Foundation - Lesson 7 : State Space Representation

1 Background

- Linear State Space Form
- Non-Linear State Space Form

2 Forming State Space

- Step 1 : Obtain Equation of Motion.
- Step 2 : Choose State Variables [ex: position, velocity ...].
- Step 3 : Take Derivative of State Vector.
- Step 4 : Write in State-Space form
- Step 5 : Write Output Equation.

2.1 Example 1

Ex : Obtain S.S from system below

- Step 1 : Obtain Equation of Motion.

$$\ddot{y} + 4\dot{y} + 3y = 3u$$

- Step 2 : Choose State Variables. We would like to know y and $\dot{y}$. Thus, Let Choose:

$$X_1 = y$$
$$X_2 = \dot{y}$$

- Step 3 : Take Derivative of State Vector.

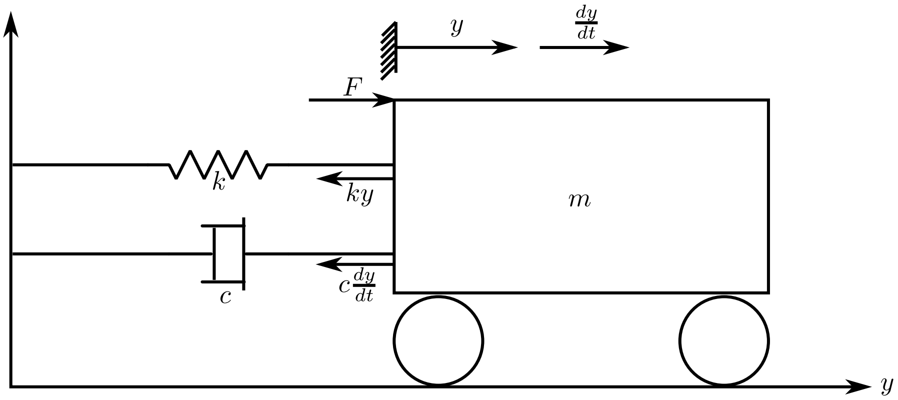
$$X_1 = y => \dot{X}_1 = \dot{y}$$
$$X_2 = \dot{y} => \dot{X}_2 = \ddot{y} = 3u - 4\dot{y} - 3y$$
$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ 3u - 4\dot{y} - 3y \end{bmatrix}$$

- Step 4 : Write in State-Space form.

$$\dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u$$

- Step 5 : Write Output Equation. We choose $y = y_{one}$ because we only interest in displacement only X_1, if we are interested in velocity X_2 as well we choose $y = y_{two}$.

$$y_{one} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ or } y_{two} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$



2.2 Example 2

Ex : Obtain S.S from system of mass, spring, damper

- Step 1 : Obtain Equation of Motion. From the 2nd law of Newton:

$$\sum \vec{F} = m\vec{a}$$
$$F - ky - c\dot{y} = m\ddot{y}$$
$$m\ddot{y} + c\dot{y} + ky = F$$
$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = \frac{F}{m}$$

- Step 2 : Choose State Variables. We would like to know y and $\dot{y}$. Thus, Let Choose:

$$X_1 = y$$
$$X_2 = \dot{y}$$

- Step 3 : Take Derivative of State Vector.

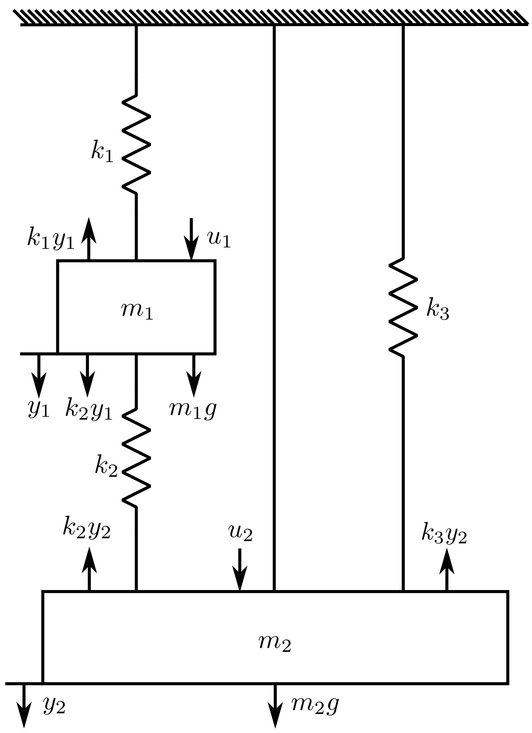
$$X_1 = y => \dot{X}_1 = \dot{y}$$
$$X_2 = \dot{y} => \dot{X}_2 = \ddot{y} = \frac{F}{m} - \frac{c}{m}\dot{y} - \frac{k}{m}y$$
$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \frac{F}{m} - \frac{c}{m}\dot{y} - \frac{k}{m}y \end{bmatrix}$$

- Step 4 : Write in State-Space form.

$$\dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

- Step 5 : Write Output Equation.

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$



2.3 Example 3

Ex : Obtain S.S from system of mass, spring with 2 vertical mass .

- Step 1 : Obtain Equation of Motion. From the 2nd law of Newton:

$$\sum \vec{F} = m\vec{a}$$

$$\begin{aligned} \text{Mass 1:} &\quad -k_1y_1 + k_2y_1 + u_1 + k_2y_2 = m_1\ddot{y}_1 \\ \text{Mass 2:} &\quad -k_3y_2 - k_2y_2 + u_2 + k_2y_1 = m_2\ddot{y}_2 \end{aligned}$$

- Step 2 : Choose State Variables. We would like to know y and $\dot{y}$. Thus, Let Choose:

$$\begin{aligned} X_1 &= y_1 \\ X_2 &= \dot{y}_1 \\ X_3 &= y_2 \\ X_4 &= \dot{y}_2 \end{aligned}$$

- Step 3 : Take Derivative of State Vector.

$$X_1 = y_1 => \dot{X}_1 = \dot{y}_1$$

$$X_2 = \dot{y}_1 => \dot{X}_2 = \ddot{y}_1 = -\frac{k_1}{m_1}y_1 + \frac{k_2}{m_1}y_1 + \frac{1}{m_1}u_1 + \frac{k_2}{m_1}y_2 = \frac{k_2 - k_1}{m_1}y_1 + \frac{1}{m_1}u_1 + \frac{k_2}{m_1}y_2$$

$$X_3 = y_2 => \dot{X}_3 = \dot{y}_2$$

$$X_4 = \dot{y}_2 => \dot{X}_4 = \ddot{y}_2 = -\frac{k_3}{m_2}y_2 - \frac{k_2}{m_2}y_2 + \frac{1}{m_2}u_2 + \frac{k_2}{m_2}y_1 = \frac{-k_3 - k_2}{m_2}y_2 + \frac{1}{m_2}u_2 + \frac{k_2}{m_2}y_1$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 \\ \frac{k_2-k_1}{m_1}y_1 + \frac{1}{m_1}u_1 + \frac{k_2}{m_1}y_2 \\ \dot{y}_2 \\ \frac{-k_3-k_2}{m_2}y_2 + \frac{1}{m_2}u_2 + \frac{k_2}{m_2}y_1 \end{bmatrix}$$

- Step 4 : Write in State-Space form.

$$\dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{k_2-k_1}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \frac{-k_3-k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- Step 5 : Write Output Equation.

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

2.4 Example 4

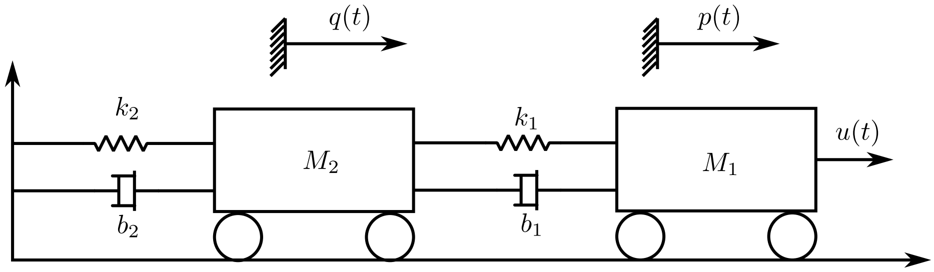
Example : Solve system of single mass and spring and force using Matlab. MATLAB Numerical Method using ode45(Runge Kutta)

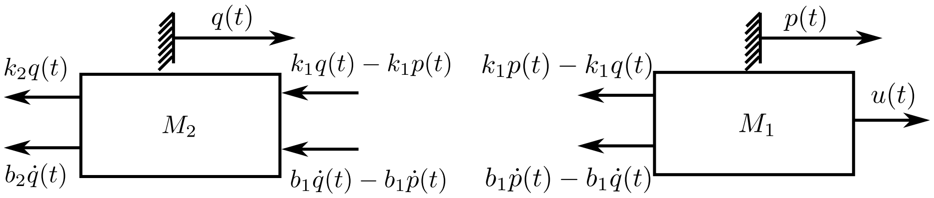
```
[t,x] = ode45(@f,tspan,x_0)
t = time
x = state vector
ode45 = solver
f = function
tspan = t_0 -> t_f
x_0 = initial condition


Example:

tspan = [0,10];
x_0 = [0,0];

function dx = model(t,x)
% dx = Ax+Bu
k = 0.01;m=1;u=2;
A = [0 1;-k/m 0];
B = [0;1/m];
dx = A*x + B*u;

[t,x] = ode45(@model,tspan,x_0);
plot(t,x(:;1))
hold on
plot(t,x(:;2))
legend('displacement','velocity')
```

2.5 Example 5

Ex : Obtain S.S from system of mass, spring, damper with 2 horizontal mass . Equation of Motion

$$\sum \vec{F} = m\vec{a}$$





$$\text{Mass 1: } m_1\ddot{p}(t) + b_1\dot{p}(t) + k_1p(t) = u(t) + k_1q(t) + b_1\dot{q}(t)$$
$$\text{Mass 2: } m_2\ddot{q}(t) + (k_1+k_2)q(t) + (b_1+b_2)\dot{q}(t) = k_1p(t) + b_1\dot{p}(t)$$
$$\ddot{p}(t) = \frac{1}{m_1}u(t) + \frac{k_1}{m_1}q(t) + \frac{b_1}{m_1}\dot{q}(t) - \frac{b_1}{m_1}\dot{p}(t) - \frac{k_1}{m_1}p(t)$$
$$\ddot{q}(t) = \frac{k_1}{m_2}p(t) + \frac{b_1}{m_2}\dot{p}(t) - \frac{(k_1+k_2)}{m_2}q(t) - \frac{(b_1+b_2)}{m_2}\dot{q}(t)$$

Let:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{bmatrix} => \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \ddot{p} \\ \ddot{q} \end{bmatrix}$$

Thus, we get state space form:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & \frac{b_1}{m_2} & -\frac{(b_1+b_2)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

3 State Space of Scalar Differential Equation System

3.1 Case 1

Consider equation below:

$$y^{(n)} + a_1y^{(n-1)} + ... + a_{n-1}y' + a_ny = u \leftarrow \text{Input has not derivative}$$

Let:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{n-1} \\ y^n \end{bmatrix}$$

Thus

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ \vdots \\ y^n \\ -a_0x_1 - a_1x_2 ... - a_nx_n + u \end{bmatrix} = \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \\ -a_0x_1 - a_1x_2 ... - a_nx_n + u \end{bmatrix}$$

Arrange into SS form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & ... & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We have a corresponding Transfer Function is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1s^{n-1} + ... + a_{n-1}s + a_n}$$

3.2 Case 2

Consider equation below:

$$y^{(n)} + a_1y^{(n-1)} + ... + a_{n-1}y' + a_ny = \beta_0u^n + \beta_1u^{n-1} + ... + \beta_nu \leftarrow \text{Input has derivative}$$

Let:

$$\begin{aligned} x_1 &= y - \beta_0u \\ x_2 &= y' - \beta_0u' - \beta_1u = x_1' - \beta_1u \\ &\vdots \\ x_n &= y^{n-1} - \beta_0u^{n-1} - ... - \beta_{n-1}u = x_{n-1}' - \beta_{n-1}u \end{aligned}$$

Where $\beta_0, \beta_1, ..., \beta_{n-1}$ are determined from:

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1\beta_0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ &\vdots \end{aligned}$$

Arrange into SS form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & ... & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

We have a corresponding Transfer Function is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

4 Transfer Function to State Space

Example:

$$\frac{Y(s)}{U(s)} = \frac{100}{s^4 + 20s^3 + 10s^2 + 7s + 100}$$

$$(s^4 + 20s^3 + 10s^2 + 7s + 100)Y(s) = 100U(s)$$

Taking Inverse Laplace Transform

$$y^{(4)} + 20y^{(3)} + 10y'' + 7y' + 100y = 100u$$

Let:

$$\begin{aligned}
&x_1 = y => \dot{x}_1 = y' = x_2 \\
&x_2 = y' => \dot{x}_2 = y'' = x_3 \\
&x_3 = y'' => \dot{x}_3 = y^{(3)} = x_4 \\
&x_3 = y''' => \dot{x}_4 = y^{(4)} = 100u - 20y^{(3)} - 10y'' - 7y' - 100y
\end{aligned}$$

State Space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -7 & -10 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

5 State Space to Transfer Function

We have a Transfer Function:

$$\frac{Y(s)}{U(s)} = G(s)$$

with state space in form of:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Let have a Laplace transform of SS:

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

Assuming $x(0) = 0IC$, we get:

$$\begin{aligned} sX(s) - AX(s) &= BU(s) \\ (sI - A)X(s) &= BU(s) \\ (sI - A)^{-1}(sI - A)X(s) &= (sI - A)^{-1}BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) \end{aligned}$$

Substitute into $Y(s)$

$$\begin{aligned} Y(s) &= C[(sI - A)^{-1}BU(s)] + DU(s) \\ Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ Y(s) &= [C(sI - A)^{-1}B + D]U(s) \end{aligned}$$

Thus the Transfer function can be found by:

$$G(s) = C(sI - A)^{-1}B + D$$

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} [\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix}]^{-1} \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} + 0$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ -5 & -25 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{(s^2+5s+25)}{(s^3+5s^2+25s-5)} & \frac{(-s-5)}{(s^3+5s^2+25s-5)} & \frac{1}{(s^3+5s^2+25s-5)} \\ \frac{-5}{(s^3+5s^2+25s-5)} & \frac{(s^2+5s)}{(s^3+5s^2+25s-5)} & \frac{-s}{(s^3+5s^2+25s-5)} \\ \frac{5s}{(s^3+5s^2+25s-5)} & \frac{(25s-5)}{(s^3+5s^2+25s-5)} & \frac{s^2}{(s^3+5s^2+25s-5)} \end{bmatrix} \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{(-25s-245)}{(s^3+5s^2+25s-5)} \\ \frac{(25s^2+245s)}{(s^3+5s^2+25s-5)} \\ \frac{(-120s^2+625s-125)}{(s^3+5s^2+25s-5)} \end{bmatrix}$$

$$G(s) = \frac{(-25s-245)}{(s^3+5s^2+25s-5)}$$

Thus

$$G(s) = \frac{25s+245}{s^3+5s^2+25s+5}$$