Claim. *Suppose $L \subseteq \{a, b\}^*$ is defined recursively as follows:*

$$\Lambda \in L; \forall x \in L \; ax, axb \in L.$$

Show that $L = L_0$, where $L_0 = \{a^i b^j | i \geq j\}$.

Proof sketch. Proving $L = L_0$ means that we need to prove both

$$L \subseteq L_0, \text{ and } L_0 \subseteq L.$$

$\forall x \in L \;\; x \in L_0$ $\qquad\qquad$ $\forall x \in L_0 \;\; x \in L$

Intuition:

1. In *L* strings are generated with more a's than b's
2. In L_0 strings look like $\underbrace{aaaa\ldots aaaa}_{i}\underset{\geq}{}\underbrace{bb\ldots bb}_{j}$

Claim. *Suppose $L \subseteq \{a, b\}^*$ is defined recursively as follows:*

$$\Lambda \in L; \text{ for every } x \in L, \text{ both } ax \text{ and } axb \text{ are in } L.$$

Show that $L = L_0$, where $L_0 = \{a^i b^j | i \geq j\}$.

Proof sketch.

Part 2 ($L_0 \subseteq L$). We show by induction on n

$$\forall n \in \mathbb{N}, \text{ if } y \in L_0, \text{ and } |y| = n, \text{ then } y \in L.$$

Basis step: $y \in L_0$, and $|y| = 0 \Rightarrow y \in L$. This is true because if $|y| = 0$ then $y = \Lambda$, and $\Lambda \in L$.

Induction Hypothesis: $k \in \mathbb{N}, \forall y \in L_0$ such that

$$|y| \leq k \Rightarrow y \in L$$

Claim. *Suppose $L \subseteq \{a,b\}^*$ is defined recursively as follows:*

$$\Lambda \in L; \text{ for every } x \in L, \text{ both } ax \text{ and } axb \text{ are in } L.$$

Show that $L = L_0$, where $L_0 = \{a^i b^j | i \geq j\}$.

Proof sketch (cont).

IS: $y \in L_0$, $|y| = k+1 \Rightarrow y = a^i b^j$, and $i + j = k + 1$.

- $y \neq \Lambda$ because $k \geq 0$ and $|y| = k+1 \Rightarrow$ we must show that $y = ax$ or $y = axb$ for some $x \in L$.

Claim. *Suppose* $L \subseteq \{a,b\}^*$ *is defined recursively as follows:*

$$\Lambda \in L; \text{ for every } x \in L, \text{ both } ax \text{ and } axb \text{ are in } L.$$

Show that $L = L_0$, *where* $L_0 = \{a^i b^j | i \geq j\}$.

Proof sketch (cont).

IS: $y \in L_0$, $|y| = k+1 \;\Rightarrow\; y = a^i b^j$, and $i + j = k + 1$.

- $y \neq \Lambda$ because $k \geq 0$ and $|y| = k+1 \;\Rightarrow\;$ we must show that $y = ax$ or $y = axb$ for some $x \in L$.
- We know that $i + j > 0$, and $i \geq j \;\Rightarrow\; i > 0$, i.e., $y = ax$ for some x; if $j > 0$ then $y = axb$ for some x.

Claim. *Suppose $L \subseteq \{a, b\}^*$ is defined recursively as follows:*

$$\Lambda \in L; \text{ for every } x \in L, \text{ both } ax \text{ and } axb \text{ are in } L.$$

Show that $L = L_0$, where $L_0 = \{a^i b^j | i \geq j\}$.

Proof sketch (cont).

IS: $y \in L_0$, $|y| = k + 1 \Rightarrow y = a^i b^j$, and $i + j = k + 1$.

- $y \neq \Lambda$ because $k \geq 0$ and $|y| = k + 1 \Rightarrow$ we must show that $y = ax$ or $y = axb$ for some $x \in L$.
- We know that $i + j > 0$, and $i \geq j \Rightarrow i > 0$, i.e., $y = ax$ for some x; if $j > 0$ then $y = axb$ for some x.
- If $j > 0$ then $y = axb$, where $x = a^{i-1}b^{j-1}$, and $i - 1 \geq j - 1$, i.e., $x \in L_0$ and by IH $x \in L$ and then also $y \in L$. Case $j = 0$ is similar.

>>> Part 1 is easy: complete a home.

Summary

- In typical proofs by mathematical induction we choose an integer that is
  - The length of string
  - The number of substrings whose concatenation gives string *x*
  - The exponent of the language in the * of some expression

- Typical proofs by structural induction (SI):
  - SI doesn't work without recursive definition (RD) of the language
  - The basis of SI corresponds to the basis of the RD
  - Formulate induction hypothesis on all input elements of the recursive rules
  - Prove induction step on all recursive rules in RD
  - Break down the problem you need to prove in induction step to easier problems in which you can apply the hypothesis, basis statement, etc.

Claim. *Suppose that* $x, y \in \{a, b\}^*$ *and neither* Λ. *Show that*

$xy = yx \;\Rightarrow\; \exists z \in \{a, b\}^*,$ *and* $i, j \in \mathbb{N}$, *such that* $x = z^i$, *and* $y = z^j$.

Proof sketch. Let d be the greatest common divisor of $|x|$, and $|y|$. We rewrite x and y as

$$x = x_1 x_2 \cdots x_p, \text{ and } y = y_1 y_2 \cdots y_q,$$

where all $|x_i| = |y_j| = d$ for all i, j.

Since $xy = yx$ then $x^q y^p = y^p x^q$ (begin with $x^q y^p$, and run repeated transpositions, i.e., switch x and y).

$$xx \ldots xyy \ldots y = xx \ldots yxy \ldots y = \cdots = yy \ldots yxx \ldots x$$

Claim. *Suppose that $x, y \in \{a, b\}^*$ and neither Λ. Show that*

$$xy = yx \Rightarrow \exists z \in \{a, b\}^*, \text{ and } i, j \in \mathbb{N}, \text{ such that } x = z^i, \text{ and } y = z^j.$$

Proof sketch.

Both $x^q y^p$, and $y^p x^q$ have the same length $2pqd$,e.g.,

$$|x^q y^p| = |x_1 \cdots x_p \cdots q \text{ times } \cdots x_1 \cdots x_p y^q| = \cdots = 2pqd.$$

- In both cases, prefixes x^q (of $x^q y^p$), and y^p (of $y^p x^q$) have the same length and thus are equal.
- If $x^q = (x_1 \cdots x_p)^q$ then x_1 appears in positions 1, $pd+1$, $2pd+1$, ... , $(q-1)pd + 1$.
- In y^p, the substring y_{r_i} of length d can be found at $ipd+1$, where $r_i = ip + 1 \; mod \; q$. Since p and q have no common factors, all r_i are different. Then, it follows that all y_i are equal (z). Same for x_i.