Example 1.2:

Approximate $f'(t)$ based on $f(t)$, $f(t-\Delta t)$, and $f(t-2\Delta t)$, of the form $f'(t) = af(t) + bf(t-\Delta t) + cf(t-2\Delta t)$.

It is known that:

- $f(t-\Delta t) = f(t) - f'(t)\Delta t + \frac{1}{2} f''(t)\Delta t^2 - \frac{1}{6} f'''(t)\Delta t^3 + O(h^4)$.
- $f(t-2\Delta t) = f(t) - 2\Delta t f'(t) + \frac{1}{2}(2\Delta t)^2 f''(t) - \frac{1}{6}(2\Delta t)^3 f'''(t) + O(h^4)$.

Thus,

$$f'(t) = af(t) + bf(t) - bf'(t)\Delta t + \frac{b}{2} f''(t)\Delta t^2 - \frac{b}{6} f'''(t)\Delta t^3 + cf(t) - 2c\Delta t f'(t) + \frac{c}{2}(2\Delta t)^2 f''(t) - \frac{c}{6}(2\Delta t)^3 f'''(t)$$

$$= (a+b+c)f(t) - (b+2c)f'(t)\Delta t + \frac{1}{2}(b+4c)f''(t)\Delta t^2 - \frac{1}{6}(b+8c)\Delta t^3 f'''(t) + \dots$$

We need

$$a + b + c = 0 \quad \text{------} \quad 1$$

$$b + 2c = -\frac{1}{\Delta t} \quad \text{------} \quad 2$$

$$b + 4c = 0 \quad \text{------} \quad 3$$

~~to~~ be agreed with $f'(t)$.

By substituting ③ in ②:

$$-4c + 2c = -\frac{1}{\Delta t}$$

$$\Rightarrow c = \frac{1}{2\Delta t} \quad \text{------} \quad ④ \qquad \Rightarrow b = -4c = \frac{-4}{2\Delta t} = \frac{-2}{\Delta t}$$

By substituting 3 & 4 in 1 :

$$a - \frac{4}{2\Delta t} + \frac{1}{2\Delta t} = 0$$

$$\Rightarrow a = \frac{3}{2\Delta t}$$

$$\therefore f'(t) = \frac{3}{2\Delta t} f(t) - \frac{2}{\Delta t} f(t - \Delta t) + \frac{1}{2\Delta t} f(t - 2\Delta t)$$

$$= \frac{1}{2\Delta t} \left[3f(t) - 4f(t - \Delta t) + f(t - 2\Delta t) \right]$$