\right)$

$$\int_\Omega u' v\,dx + \int_\Omega u' v'\,dx = \int_\Omega f(x,t)\,v\,dx$$

$$\int_0^1 \left[\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}\right] v(x)\,dx + \int_0^1 u'(x) v'(x)\,dx = \int_0^1 f(x,t)\,v(x)\,dx$$

Galerkin: $\qquad (t+\Delta t) \rightarrow n+1 \qquad t \rightarrow n$

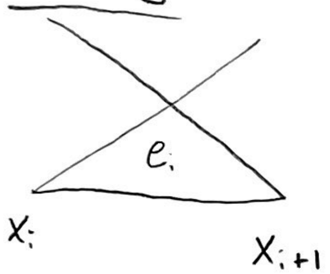
$U(x) = \sum_j U_j \phi_j(x) \qquad V(x) = \phi_i(x) \quad \rightarrow \quad \phi_i(x) = \phi_j(x) = \sin(\pi x)$

$$\frac{1}{\Delta t}\int_0^1 \sum_j U_j^{n+1} \phi_i \phi_j\,dx - \frac{1}{\Delta t}\int_0^1 \sum_j U_j^{n} \phi_i \phi_j\,dx + \int_0^1 \sum_j U_j^{n} \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,dx = \int_0^1 f(x,t)\,\phi_i\,dx$$

stiffness Matrix

mass matrix

mapping



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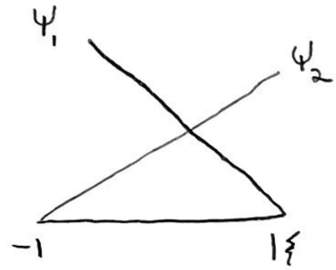

$e_i \rightarrow (0,1) \qquad \hat{e}_i \rightarrow (-1,1)$

$$\int_{e_i} \phi_i \phi_j\,dx \rightarrow \int_{\hat{e}_i} \psi_\ell \psi_m \left|\det J^{e_i}\right| d\xi$$

$\psi_\ell = \dfrac{1-\xi}{2} \qquad \psi_\ell' = -1/2$

$\psi_m = \dfrac{1+\xi}{2} \qquad \psi_m' = 1/2$

$$\int_{e_i} \phi_i' \phi_j'\,dx \rightarrow \int_{\hat{e}_i} \psi_\ell' \psi_m' \left|\det J^{e_i}\right| d\xi$$

Creating the Mass and Stiffness Matrices

$$K = \int_{-1}^{1} \left(\frac{\partial \psi_i}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \psi_j}{\partial \xi}\frac{\partial \xi}{\partial x}\right)\frac{\partial x}{\partial \xi}\,\partial \xi$$

$$\frac{\partial \xi}{\partial x} = \frac{2}{h} \qquad \frac{\partial x}{\partial \xi} = \frac{h}{2}$$

$$K = \int_{-1}^{1} \left(\tfrac{1}{2}\cdot\frac{2}{h}\right)\left(-\tfrac{1}{2}\cdot\frac{2}{h}\right)\left(\frac{h}{2}\right)\partial \xi = \boxed{\frac{-1}{2h}\int_{-1}^{1}\partial \xi \rightarrow \text{For each element}}$$

$$M = \int_{-1}^{1} \psi_i \psi_j \det J\,\partial \xi = \int_{-1}^{1}\left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{h}{2}\,d\xi = \boxed{\frac{h}{8}\int_{-1}^{1}(1-\xi)^2\,d\xi}$$

↳ For each element