Lesson 1 : DC Motor

1 Background

DC motor is a mechatronic product that consist of two parts: the mechanical part and the electrical part. A typical dc motor used by a robot is constructed by: a dc motor, a wheel encoder (for measuring rotation pulse of motor), and a gear box (for reducing the speed of motor).

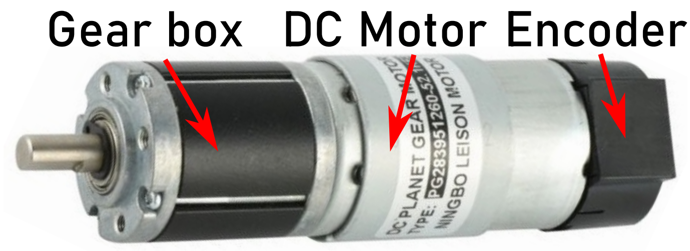

Figure 1: Typical dc motor

2 Modelling

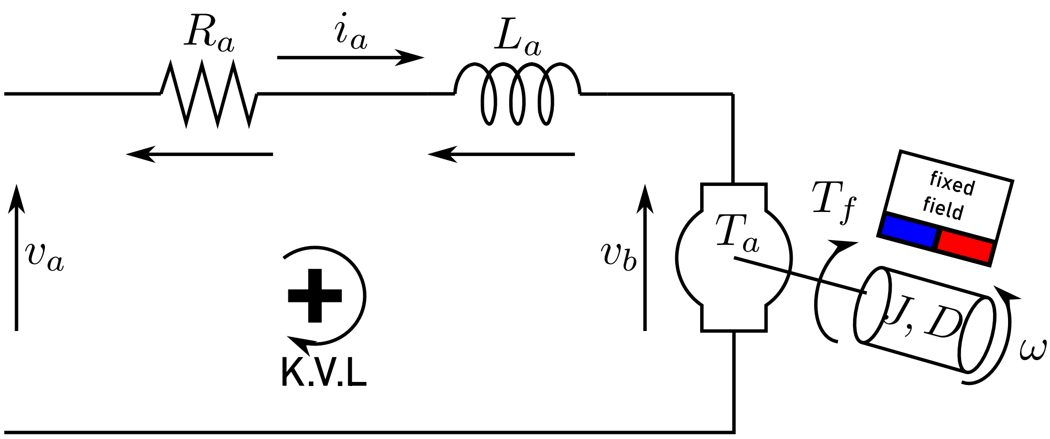

Figure 2: dc motor model

2.1 Electrical Part

$$v_b(t) = K_b\dot{\theta}(t) = K_b\omega(t) \tag{1}$$

Where:

- $v_b(t)$ is voltage at terminal conductor of motor
- K_b is back emf constant
- $\dot{\theta} = \omega$ is angular velocity of motor

$$T_a = K_t i_a(t) \tag{2}$$

Where:

- T_a is rotor torque
- K_t is motor torque
- i_a is the current draw by motor

By applying Kirchoff Voltage Law to the circuit loop in Figure 2

- $v_a(t)$ is input voltage from power source
- $v_{resistance} = R_a i_a(t)$ is voltage across resistance
- $v_{inductor} = L_a \frac{di_a(t)}{dt}$ is voltage across inductor

$$v_a(t) - v_b(t) - R_a i_a(t) - L_a \frac{di_a(t)}{dt} = 0$$

$$\Rightarrow v_a(t) = v_b(t) + R_a i_a(t) + L_a \frac{di_a(t)}{dt} \tag{3}$$

Substitute Equation 1 into Equation 3, we get:

$$v_a(t) = K_b \omega(t) + R_a i_a(t) + L_a \frac{di_a(t)}{dt} \tag{4}$$

In practical dc motor the L_a is very small ($L_a \approx 0$) and neglectable.

$$v_a(t) = K_b \omega(t) + R_a i_a(t)$$

$$\Rightarrow \boxed{i_a(t) = \frac{v_a(t) - K_b \omega(t)}{R_a}} \tag{5}$$

2.2 Mechanical Part

$$T_a = T_f + J\dot{\omega}(t) \tag{6}$$

Where:

- T_f is torque of coulomb friction and viscous friction
- J is moment of inertia

We know that:

$$T_f = T_c sign[\omega(t)] + D\omega(t) \tag{7}$$

Where:

- T_c is coulomb friction torque
- D is coefficient viscous friction

Substitute Equation 7 to Equation 6, we get:

$$T_a = T_c sign[\omega(t)] + D\omega(t) + J\dot{\omega}(t)$$

2.2.1 Approximation of coulomb friction to zero $T_c \approx 0$

$$T_a = D\omega(t) + J\dot{\omega}(t) \tag{8}$$

From Equation 2: $T_a = K_t i_a(t)$ substitute to Equation 8:

$$K_t i_a(t) = D\omega(t) + J\dot{\omega}(t)$$

$$\Rightarrow \boxed{i_a(t) = \frac{D\omega(t) + J\dot{\omega}(t)}{K_t}} \tag{9}$$

2.2.2 Keep coulomb friction T_c

$$\Rightarrow \boxed{i_a(t) = \frac{T_c sign[\omega(t)] + D\omega(t) + J\dot{\omega}(t)}{K_t}} \tag{10}$$

3 Electrical and Mechanical combine

3.1 Approximation of coulomb friction to zero $T_c \approx 0$

From Equation 5 and Equation 9: Put it side by side:

$$i_a(t) = i_a(t)$$

$$\frac{v_a(t) - K_b\omega(t)}{R_a} = \frac{D\omega(t) + J\dot{\omega}(t)}{K_t}$$

Get $\dot{\omega}(t)$:

$$\dot{\omega}(t) = \frac{\frac{(v_a(t)-K_b\omega(t))K_t}{R_a} - D\omega(t)}{J}$$

$$\dot{\omega}(t) = \frac{(v_a(t) - K_b\omega(t))K_t}{R_a J} - \frac{D\omega(t)}{J}$$

$$\dot{\omega}(t) = \frac{v_a(t)K_t - K_b K_t \omega(t)}{R_a J} - \frac{D\omega(t)}{J}$$

Separate $\omega(t)$ and $v_a(t)$:

$$\boxed{\dot{\omega}(t) = -(\frac{K_b K_t + DR_a}{R_a J})\omega(t) + \frac{K_t}{R_a J} v_a(t)} \tag{11}$$

Let:

- $a = (\frac{K_b K_t + DR_a}{R_a J})\omega(t) \qquad [1/s]$
- $b = \frac{K_t}{R_a J} \qquad [rad/s^2/V]$

We get lamped Parameter in a simplified form as:

$$\Rightarrow \boxed{\dot{\omega}(t) = -a\omega(t) + bv_a(t)} \tag{12}$$

3.2 Keep coulomb friction T_c

$$\boxed{\dot{\omega}(t) = -(\frac{K_b K_t + DR_a}{R_a J})\omega(t) + \frac{K_t}{R_a J} v_a(t) - \frac{T_c}{J} sign(\omega(t))} \tag{13}$$

Let:

- $a = (\frac{K_b K_t + DR_a}{R_a J})$ $[1/s]$
- $b = \frac{K_t}{R_a J}$ $[rad/s^2/V]$
- $c = \frac{T_c}{J}$ $[.]$

We get lamped Parameter in a simplified form as:

$$\Rightarrow \boxed{\dot{\omega}(t) = -a\omega(t) + bv_a(t) - csign(\omega(t))} \tag{14}$$

4 Simulation

In general, the equation $\dot{\omega}(t) = -a\omega(t) + bv_a(t) - csign(\omega(t)$ is used to represent all the dc motor in the market. By modifying the parameters a, b, c will result in different dc motor.

From equation $\dot{\omega}(t) = -a\omega(t) + bv_a(t) - csign(\omega(t)$

- $\dot{\omega}(t)$ is the angular acceleration of dc motor and is the output of the system
- $\omega(t)$ is the angular velocity of dc motor and is the output of the system
- $v_a(t)$ is the input voltage to dc motor and is the input of the system

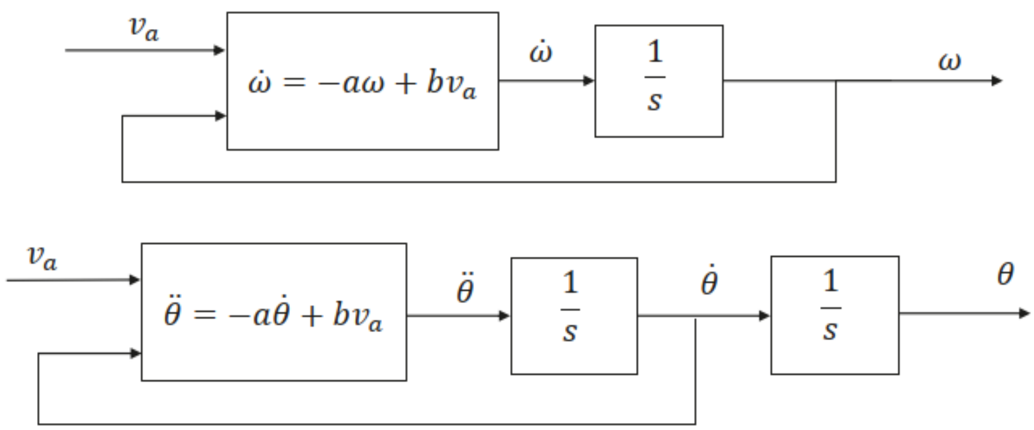

Figure 3: Simulation Flow

4.1 SIMULINK

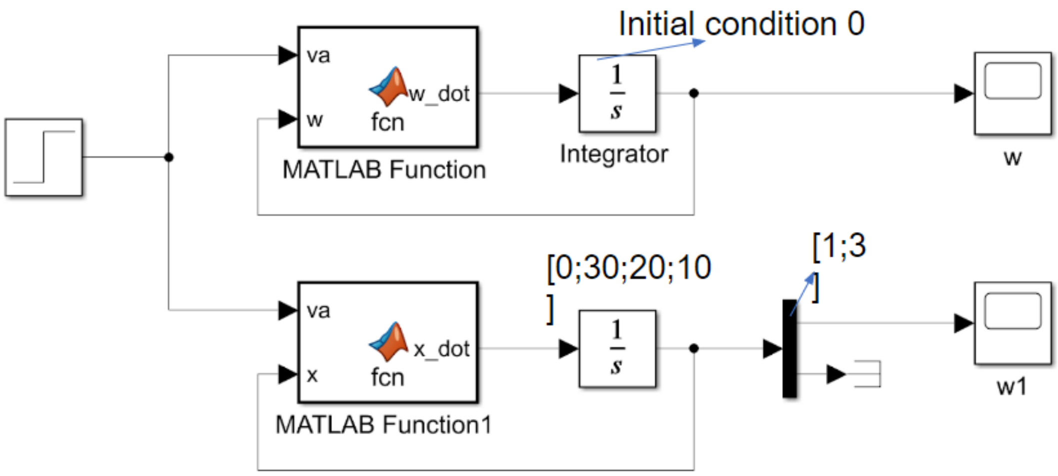

Figure 4: SIMULINK Simulation Flow

5 DC Motor 2nd Order Model (L_a is not Neglected)

5.1 No Friction

From Equation 4 and Equation 9, we have:

$$
\begin{aligned}
v_a(t) &= K_b\omega(t) + R_a i_a(t) + L_a\frac{di_a(t)}{dt} \\
i_a(t) &= \frac{D\omega(t) + J\dot{\omega}(t)}{K_t} \\
\frac{di_a(t)}{d(t)} &= \frac{d}{dt}\left(\frac{D\omega(t) + J\dot{\omega}(t)}{K_t}\right) \\
&= \frac{1}{K_t}\frac{d}{dt}(D\omega(t) + J\dot{\omega}(t)) \\
&= \frac{1}{K_t}\frac{d}{dt}(D\omega(t)) + \frac{d}{dt}(J\dot{\omega}(t)) \\
&= \frac{1}{K_t}(D\dot{\omega}(t) + J\ddot{\omega}(t))
\end{aligned}
$$

We get:

$$
v_a(t) = K_b\omega(t) + R_a\frac{D\omega(t) + J\dot{\omega}(t)}{K_t} + L_a\frac{1}{K_t}(D\dot{\omega}(t) + J\ddot{\omega}(t)) \tag{15}
$$

$$
\begin{aligned}
K_t v_a(t) &= K_t K_b\omega(t) + R_a(D\omega(t) + J\dot{\omega}(t)) + L_a(D\dot{\omega}(t) + J\ddot{\omega}(t)) \\
K_t v_a(t) &= K_t K_b\omega(t) + R_a D\omega(t) + R_a J\dot{\omega}(t) + L_a D\dot{\omega}(t) + L_a J\ddot{\omega}(t) \\
K_t v_a(t) &= (K_t K_b + R_a D)\omega(t) + (R_a J + L_a D)\dot{\omega}(t) + L_a J\ddot{\omega}(t) \\
L_a J\ddot{\omega}(t) &= -(K_t K_b + R_a D)\omega(t) - (R_a J + L_a D)\dot{\omega}(t) + K_t v_a(t) \\
\ddot{\omega}(t) &= -\frac{R_a J + L_a D}{L_a J}\dot{\omega}(t) - \frac{K_t K_b + R_a D}{L_a J}\omega(t) + \frac{K_t}{L_a J}v_a(t)
\end{aligned}
$$

Let:

- $a = \frac{R_a J + L_a D}{L_a J}$ [.]
- $b = \frac{K_t K_b + R_a D}{L_a J}$ [.]
- $c = \frac{K_t}{L_a J}$ [.]

We get lamped Parameter in a simplified form as:

$$\Rightarrow \boxed{\ddot{\omega}(t) = -a\dot{\omega}(t) - b\omega(t) + cv_a(t)} \tag{16}$$

5.2 With Friction

From Equation 4 and Equation 10, we have:

$$\begin{aligned} v_a(t) &= K_b\omega(t) + R_a i_a(t) + L_a\frac{di_a(t)}{dt} \\ i_a(t) &= \frac{T_c sign[\omega(t)] + D\omega(t) + J\dot{\omega}(t)}{K_t} \end{aligned}$$

$$\begin{aligned} \frac{di_a(t)}{d(t)} &= \frac{d}{dt}\left(\frac{T_c sign[\omega(t)] + D\omega(t) + J\dot{\omega}(t)}{K_t}\right) \\ &= \frac{1}{K_t}(D\dot{\omega}(t) + J\ddot{\omega}(t)) \leftarrow \text{derivative of sign function is 0} \end{aligned}$$

We get:

$$v_a(t) = K_b\omega(t) + R_a\frac{T_c sign[\omega(t)] + D\omega(t) + J\dot{\omega}(t)}{K_t} + L_a\frac{1}{K_t}(D\dot{\omega}(t) + J\ddot{\omega}(t)) \tag{17}$$

$$\begin{aligned} K_t v_a(t) &= K_t K_b\omega(t) + R_a T_c sign[\omega(t)] + R_a D\omega(t) + R_a J\dot{\omega}(t)) + L_a D\dot{\omega}(t) + L_a J\ddot{\omega}(t) \\ \ddot{\omega}(t) &= -\frac{R_a J + L_a D}{L_a J}\dot{\omega}(t) - \frac{K_t K_b + R_a D}{L_a J}\omega(t) + \frac{K_t}{L_a J}v_a(t) - \frac{R_a T_c}{L_a J}sign[\omega(t)] \end{aligned}$$

Let:

- $a = \frac{R_a J + L_a D}{L_a J}$ [.]
- $b = \frac{K_t K_b + R_a D}{L_a J}$ [.]
- $c = \frac{K_t}{L_a J}$ [.]
- $d = \frac{R_a T_c}{L_a J}$ [.]

We get lamped Parameter in a simplified form as:

$$\Rightarrow \boxed{\ddot{\omega}(t) = -a\dot{\omega}(t) - b\omega(t) + cv_a(t) - dsign(\omega(t))} \tag{18}$$