Time-per-Revolution in the Prime-Driven Torus

Abstract

Periodic parameter spaces and their resonance phenomena are captured by toroidal windings. We show that:

1. The classical Tonnetz is the case $n = 2$, a 2-torus T^2_{Tonnetz} with rational angular increments encoding musical fifths and thirds.
2. The Prime-FM model is the case $n = 8$, an 8-torus T^8_{PrimeFM} with twist fractions derived from prime-gap ratios.
3. Both admit the same linear-flow description $\theta(t) = \theta(0) + t\,\omega \pmod{2\pi}$, differing only in the choice of ω.

1 Definitions

Definition 1 (n-Torus)**.**

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} = \mathbb{R}^n \,/\, (2\pi\mathbb{Z})^n,$$

with coordinates $\theta = (\theta_1, \ldots, \theta_n) \bmod 2\pi$*.*

Definition 2 (Linear Flow)**.** *Given* $\omega \in \mathbb{R}^n$*, the discrete flow* $\Phi : \mathbb{Z} \to T^n$ *is*

$$\Phi(t) = \theta(0) + t\,\omega \pmod{2\pi}.$$

2 Periodicity versus Density

Proposition 1. *If each ratio* $\omega_i/\omega_j \in \mathbb{Q}$*, then* $\Phi(t)$ *is periodic. Otherwise, its orbit is dense in an embedded sub-torus of* T^n*.*

Proof. Standard results on linear flows on tori (Kronecker's theorem). □

3 Tonnetz as T^2_{Tonnetz}

Define two angular velocities on $\mathbb{Z}_{12} \cong S^1$:

$$\omega_F = 2\pi\frac{7}{12}, \quad \omega_M = 2\pi\frac{4}{12}.$$

Then

$$T^2_{\text{Tonnetz}} = S^1_F \times S^1_M, \quad \Phi_{\text{Tonnetz}}(t) = \bigl(t\,\omega_F,\; t\,\omega_M\bigr) \pmod{2\pi}.$$

Since $\omega_F/\omega_M = 7/4 \in \mathbb{Q}$, all orbits close after at most 12 steps.

4 Prime-FM Model as $\boldsymbol{T}^8_{\text{PrimeFM}}$

Let $p_1, \ldots, p_8$ be eight successive primes and set

$$\phi_i = \frac{p_i}{\sum_{j=1}^{8} p_j}, \quad \omega_i = 2\pi\, \phi_i.$$

Then

$$T^8_{\text{PrimeFM}} = \prod_{i=1}^{8} S^1_i, \quad \Phi_{\text{PrimeFM}}(t) = \big(t\,\omega_1, \ldots, t\,\omega_8\big) \pmod{2\pi}.$$

Each $\omega_i/\omega_j = p_i/p_j \in \mathbb{Q}$, so the flow is periodic with period $\sum_j p_j$.

5 Unified Theorem and Proof

Theorem 1. *Both the Tonnetz and the Prime-FM constructions are instances of the linear flow $\Phi(t) = \theta(0) + t\,\omega \pmod{2\pi}$ on T^n, differing only in n and the choice of ω.*

Proof.

1. By definition $T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$.
2. Any $\omega \in \mathbb{R}^n$ yields $\Phi(t) = t\,\omega \pmod{2\pi}$.
3. Tonnetz: $n = 2$, $\omega = (2\pi\frac{7}{12}, 2\pi\frac{4}{12})$.
4. Prime-FM: $n = 8$, $\omega_i = 2\pi\, p_i/\sum_j p_j$.
5. Conclusion: Same underlying toroidal-winding framework.

□

6 Conclusion

The classical Tonnetz ($n = 2$, rational intervals) and the Prime-FM 8-torus ($n = 8$, prime-ratio twists) both realize toroidal resonance as linear flows on T^n. Rational vs. irrational-ratio regimes govern closed orbits versus dense quasi-periodicity, unifying musical and number-theoretic resonances in one geometric paradigm.