Claim. *Let $M = (Q, \Sigma, q_0, A, \delta)$ be an NFA. Show that if S and T are subsets of Q for which $S \subseteq T$, then $\Lambda(S) \subseteq \Lambda(T)$.*

Proof. We need to show that if $s \in \Lambda(S)$ then $s \in \Lambda(T)$.

- $\forall s \in S$, it is true that $s \in T$ and then (by def of $\Lambda(T)$) $s \in \Lambda(T)$
- if $s \notin S$ then it was added to $\Lambda(S)$ by the recursive rule

$$\delta(q, \Lambda) \subseteq \Lambda(S) \text{ for some } q \in \Lambda(S)$$

  . We prove by induction on the number of recursive applications of $\delta(\cdot, \Lambda)$, i.e., by structural induction.

- IH: all elements in $\Lambda(S)$ that were added by less than k applications of $\delta(\cdot, \Lambda)$ are in $\Lambda(T)$
- IS: We add s to $\Lambda(S)$ by kth application of δ. W.l.o.g., there exists $q \in \Lambda(S)$ (such that $s \in \delta(q, \Lambda)$) that was added by less than k applications of δ to $\Lambda(S)$ and by IH $q \in \Lambda(T)$. By def of $\Lambda(T)$ $\delta(q, \Lambda) \subset \Lambda(T)$ because $q \in \Lambda(T)$.

$\square$

Claim. *Let* $M = (Q, \Sigma, q_0, A, \delta)$ *be an FA, and let* $M_1 = (Q, \Sigma, q_0, A, \delta_1)$ *be the NFA with no* Λ*-transitions for which*

$$\forall q \in Q \ \forall \sigma \in \Sigma \quad \delta_1(q, \sigma) = \{\delta(q, \sigma)\}.$$

Then $\forall q \in Q \ \forall \sigma \in \Sigma \quad \delta_1^*(q, x) = \{\delta^*(q, x)\}.$

Proof. The proof is by structural induction.

- Observe there are no Λ-transitions in M_1, i.e., $\delta_1^*(q, \Lambda) = \{q\}$, and $\delta_1^*(q, xa) = \bigcup\{\delta_1(p, a) \mid p \in \delta_1^*(q, x)\}, \ \forall q \in Q, \ \forall a \in \Sigma.$
- BS: $\delta_1^*(q, \Lambda) = \{q\}$, and $\delta^*(q, \Lambda) = q$ by definition of δ^*.
- IH: Suppose that for some y, $\delta_1^*(q, y) = \{\delta^*(q, y)\}$, for every q.
- IS: Then for $a \in \Sigma$,

$$\begin{aligned}
\delta_1(q, ya) &= \cup\{\delta_1(p, a) \mid p \in \delta_1^*(q, y)\} \\
&= \cup\{\delta_1(p, a) \mid p \in \{\delta^*(q, y)\}\} \\
&= \delta_1(\delta^*(q, y), a) \\
&= \{\delta(\delta^*(q, y), a)\} = \{\delta^*(q, ya)\}.
\end{aligned}$$

$\square$

Claim. *Let $M = (Q, \Sigma, q_0, A, \delta)$ be an NFA. For every $q \in Q$ and every $x, y \in \Sigma^*$,*

$$\delta^*(q, xy) = \bigcup\{\delta^*(r, y) \mid r \in \delta^*(q, x)\}.$$

Proof by structural induction is given in Exercise 3.30. Learn it!

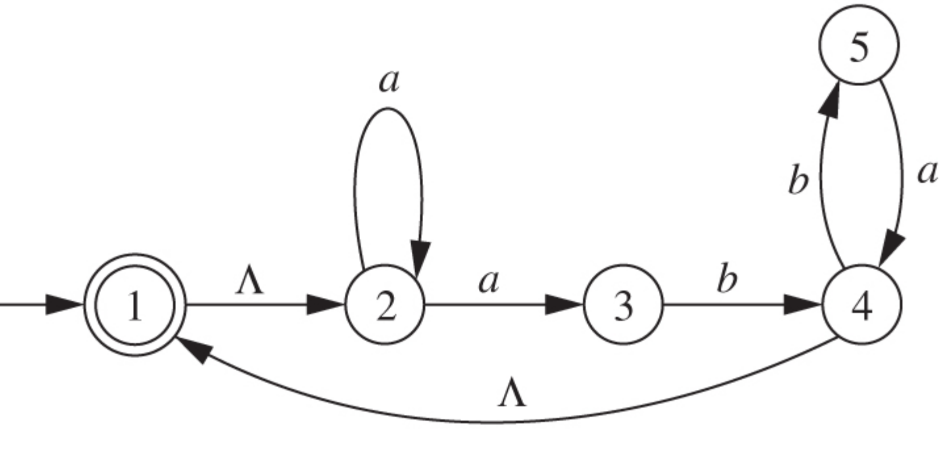
The Nondeterminism in an NFA Can Be Eliminated

- Two types of nondeterminism have arisen:
  - 1) Different arcs for the same input symbol (or no arcs), and 2) Λ-transitions
  - Both can be eliminated
- For the **second type**, introduce new transitions so that we no longer need the Λ-transitions
  - When there is no σ-transition from p to q but the NFA can go from p to q by using one or more Λ-transitions as well as σ, we introduce the σ-transition
  - The resulting NFA may have more nondeterminism of the first type, but it will have no Λ-transitions

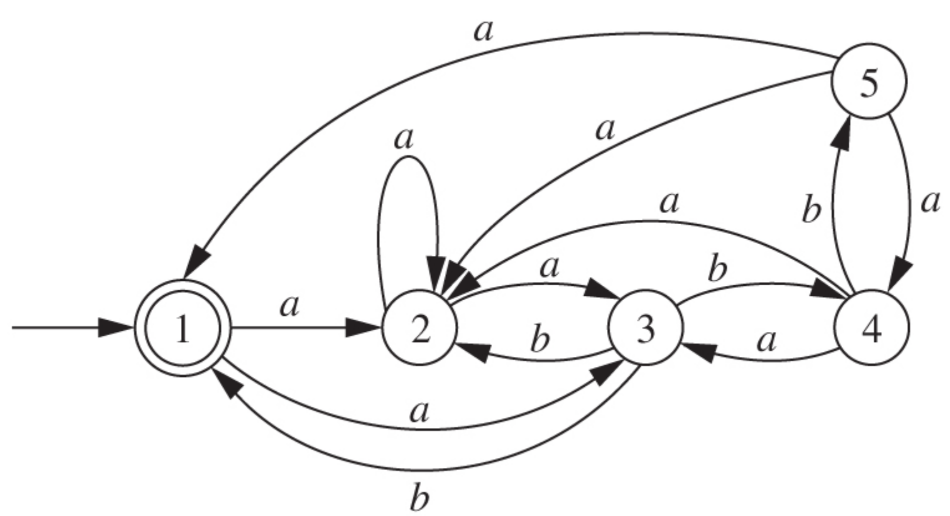
The Nondeterminism in an NFA Can Be Eliminated (cont'd.)

- **Theorem: For every language $L \subseteq \Sigma^*$ accepted by an NFA $M = (Q, \Sigma, q_0, A, \delta)$, there is an NFA M_1 with no Λ-transitions that also accepts L**
- Define $M_1 = (Q, \Sigma, q_0, A_1, \delta_1)$, where
  - for every $q \in Q$, $\delta_1(q, \Lambda) = \emptyset$, and
  - for every $q \in Q$ and every $\sigma \in \Sigma$, $\delta_1(q, \sigma) = \delta^*(q, \sigma)$
- Define $A_1 = A \cup \{q_0\}$ if $\Lambda \in L$, and $A_1 = A$ otherwise
- We can prove, by structural induction on x, that for every q and every x with $|x| \geq 1$, $\delta_1{}^*(q, x) = \delta^* (q, x)$

Homework: prove this theorem (see Theorem 3.17 in the textbook)



(a)



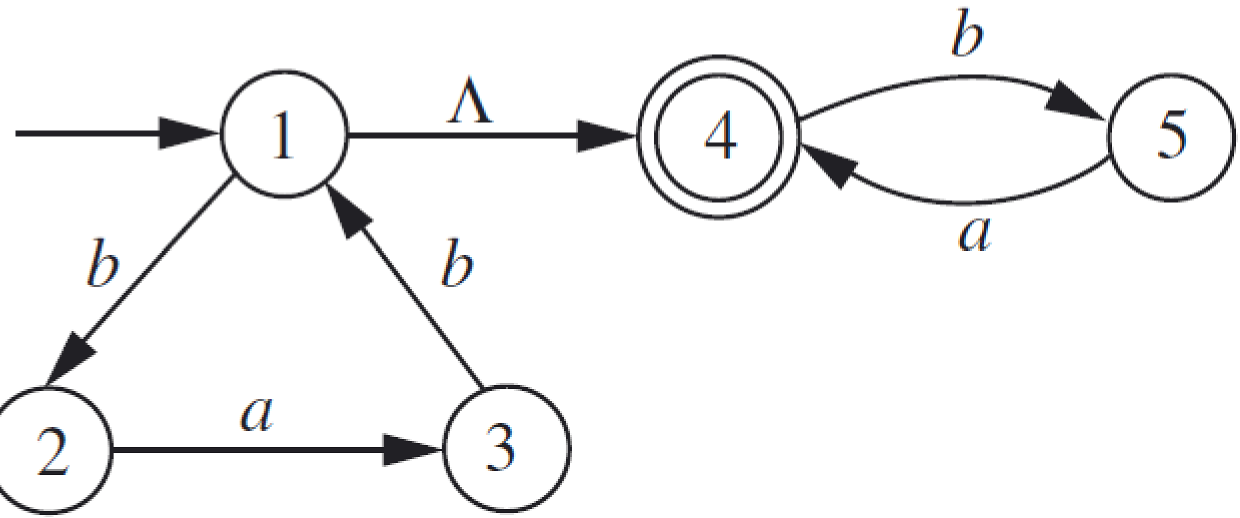
(b)

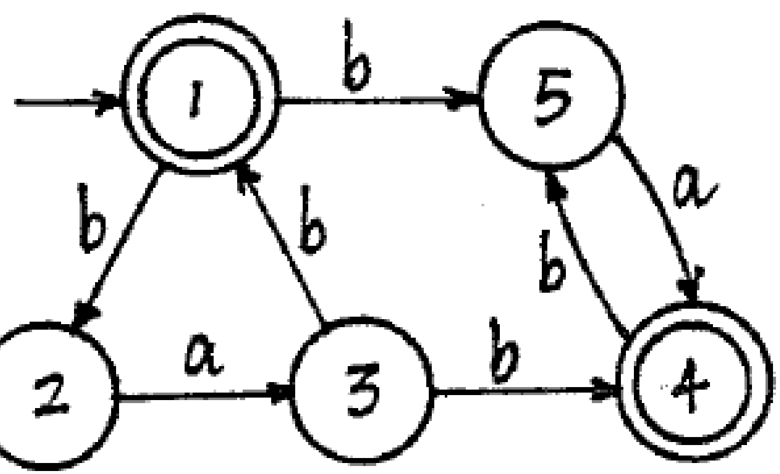
Example: Λ-transition elimination

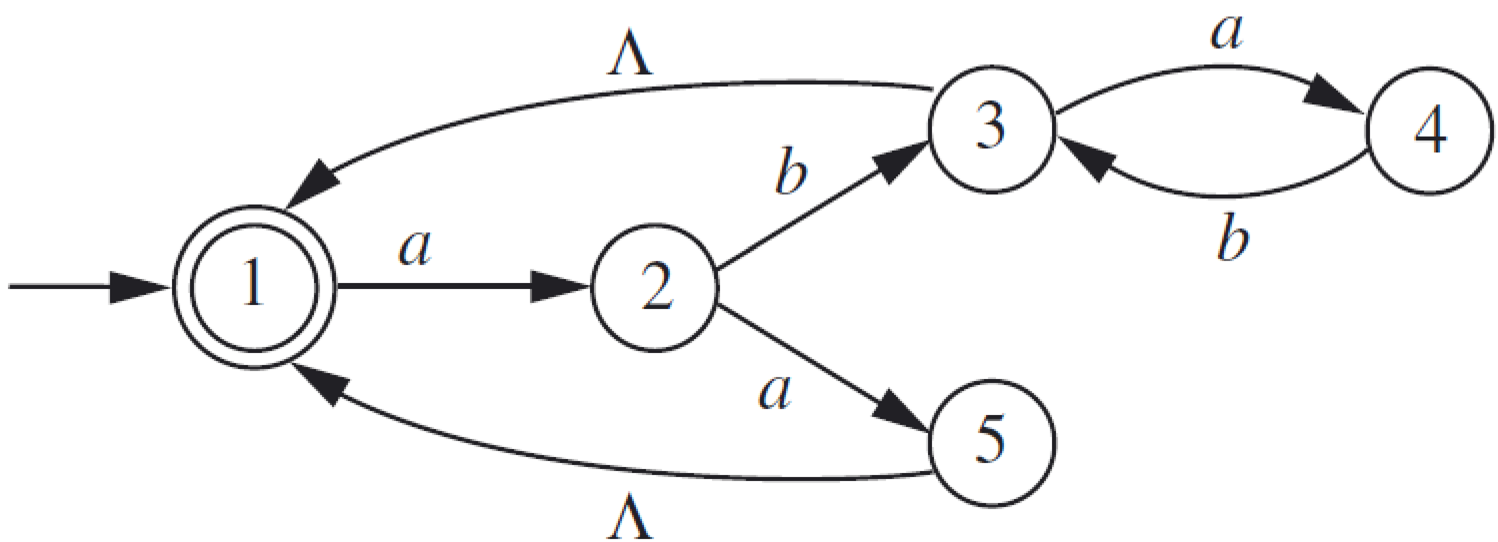
means 5 will be connected to 1, 2, and 4

q	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, \Lambda)$	$\delta^*(q, a)$	$\delta^*(q, b)$
1	∅	∅	{2}	{2, 3}	∅
2	{2, 3}	∅	∅	{2, 3}	∅
3	∅	{4}	∅	∅	{1, 2, 4}
4	∅	{5}	{1}	{2, 3}	{5}
5	{4}	∅	∅	{1, 2, 4}	∅

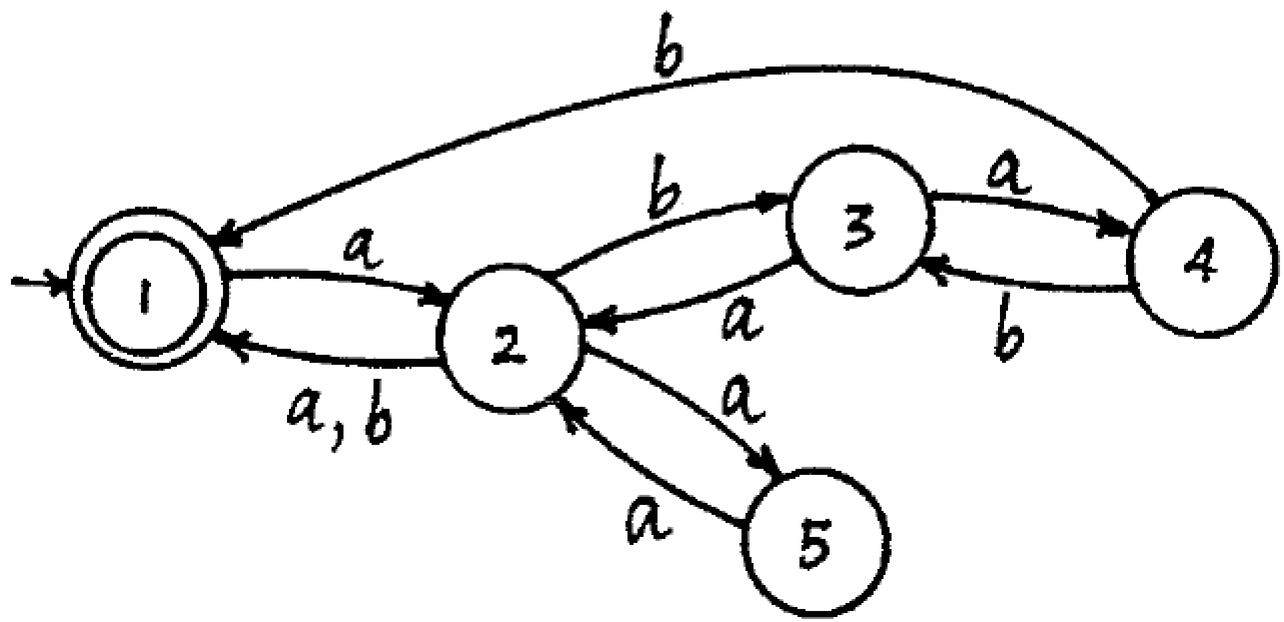
Eliminate Lambda-transition







Λ
a
3
4
b
b
a
1
2
a
5
Λ



b
b
3
a
4
a
1
2
a
b
a, b
a
a
5

The Nondeterminism in an NFA Can Be Eliminated

- **Theorem: For every language $L \subseteq \Sigma^*$ accepted by an NFA $M = (Q, \Sigma, q_0, A, \delta)$, there is an FA $M_1 = (Q_1, \Sigma, q_1, A_1, \delta_1)$ that also accepts L**
- We can assume M has no Λ-transitions. Let $Q_1 = 2^Q$ (for this reason, this is called the *subset construction*); $q_1 = \{q_0\}$; for every $q \in Q_1$ and $\sigma \in \Sigma$,

subset of Q

$$\delta_1(q, \sigma) = \cup\{\delta(p,\sigma) \mid p \in q\};$$

$$A_1 = \{q \in Q_1 \mid q \cap A \neq \emptyset\}$$

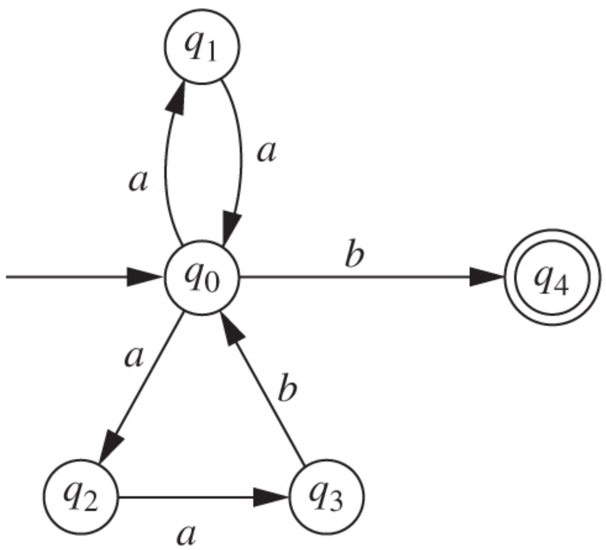
- M_1 is clearly an FA
  - It accepts the same language as M because for every $x \in \Sigma^*$, $\delta_1^*(q_1, x) = \delta^*(q_0, x)$
- The proof is by structural induction on x

Homework: prove this theorem (see Thm 3.18 in the textbook)

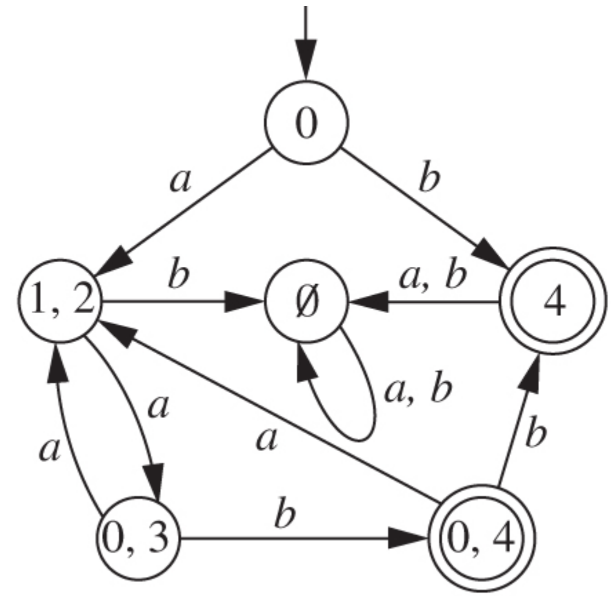
Example: Subset construction to eliminate nondeterminism

$\boldsymbol{M = (Q, \Sigma, q_0, A, \delta)}$ ⟶ $\boldsymbol{M_1 = (2^Q, \Sigma, \{q_0\}, A_1, \delta_1)}$

NFA to accept *{aa,aab}*{b}*



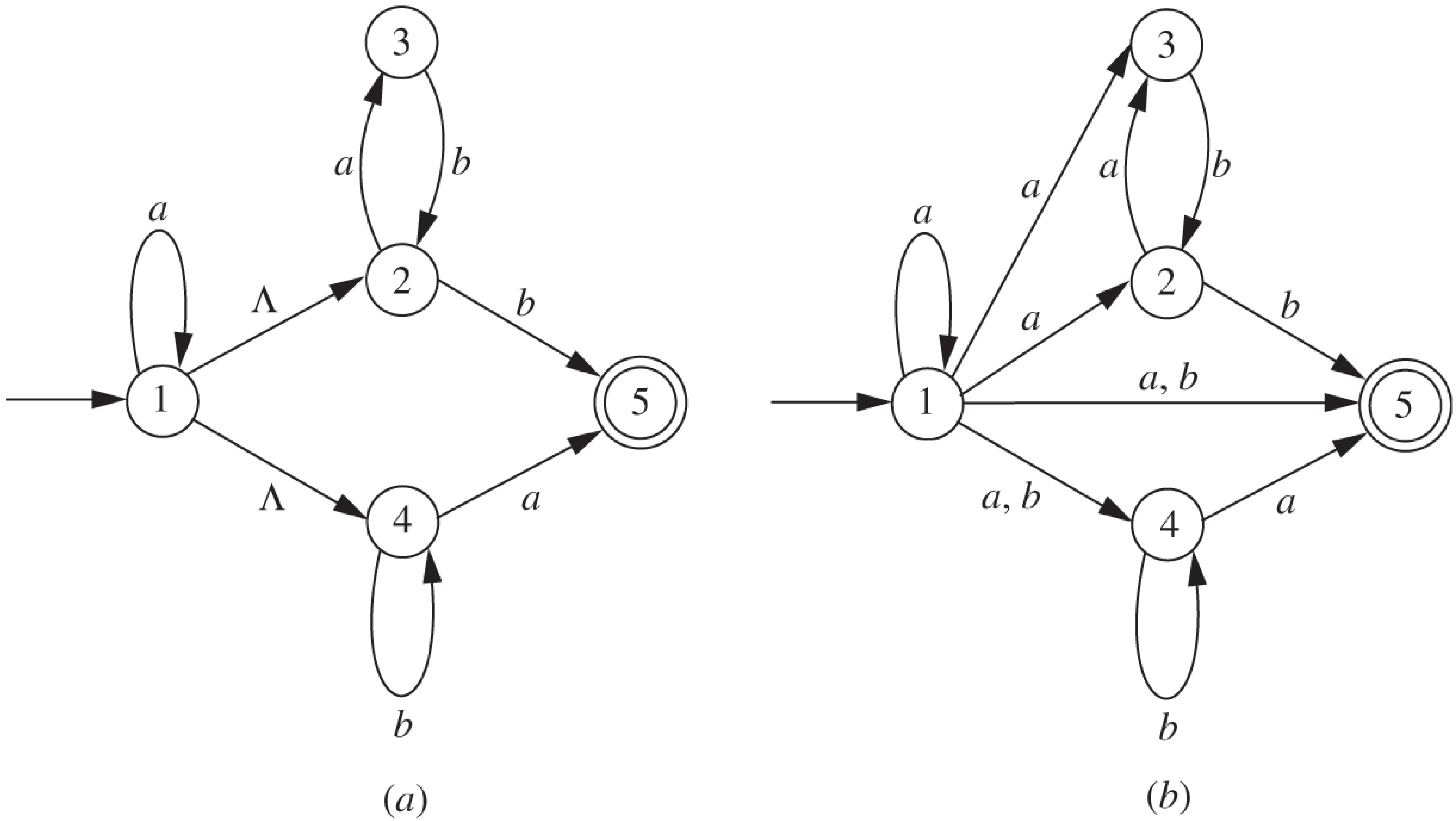
FA to accept *{aa,aab}*{b}*



- No need to generate 2^n subsets; consider only reachable states
- It is recommended to use a transition table
- Example: $\boldsymbol{\delta_1(\{1,2\},a) = \delta(1,a) \cup \delta(2,a) = \{0,3\}}$
- All reachable states that contain elements from A are in $\boldsymbol{A_1}$

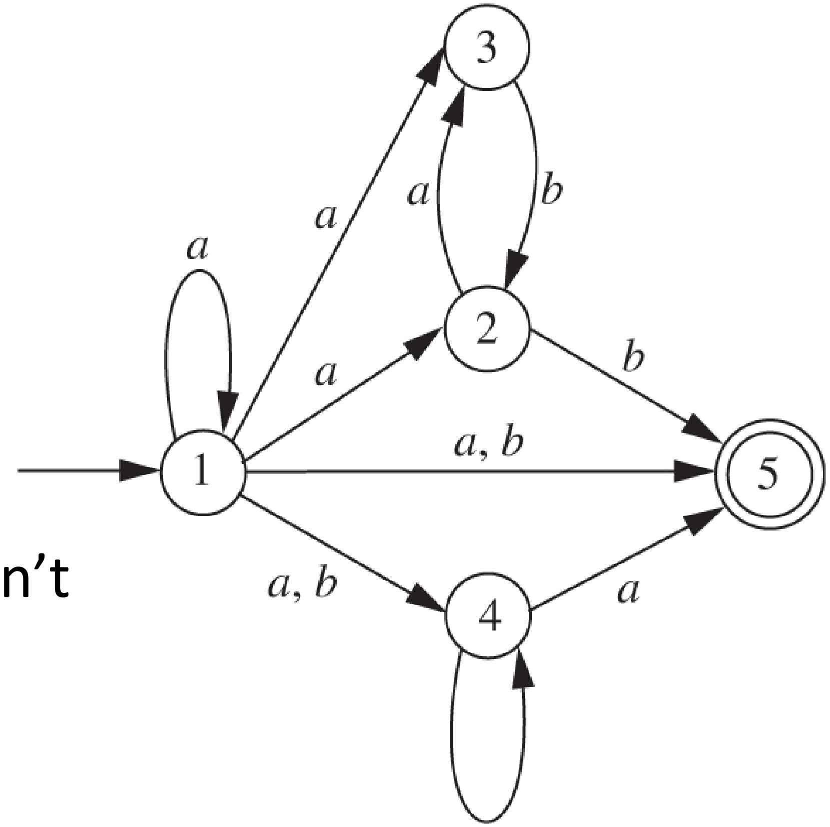
q	$\delta(q, a)$	$\delta(q, b)$
0	{1, 2}	{4}
1	{0}	Ø
2	{3}	Ø
3	Ø	{0}
4	Ø	Ø

Construct FA from NFA: 1) eliminate all lambda transitions.



Construct FA from NFA:

2) create a table of transitions.

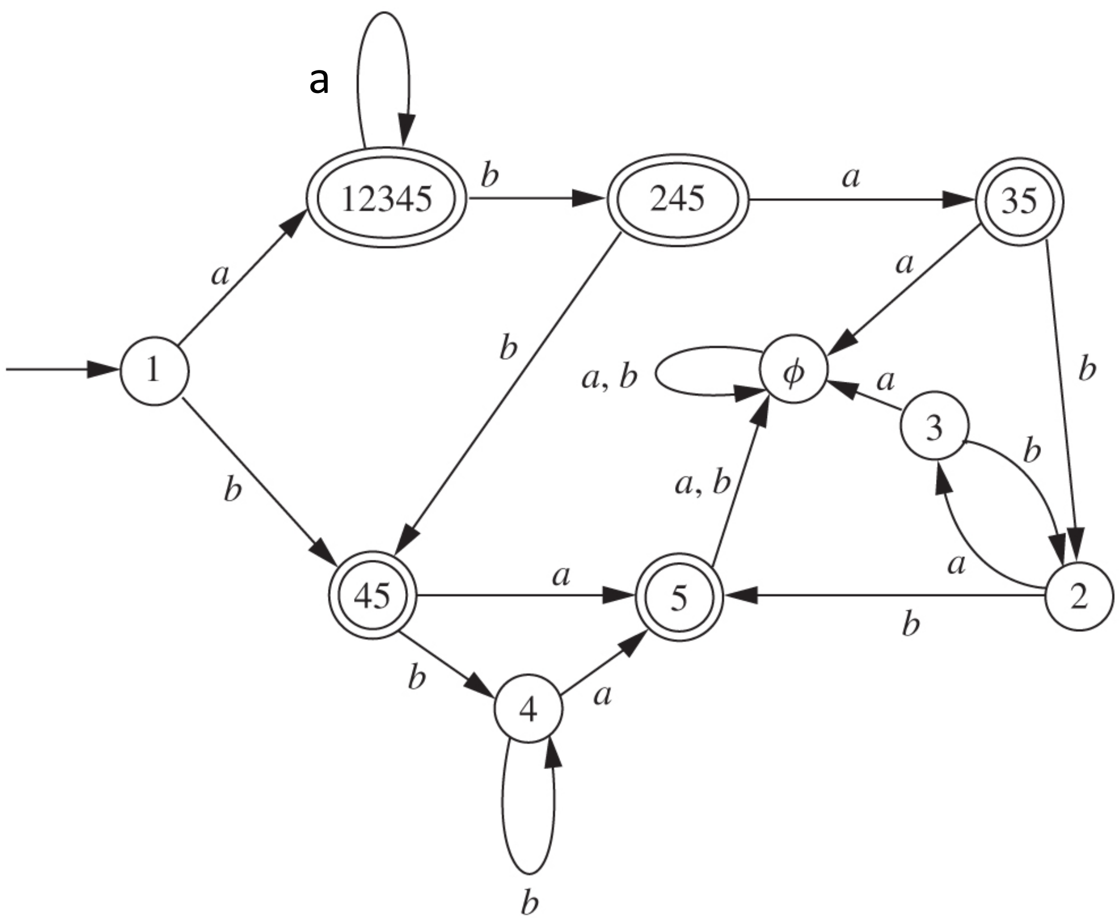


You need this only if you don't eliminate Lambda's

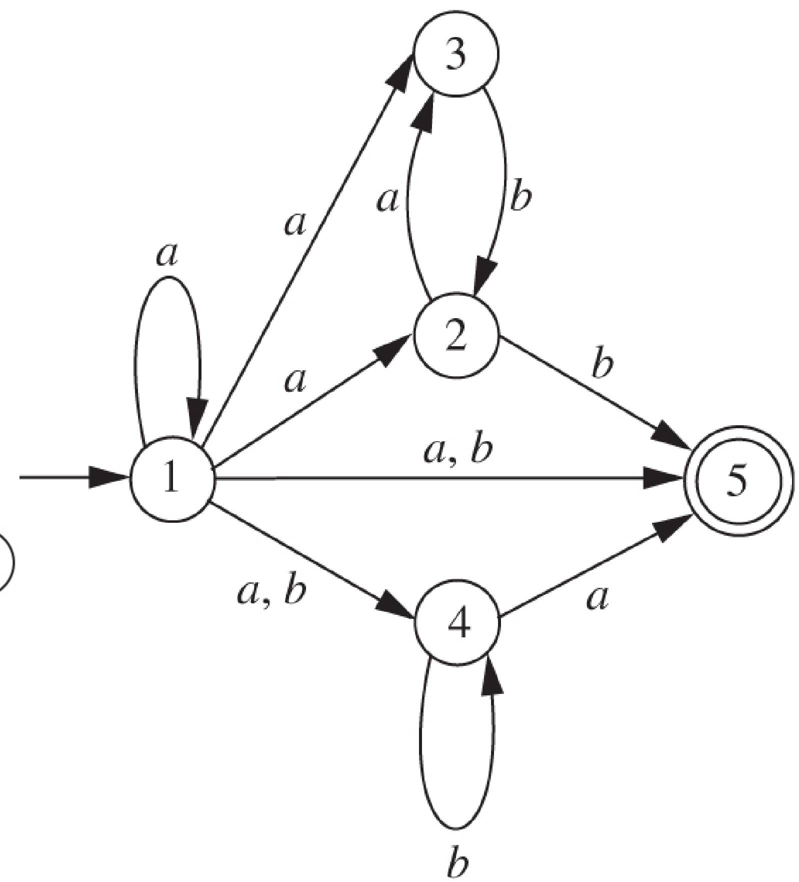
q	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, \Lambda)$	$\delta^*(q, a)$	$\delta^*(q, b)$
1	{1}	∅	{2, 4}	{1, 2, 3, 4, 5}	{4, 5}
2	{3}	{5}	∅	{3}	{5}
3	∅	{2}	∅	∅	{2}
4	{5}	{4}	∅	{5}	{4}
5	∅	∅	∅	∅	∅

3) Construct FA.

q	$\delta(q, a)$	$\delta(q, b)$	$\delta(q, \Lambda)$	$\delta^*(q, a)$	$\delta^*(q, b)$
1	{1}	Ø	{2, 4}	{1, 2, 3, 4, 5}	{4, 5}
2	{3}	{5}	Ø	{3}	{5}
3	Ø	{2}	Ø	Ø	{2}
4	{5}	{4}	Ø	{5}	{4}
5	Ø	Ø	Ø	Ø	Ø



(c)



(b)