CMPT 280

Tutorial: Timing Analysis

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Common Growth Functions

- From most slowly growing, to most quickly growing, some common growth functions:

  - $\log \log n$
  - $\log n$ (logarithmic)
  - $\sqrt{n}$
  - n (linear)
  - $n \log n$
  - n^2 (quadratic)
  - n^3 (cubic)

    ...
  - n^k (polynomial hierarchy)
  - a^n (exponential hierarchy)
  - $n!$

Growth Functions in Big-O Notation

Which of these growth functions belong to $O(n)$? $O(n^2)$? $O(n^3)$? $O(2^n)$? $O(\log n)$?

1. $5 \log n + \sqrt{n} + 1000n^2$
2. $15n \log n + 2^n - 100$
3. $42n^3 + 3n^2 + 2n + 1$
4. $7n + 1400n! + \frac{12}{7} \log n$
5. $7700n^2 \log n$
6. $\frac{8}{11}n + 8\frac{\log n}{2} + 17(n-1)$

Statement Counting

Example: arrayMax

```
Algorithm arrayMax(A, n)
Precond: A is an array of n integers.
Returns: value of largest element of A

currentMax <- A[0]
i <- 1

while (i < n)
    if ( currentMax < A[i] )
        currentMax <- A[i]
    i <- i + 1

return currentMax
```

- Loop Body: *3 or 4 statements*
    - Single loop iteration: *3 or 4 statements*
    - Loop executed $n - 1$ times.
- Best case (`if` never true): $T_{arrayMax}(n) = 2 + 3(n-1) + 1 + 1 = 3n + 1$
- Worst case (`if` always true): $T_{arrayMax}(n) = 2 + 4(n-1) + 1 + 1 = 4n$

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Active Operation

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- Active operation: `while (i < n)`
- Number of executions of active operation: n

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- Active operation: `while (i < n)`
- Number of executions of active operation: n

$$T_{arrayMax(n)} \in O(n),\, T_{arrayMax(n)} \in \Theta(n).$$

Common Analysis Cases

Linear Loops

Simple counting loops are Linear Loops:

```
for(i = 0; i < n; i++) {
    <statements>
}

i = 0;
while(i < n) {
    <statements>
    i++
}
```

Loop executes n times (n is size of input) As long as number of statements in loop body is independent of n, such a loop is $\Theta(n)$.

This loop is also linear. Why?

```
for (i = 0; i < n; i+=2)
    <statements>
end for
```

Common Analysis Cases

Logarithmic Loops

Logarithmic Loops result when the counter is is multiplied or divided each iteration:

```
for(i = 1; i < n; i = i*2)
    <loop body>
end for

for(i = n; i >= 1; i = i/2)
    <loop body>
end for
```

Claim: Each of these loops executes $f(n) = c\log_2(n)$ times where c is the number of statements in the loop body. Thus each loop is $\Theta(\log n)$.

Common Analysis Cases

Logarithmic Loops

```
for(i = 1; i < n; i = i*2)
    <loop body>
end for
```

Consider the value of i in the above loop:

- First iteration: $i = 1 = 2^0$
- Second iteration: $i = 2 = 2^1$
- Third iteration: $i = 4 = 2^2$
- j-th iteration: $i = 2^{j-1}$

Common Analysis Cases

Logarithmic Loops

```
for(i = 1; i < n; i = i*2)
    <loop body>
end for
```

- j-th iteration: $i = 2^{j-1}$

Solving the last equation for j reveals that $j = \log i + 1$. Thus, if the loop stops on the j-th iteration, we know that $i >= n$. But we also know that on the $j-1$-th iteration, $i < n$. Thus, i can only be slightly bigger than n at most. Substituting this into $j = \log i + 1$ we obtain $j = \log n + 1$, but since j must be an integer, we have round up: $j = \lceil \log n + 1 \rceil$. This is the maximum possible value for j, thus the loop executes $O(\log n)$ times.

Common Analysis Cases

Quadratic Nested Loops

Simple quadratic loops occur when the inner and outer loops each execute a fixed number of times:

```
for(i = 0; i < n; i++)
    for(j = 0; j < n; j++)
        <loop body containing c statements>
    end loop
end loop
```

Thus, total number of statements is $f(n) = c \times n \times n$ which is $\Theta(n^2)$ (Assuming no methods in the loop body with time $> O(1)$).

What if the inner loop was `for(j = 0; j < m; j++)`?

Common Analysis Cases

Dependent Quadratic Nested Loops

Dependent quadratic loops result when the number of iterations in the inner loop depends on the value of the outer loop counter:

```
for(i = 0; i < n; i++)
    for(j = 0; j < i; j++)
        <loop body containing c-1 statements>
    end loop
end loop
```

Number of loop statementsfor each value of i:

$$
\begin{aligned}
0 + c + 2c + 3c + 4c + \cdots + (n-1)c &= c \cdot (1 + 2 + 3 + \ldots + n - 1) + 1 \\
&= c \cdot \sum_{i=1}^{n-1} i + 1 \\
&= c \cdot \frac{(n-1)n}{2} + 1 \\
&= \frac{c}{2}(n^2 - n) + 1 \in \Theta(n^2)
\end{aligned}
$$

Example: Matrix Sum

```
Algorithm addMatrix( matrix1, matrix2, matrix3, n)
Precond: matrix1 and matrix2 are 2D arrays of numbers of
         size n by n
Postcond: matrix3 contains the sum of matrix1 and matrix2

for(i = 0; i < n; i++) {
    for(j = 0; j < n; j++) {
        matrix3[i][j] = matrix1[i][j] + matrix2[i][j]
    }
}
```

What is the time complexity in the worst case? Best case?

Example: Prefix Averages

```
Algorithm prefixAverages(X, n)
Precond: X is an n-element array numbers
Output: An n-element array A of numbers such that A[i]
        is the average of X[0] : X[i]

i = 0
while (i < n) {
    avg = 0
    j = 0
    while ( j <= i )  {
        avg = avg + X[j]
        j++
    }
    A[i] = avg/(i+1)
    i++
}
return A
```

What is the time complexity in the worst case? Best case?

Example: Binary Search

```
Algorithm binarySearch (arr, n, key)
Precond: arr is a sorted (ascending order) integer array of
     length n;  key is value for which to search in arr
Postcond: arr is unchanged
Return: index of position of key in arr, -1 if not found

lo = 0
hi = n-1
while ( lo <= hi )
    mid = (lo + hi) / 2
    if( key < arr[mid] )
        hi = mid - 1
    else if ( key > arr[mid] )
        lo =  mid + 1
    else
        return mid;
return -1;
```

What's the time complexity of this algorithm in the worst case? Best case?