Example 1.3:

By using Lagrange's interpolation formula, $p(x)$ is given by:

$$p(x) = \frac{[x-(\bar{x}-h)][x-(\bar{x}-2h)]}{[\bar{x}-(\bar{x}-h)][\bar{x}-(\bar{x}-2h)]}\, p(\bar{x}) + \frac{[(x-\bar{x})(x-(\bar{x}-2h)]}{[(\bar{x}-h)-\bar{x}][(\bar{x}-h)-(\bar{x}-2h)]}\, p(\bar{x}-h) + \frac{[x-\bar{x}][x-(\bar{x}-h)]}{[(\bar{x}-2h)-\bar{x}][(\bar{x}-2h)-(\bar{x}-h)]}\, p(\bar{x}-2h)$$

$$= \frac{x^2-x(2\bar{x}-3h)+(\bar{x}-h)(\bar{x}-2h)}{2h^2}\, u(\bar{x}) + \frac{x^2-x(2\bar{x}-2h)+\bar{x}(\bar{x}-2h)}{-h^2}\, u(\bar{x}-h) + \frac{x^2-x(2\bar{x}-h)+\bar{x}(\bar{x}-h)}{-2h^2}$$

$$. u(\bar{x}-2h)$$

Differentiating both sides, we get:

$$p'(x) = \frac{2x-2\bar{x}-3h}{2h^2}\, u(\bar{x}) + \frac{2x-(2\bar{x}-2h)}{-h^2}\, u(\bar{x}-h) + \frac{2x-(2\bar{x}-h)}{2h^2}\, u(\bar{x}-2h)$$

Since $u(x)$ is approximated by $p(x)$. Then, $u'(x) \approx p'(x)$

Thus,

$$u'(x) = \frac{2x-(2\bar{x}-3h)}{2h^2}\, u(\bar{x}) + \frac{2x-(2\bar{x}-2h)}{-h^2}\, u(\bar{x}-h) + \frac{2x-(2\bar{x}-h)}{2h^2}\, u(\bar{x}-2h)$$

Replacing x with $\bar{x}$, we get

$$u'(\bar{x}) = \frac{2\bar{x}-(2\bar{x}-3h)}{2h^2}\, u(\bar{x}) + \frac{2\bar{x}-(2\bar{x}-2h)}{-h^2}\, u(\bar{x}-h) + \frac{2\bar{x}-(2\bar{x}-h)}{2h^2}\, u(\bar{x}-2h)$$

$$= \left(\frac{3h}{2h^2}\right) u(\bar{x}) + \left(\frac{2h}{-h^2}\right) u(\bar{x}-h) + \left(\frac{h}{2h^2}\right) u(\bar{x}-2h)$$

$$= \frac{3}{2h}\, u(\bar{x}) - \frac{2}{h}\, u(\bar{x}-h) + \frac{1}{2h}\, u(\bar{x}-2h)$$

$$= \frac{1}{2h}\left[3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)\right]$$

Proved ✓