Clemson University, School of Computing
CPSC 3500 Foundations of Computer Science

Assignment 0: Mathematical Induction

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This assignment is not directly related to the topics of our course but it will help you to refresh on how to use mathematical induction.
Prove by induction the following claims

(a) $\forall n \in \mathbb{N} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(b) $\forall n \in \mathbb{N} \quad \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(c) Let $r \in \mathbb{R}_{\neq 1}$ (r is real, not equal 1). Prove that $\forall n \in \mathbb{N} \quad \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$

(d) $\forall n \in \mathbb{N} \quad 1 + \sum_{i=1}^{n} i \cdot i! = (n+1)!$

(e) $\forall n \in \mathbb{N}_{\geq 4} \quad n! > 2^n$

(f) Let $x \in \mathbb{R}_{>-1}$. Prove that $\forall n \in \mathbb{N} \quad (1+x)^n \geq 1 + nx$

(g) The Fibonacci function f is usually defined as follows.

$$f(0) = 0;\ f(1) = 1;\ \text{for every } n \in \mathbb{N}_{>1},\ f(n) = f(n-1) + f(n-2).$$

Here we need to give both the values $f(0)$ and $f(1)$ in the first part of the definition, and for each larger n, $f(n)$ is defined using both $f(n-1)$ and $f(n-2)$. Use induction to show that for every $n \in \mathbb{N}$, $f(n) \leq (5/3)^n$. (Note that in the induction step, you can use the recursive formula only if $n > 1$; checking the case $n = 1$ separately is comparable to performing a second basis step.)

Solutions to Assignment 1

Solution to Problem 1

(a) Proof by induction on n

- Basis step: $n = 1$, $1 = 1(1+1)/2$
- Hypothesis: Suppose the claim is true for some $n = k$, $\sum_{i=1}^{k} i = k(k+1)/2$
- Induction step: We need to prove the claim for $n = k+1$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2$$

  Thus the claim is true for all n.

(b) Proof by induction on n

- Basis step: $n = 1$, $\dfrac{1}{1(1+1)} = \dfrac{1}{1+1}$
- Hypothesis: Suppose the claim is true for some $n = k$, $\sum_{i=1}^{k} \dfrac{1}{i(i+1)} = \dfrac{k}{k+1}$
- Induction step: We need to prove the claim for $n = k+1$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{1}{k+1}\left(k + \frac{1}{k+2}\right) = \frac{k+1}{k+2}$$

  Thus the claim is true for all n.

(d) Proof by induction on n

- Basis step: $n = 1$, $1 + 1 = 2!$
- Hypothesis: Suppose the claim is true for some $n = k$, $\sum_{i=1}^{k} i \cdot i! = (k+1)!$
- Induction step: We need to prove the claim for $n = k+1$

$$\sum_{i=1}^{k+1} i \cdot i! = \sum_{i=1}^{k} i \cdot i! + (k+1)(k+1)! = (k+1)! + (k+1)(k+1)! = (k+2)!$$

  Thus the claim is true for all n.

(f)

Induction step:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \\
&= 1 + nx + x + nx^2 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x
\end{aligned}$$

(g)
In the basis step we must show that $f(0) \leq (5/3)^0$, and this is true. The induction hypothesis is that $k \geq 0$, and for every n with $0 \leq n \leq k$, $f(n) \leq (5/3)^n$. In the induction step we will show $f(k+1) \leq (5/3)^{k+1}$. If $k+1 = 1$, this is true by def $f(1) = 1$. Otherwise,

$$\begin{aligned}
f(k+1) &= f(k) + f(k-1) \\
&\leq (5/3)^k + (5/3)^{k-1} \qquad \text{by induction hypothesis} \\
&= (5/3)^{k-1}(5/3 + 1) \\
&= (5/3)^{k-1}(8/3) \\
&= (5/3)^{k-1}(24/9) \\
&< (5/3)^{k-1}(25/9) \\
&= (5/3)^{k+1}
\end{aligned}$$