Toroidal Realization of the Standard Model Cartan

The full Cartan torus of E_8 is an eight–torus:

$$T^8 \;=\; S^1_1 \times S^1_2 \times \cdots \times S^1_8.$$

We identify four of these circles as the gauge Cartan

$$T^4_{\text{gauge}} \;=\; T^2_{SU(3)} \times T^1_{SU(2)} \times T^1_{U(1)_Y},$$

where

$$\begin{aligned}
\operatorname{rank} SU(3) &= 2, \quad T^2_{SU(3)} = S^1_a \times S^1_b,\\
\operatorname{rank} SU(2) &= 1, \quad T^1_{SU(2)} = S^1_c,\\
\operatorname{rank} U(1)_Y &= 1, \quad T^1_{U(1)_Y} = S^1_d.
\end{aligned}$$

The complementary "matter torus" is then

$$T^4_{\text{matter}} \;=\; S^1_e \times S^1_f \times S^1_g \times S^1_h,$$

so that

$$T^8 \;\cong\; \underbrace{T^2_{SU(3)} \times T^1_{SU(2)} \times T^1_{U(1)_Y}}_{T^4_{\text{gauge}}} \times T^4_{\text{matter}}.$$

U(1) Line Bundle over T^4

On the abelian hypercharge bundle $L \to T^4$, the connection 1-form A and curvature 2-form F satisfy

$$F \;=\; dA + A \wedge A \;=\; dA,$$

since $A \wedge A = 0$ for $U(1)$. The bundle is classified by its first Chern class

$$c_1(L) \;=\; \frac{1}{2\pi}\int_{T^4} F \;=\; \sum_{i<j} n_{ij}\, dx^i \wedge dx^j,$$

with integers $n_{ij} \in \mathbb{Z}$ giving the quantized hypercharge flux on each of the six independent 2-cycles of T^4.

Gauge vs. Matter Weights

Let $\lambda \in \Lambda_{E_8}$ be a root or weight vector in the 8-dimensional Cartan lattice. Decompose its components as

$$\lambda \;=\; (\lambda_a, \lambda_b\,;\, \lambda_c\,;\, \lambda_d\,;\, \lambda_e, \lambda_f, \lambda_g, \lambda_h).$$

- *Gauge bosons:* non-zero $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ (adjoint of $SU(3) \times SU(2) \times U(1)_Y$), zero on $\lambda_{e\text{–}h}$.
- *Matter fields:* $\lambda_a = \lambda_b = \lambda_c = \lambda_d = 0$, non-zero $\lambda_e, \ldots, \lambda_h$.

Thus the SM gauge group $SU(3) \times SU(2) \times U(1)_Y$ "uses" four of the eight circles, and the remaining four furnish the geometric realization of the chiral matter multiplets.