


**Claim.** Suppose  $L \subseteq \{a, b\}^*$  is defined recursively as follows:


$$\Lambda \in L; \forall x \in L \ ax, axb \in L.$$

Show that  $L = L_0$ , where  $L_0 = \{a^i b^j \mid i \geq j\}$ .

*Proof sketch.* Proving  $L = L_0$  means that we need to prove both

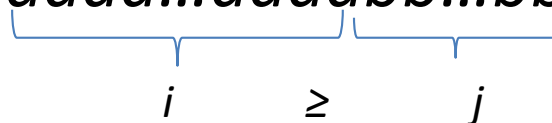
$$L \subseteq L_0, \text{ and } L_0 \subseteq L.$$

$$\forall x \in L \quad x \in L_0$$


$$\forall x \in L_0 \quad x \in L$$


Intuition:

1. In  $L$  strings are generated with more a's than b's
2. In  $L_0$  strings look like  $aaaa...aaaabb...bb$



$$\underbrace{\hspace{1.5cm}}_i \geq \underbrace{\hspace{1.5cm}}_j$$

**Claim.** Suppose  $L \subseteq \{a, b\}^*$  is defined recursively as follows:

$\Lambda \in L$ ; for every  $x \in L$ , both  $ax$  and  $axb$  are in  $L$ .

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*Proof sketch.*

Part 2 ( $L_0 \subseteq L$ ). We show by induction on  $n$

$\forall n \in \mathbb{N}$ , if  $y \in L_0$ , and  $|y| = n$ , then  $y \in L$ .

Basis step:  $y \in L_0$ , and  $|y| = 0 \Rightarrow y \in L$ . This is true because if  $|y| = 0$  then  $y = \Lambda$ , and  $\Lambda \in L$ .

Induction Hypothesis:  $k \in \mathbb{N}$ ,  $\forall y \in L_0$  such that

$$|y| \leq k \Rightarrow y \in L$$

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*Proof sketch (cont).*

IS:  $y \in L_0$ ,  $|y| = k + 1 \Rightarrow y = a^i b^j$ , and  $i + j = k + 1$ .

- $y \neq \Lambda$  because  $k \geq 0$  and  $|y| = k + 1 \Rightarrow$   
we must show that  $y = ax$  or  $y = axb$  for some  $x \in L$ .

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- We know that  $i + j > 0$ , and  $i \geq j \Rightarrow i > 0$ , i.e.,  $y = ax$  for some  $x$ ; if  $j > 0$  then  $y = axb$  for some  $x$ .

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- If  $j > 0$  then  $y = axb$ , where  $x = a^{i-1} b^{j-1}$ , and  $i - 1 \geq j - 1$ , i.e.,  $x \in L_0$  and by IH  $x \in L$  and then also  $y \in L$ . Case  $j = 0$  is similar.

# Summary

- In typical proofs by mathematical induction we choose an integer that is
  - The length of string
  - The number of substrings whose concatenation gives string  $x$
  - The exponent of the language in the  $*$  of some expression
- Typical proofs by structural induction (SI):
  - SI doesn't work without recursive definition (RD) of the language
  - The basis of SI corresponds to the basis of the RD
  - Formulate induction hypothesis on all input elements of the recursive rules
  - Prove induction step on all recursive rules in RD
  - Break down the problem you need to prove in induction step to easier problems in which you can apply the hypothesis, basis statement, etc.

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$xy = yx \Rightarrow \exists z \in \{a, b\}^*, \text{ and } i, j \in \mathbb{N}, \text{ such that } x = z^i, \text{ and } y = z^j.$

*Proof sketch.* Let  $d$  be the greatest common divisor of  $|x|$ , and  $|y|$ . We rewrite  $x$  and  $y$  as

$$x = x_1x_2 \cdots x_p, \text{ and } y = y_1y_2 \cdots y_q,$$

where all  $|x_i| = |y_j| = d$  for all  $i, j$ .

Since  $xy = yx$  then  $x^qy^p = y^px^q$  (begin with  $x^qy^p$ , and run repeated transpositions, i.e., switch  $x$  and  $y$ ).

$$xx \dots xy y \dots y = xx \dots yxy \dots y = \dots = yy \dots yxx \dots x$$

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*Proof sketch.*

Both  $x^q y^p$ , and  $y^p x^q$  have the same length  $2pqd$ , e.g.,

$$|x^q y^p| = |x_1 \cdots x_p \cdots q \text{ times } \cdots x_1 \cdots x_p y^q| = \cdots = 2pqd.$$

- In both cases, prefixes  $x^q$  (of  $x^q y^p$ ), and  $y^p$  (of  $y^p x^q$ ) have the same length and thus are equal.
- If  $x^q = (x_1 \cdots x_p)^q$  then  $x_1$  appears in positions  $1, pd+1, 2pd+1, \dots, (q-1)pd+1$ .
- In  $y^p$ , the substring  $y_{r_i}$  of length  $d$  can be found at  $ipd+1$ , where  $r_i = ip + 1 \pmod q$ . Since  $p$  and  $q$  have no common factors, all  $r_i$  are different. Then, it follows that all  $y_i$  are equal ( $z$ ). Same for  $x_i$ .