Basic Ingredients of CFD:

1- The Mathematical Model

A set of DEs that are describing the Physics of the model & the BCs.

• Simplifications must be performed in order to obtain solutions. In order to do so, the target should be understood.

2- The Discretization Method:

→ Descretizing the mathematical model: infinite dimentional operator → finite dimentional one (linear system of algebraic equations).

Approximating PDEs by a system of algebraic equations methods:

• FDM
• FVM
• FEM

the mostly used ones

→ Discretizing the Geometry:

To build a Mesh/Grid that defines the geometry.

3- Analyze the Numerical Scheme:

4 conditions should be satisfied to have a valid numeri scheme: Accuracy - Stability - Convergence - Consistency.

4- Solve :

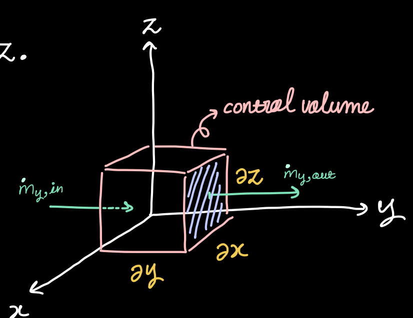
Done by computer. It could be TD or TI solution.

5- Post-Processing :

visulaization of the solution.

Continuity equations : (1D)

It is a formulation of the mass conservation law. To derive this formulation, we consider a control volume with length Δx and width Δy and height Δz.



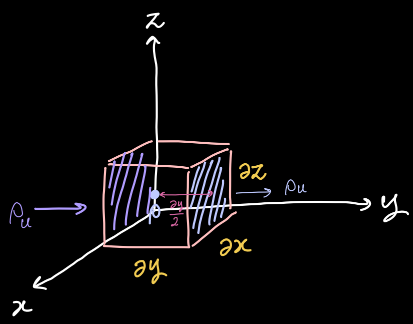
For a compressible flow, along y axis, the change in mass flux dm_y is defined as: $dm = \frac{\partial}{\partial y}(\rho v_y)\, dy$. If the mass in the volume element generally changes over time, then ρ also does: $\dot{m}_{cv} = \frac{\partial \rho}{\partial t} dV$

The sum of $\dot{m}_{cvs}$ is given by: $\frac{\partial}{\partial t}\int_{cv} \rho\, dv_y$ "the change of mass inside the control volume"

- flux of mass across the sum of control surfaces (cs): $\frac{\partial}{\partial t}\int_{cs} \rho \vec{V} \hat{n}\, dA$

According to mass conservation: $\frac{\partial}{\partial t}\int_{cs} \rho \vec{V} \hat{n}\, dA + \frac{\partial}{\partial t}\int_{cv} \rho\, dv_y = 0$

the differential form of $\frac{\partial}{\partial t}\int_{cv} \rho\, dv_y$: $\frac{\partial \rho}{\partial t} = \partial x \partial y \partial z$ .... (1)



The density at the center equal ρ. Thus, we can obtain the flow of the mass out of the shaded face by Taylor expansion: $\left[\rho_u + \frac{\partial}{\partial y}\rho_u \frac{\partial y}{2}\right]\partial x\partial z$

$$\left[\rho u - \frac{\partial}{\partial y}\rho u \frac{\partial y}{2}\right]\partial x\partial z$$

∴ The net mass flow through the y-axis:

$$\left[\cancel{\rho_u} + \frac{\partial}{\partial y}\rho_u \frac{\partial y}{2}\right]\partial x\partial z - \left[\cancel{\rho u} - \frac{\partial}{\partial y}\rho u \frac{\partial y}{2}\right]\partial x\partial z$$

$$\Rightarrow \cancel{2}\frac{\partial}{\partial y}\rho u \frac{\partial y}{\cancel{2}}\partial x\partial z = \frac{\partial}{\partial y}\rho u\, \partial y\partial x\partial z \text{ ------ } (2)$$

And the same for the x and z-axies: $\frac{\partial}{\partial y}\rho_u \partial y\partial x\partial z + \frac{\partial}{\partial x}\rho_v \partial y\partial x\partial z + \frac{\partial}{\partial z}\rho_w \partial y\partial x\partial z$

The sum of (1) & (2) gives us the differential form of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y}\rho_u + \frac{\partial}{\partial z}\rho_w + \frac{\partial}{\partial x}\rho_v = 0$$

Conservation of Momentum (deriving Navier Stokes Equations):

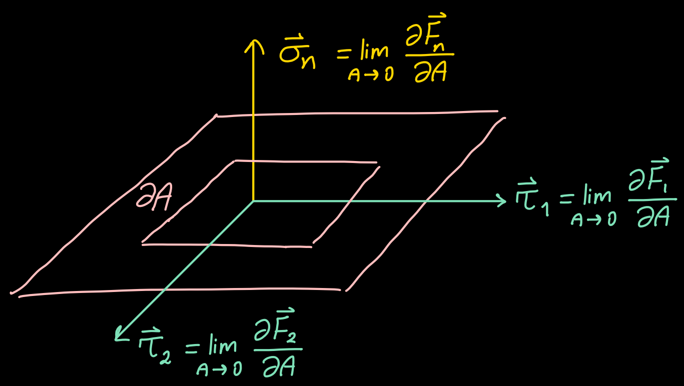
The force of a system: $\frac{\partial}{\partial t}\int_{sys}\vec{V}\,dm = F$

⇒ For a C.V.: The sum of momentum + the sum of momentum flow:

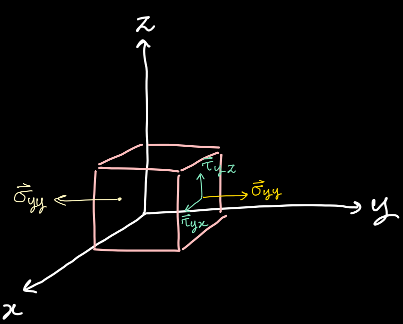
$$\Sigma \vec{F}_{cv} = \frac{\partial}{\partial t}\int_{cv}\vec{V}\rho\, dV + \frac{\partial}{\partial t}\int_{cs}\vec{V}\rho\vec{V}\hat{n}\, dA$$

For infinitesimal mass dm: $\partial\vec{F} = \frac{\partial\vec{V}}{\partial t}dm = \vec{a}\,dm$ ⇒ for body forces: $\partial\vec{F}_b = \vec{g}\,dm$

- Surface forces: $\underbrace{\text{Normal}}_{\text{normal stress}}$ & $\underbrace{\text{tangential}}_{\text{shear stress}}$ to the element area:

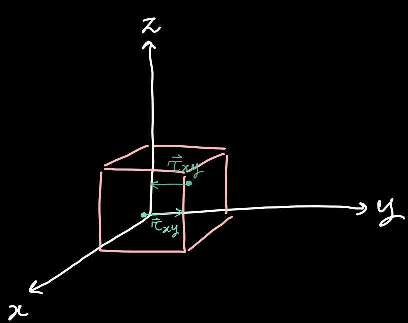


Back to our fluid element :



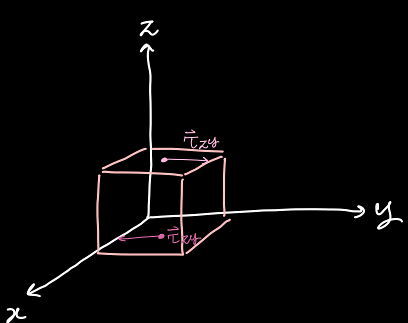
$$\left[\vec{\sigma}_{yy} - \frac{\partial}{\partial y}\vec{\sigma}_{yy}\frac{\partial y}{2}\right]\partial x \partial z$$

$$\left[\vec{\sigma}_{yy} + \frac{\partial}{\partial y}\vec{\sigma}_{yy}\frac{\partial y}{2}\right]\partial x \partial z$$



$$\left[\vec{\tau}_{xy} - \frac{\partial}{\partial x}\vec{\tau}_{xy}\frac{\partial x}{2}\right]\partial z\, \partial y$$

$$\left[\vec{\tau}_{xy} + \frac{\partial}{\partial x}\vec{\tau}_{xy}\frac{\partial x}{2}\right]\partial z\, \partial y$$



$$\left[\vec{\tau}_{xy} + \frac{\partial}{\partial x}\vec{\tau}_{xy}\frac{\partial x}{2}\right]\partial z\, \partial y$$

$$\left[\vec{\tau}_{xy} - \frac{\partial}{\partial x}\vec{\tau}_{xy}\frac{\partial x}{2}\right]\partial z\, \partial y$$

$$\Rightarrow \partial F_{sy} = \left[\vec{\sigma}_{yy}\ \frac{\partial}{\partial y}\ \vec{\tau}_{zy}\ \frac{\partial}{\partial z}\ \vec{\tau}_{xy}\ \frac{\partial}{\partial x}\right]\partial x \partial y \partial z$$

$$\Rightarrow \partial F_{sz} = \left[\vec{\sigma}_{zz}\ \frac{\partial}{\partial z}\ \vec{\tau}_{yz}\ \frac{\partial}{\partial y}\ \vec{\tau}_{xz}\ \frac{\partial}{\partial x}\right]\partial x \partial y \partial z$$

$$\Rightarrow \partial F_{xy} = \left[\vec{\sigma}_{xx}\ \frac{\partial}{\partial x}\ \vec{\tau}_{zx}\ \frac{\partial}{\partial z}\ \vec{\tau}_{yx}\ \frac{\partial}{\partial y}\right]\partial x \partial y \partial z$$

Equations of Motion:

$$\rho g_y \left[\vec{\sigma}_{yy} \frac{\partial}{\partial y} \vec{\tau}_{zy} \frac{\partial}{\partial z} \vec{\tau}_{xy} \frac{\partial}{\partial x} \right] \partial x \partial y \partial z = \rho \left[\frac{\partial u}{\partial t} u \frac{\partial u}{\partial y} v \frac{\partial u}{\partial x} w \frac{\partial u}{\partial z} \right]$$

$$\rho g_z \left[\vec{\sigma}_{zz} \frac{\partial}{\partial z} \vec{\tau}_{yz} \frac{\partial}{\partial y} \vec{\tau}_{xz} \frac{\partial}{\partial x} \right] \partial x \partial y \partial z = \rho \left[\frac{\partial w}{\partial t} u \frac{\partial w}{\partial y} v \frac{\partial w}{\partial x} w \frac{\partial w}{\partial z} \right]$$

$$\rho g_x \left[\vec{\sigma}_{xx} \frac{\partial}{\partial x} \vec{\tau}_{zx} \frac{\partial}{\partial z} \vec{\tau}_{yx} \frac{\partial}{\partial y} \right] \partial x \partial y \partial z = \rho \left[\frac{\partial v}{\partial t} u \frac{\partial v}{\partial y} v \frac{\partial v}{\partial x} w \frac{\partial v}{\partial z} \right]$$

↳ unsteady term

Therefore, we have 3 EOMs & 1 equation of continuity, as well as many variables: u, v, w, and all the stresses.

Navier Stokes Equations:

3 assumptions: Newtonian fluid - Isothermal flow - Incompressible flow.

$\Rightarrow \vec{\sigma}_{yy} = -p + 2\mu \frac{\partial u}{\partial y}$ $\quad \vec{\sigma}_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$ $\quad \vec{\sigma}_{xx} = -p + 2\mu \frac{\partial v}{\partial x}$, where μ is the viscosity

Incompressible flow means $\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} + \frac{\partial v}{\partial x} = 0$ "constant density"

$\therefore \vec{\sigma}_{yy} + \vec{\sigma}_{xx} + \vec{\sigma}_{zz} = -3p$

- For Shear Stresses:

$$\vec{\tau}_{yz} = \vec{\tau}_{zy} = \mu \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \right]$$

$$\vec{\tau}_{xz} = \vec{\tau}_{zx} = \mu \left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \right]$$

$$\vec{\tau}_{yx} = \vec{\tau}_{xy} = \mu \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]$$

$\Rightarrow$ By substituting in y direction :

$$\vec{\sigma}_{yy} \frac{\partial}{\partial y} \vec{\tau}_{zy} \frac{\partial}{\partial z} \vec{\tau}_{xy} \frac{\partial}{\partial x} = -p \frac{\partial}{\partial y} + 2\mu \frac{\partial^2 u}{\partial y^2} + \mu \left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial y} \right) + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right)$$

$$= -p \frac{\partial}{\partial y} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} \right) + \mu \frac{\partial}{\partial y} \underbrace{\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} + \frac{\partial v}{\partial x} \right)}_{=0}$$

$$= -p \frac{\partial}{\partial y} + \mu \underbrace{\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} \right)}_{\text{Laplacian}}$$

Thus, the EOM in y direction :

$$\rho g_x - p \frac{\partial}{\partial y} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} \right) = \rho \left[\frac{\partial u}{\partial t} u \frac{\partial u}{\partial y} v \frac{\partial u}{\partial x} w \frac{\partial u}{\partial z} \right]$$

in vector form: $\rho g - \nabla p + \mu \nabla^2 \vec{V} = \rho \frac{\partial u}{\partial t} \vec{V} (\vec{V} \cdot \nabla) \vec{V}$

Now we have 3 scalar equation & a continuity equation $\rightarrow$ 5 variables : u, v, w, p, ρ.

In order to close the system & solve, a fifth state equation should be added.

$$\rho g - \boxed{\frac{\nabla p}{\rho}} + \overbrace{\mu \nabla^2 \vec{V}}^{\text{viscousity diffusion (nonlinear)}} = \rho \boxed{\frac{\partial u}{\partial t} \vec{V} \underbrace{(\vec{V} \cdot \nabla) \vec{V}}_{\text{convective acceleration (nonlinear)}}} \rightarrow \text{Acceleration term}$$

pressure gradient

unsteady acceleration

Consider that $\mu = 0$. Then, NS equations become: $\rho g - \frac{\nabla p}{\rho} = \frac{\partial u}{\partial t} \vec{v} (\vec{v} \cdot \nabla) \vec{v}$

"Eulerian equation"

-nonlinear-