Foundation - Lesson 4 : 2nd Order ODE Standard Form Natural Frequency and Damping Ratio

1 Spring Mass Damper Modeling

(Free Body Diagram)

$$
\begin{aligned}
-kx(t) - b\dot{x}(t) &= m\ddot{x}(t) \\
m\ddot{x}(t) + kx(t) + b\dot{x}(t) &= 0 \\
\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x(t) &= 0
\end{aligned}
$$

Let have the differential model above to look like the General Standard Form of :

$$\boxed{\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0}$$

Where :

- ζ is damping ratio
- ω_n is natural frequency

Thus, we have :

- $\frac{b}{m} = 2\zeta\omega_n \rightarrow \zeta = \frac{b}{2\sqrt{km}}$
- $\frac{k}{m} = \omega_n^2 \rightarrow \omega_n = \sqrt{\frac{k}{m}}$

Let solve the Standard Form $x(t)$:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

Using Laplace Transform :

$$
\begin{aligned}
s^2X(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n(sX(s) - x(0)) + \omega_n^2 X(s) &= 0 \\
s^2X(s) - sx_0 - \dot{x}_0 + 2\zeta\omega_n(sX(s) - x_0) + \omega_n^2 X(s) &= 0
\end{aligned}
$$

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let Find the root of the Denominator of $X(s)$. From solving the 2nd order quadratic formula, we have the root :

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

From the root, we can see that there are 3 cases:

- Distinct Real Root
- Double Real Root
- Complex Root

1.1 Distinct Real Roots

To have the Distinct Real Root Case, We need:

$$\begin{aligned}(2\zeta\omega_n)^2 - 4\omega_n^2 &> 0\\ 4\zeta^2\omega_n^2 - 4\omega_n^2 &> 0\\ 4\omega_n^2(\zeta^2 - 1) &> 0\\ (\zeta^2 - 1) &> 0\\ \zeta^2 &> 1\\ \zeta &> 1\end{aligned}$$

We get Over-damped Case from the damping ratio of $\boxed{\zeta > 1}$

1.2 Double Real Root

To have the Double Real Root Case, We need:

$$\begin{aligned}(2\zeta\omega_n)^2 - 4\omega_n^2 &= 0\\ 4\zeta^2\omega_n^2 - 4\omega_n^2 &= 0\\ 4\omega_n^2(\zeta^2 - 1) &= 0\\ (\zeta^2 - 1) &= 0\\ \zeta^2 &= 1\\ \zeta &= 1\end{aligned}$$

We get Critically damped Case from the damping ratio of $\boxed{\zeta = 1}$
From the mathematical perspective, the damping ratio is unity (1) mean Critically damped. Where some people from control perspective prefer the damping ratio of $\frac{1}{\sqrt{2}}$ to be Critically damped.

1.3 Complex Root

To have the Complex Root Case, We need:

$$\begin{aligned}(2\zeta\omega_n)^2 - 4\omega_n^2 &< 0\\ 4\zeta^2\omega_n^2 - 4\omega_n^2 &< 0\\ 4\omega_n^2(\zeta^2 - 1) &< 0\\ (\zeta^2 - 1) &< 0\\ \zeta^2 &< 1\\ \zeta &< 1\end{aligned}$$

2 Discussion of Each Cases

2.1 Over-damped Case ($\zeta > 1$)

Above equation can be written as:

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{a_1}{s + r_1} + \frac{a_2}{s + r_2}$$

After using Partial Fraction Decomposition, we get:

$$a_1 = \frac{-\dot{x}_0 + x_0(-\zeta\omega_n + \sqrt{(\zeta^2-1)\omega_n^2})}{2\sqrt{(\zeta^2-1)\omega_n^2}}$$

$$a_2 = \frac{\dot{x}_0 + x_0(\zeta\omega_n + \sqrt{(\zeta^2-1)\omega_n^2})}{2\sqrt{(\zeta^2-1)\omega_n^2}}$$

Thus the solution of differential equation $\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$ where $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ is :

$$x(t) = a_1 e^{r_1 t} + a_2 e^{r_2 t}$$

Where:

$$a_1 = \frac{-\dot{x}_0 + x_0(-\zeta\omega_n + \sqrt{(\zeta^2-1)\omega_n^2})}{2\sqrt{(\zeta^2-1)\omega_n^2}}$$
$$a_2 = \frac{\dot{x}_0 + x_0(\zeta\omega_n + \sqrt{(\zeta^2-1)\omega_n^2})}{2\sqrt{(\zeta^2-1)\omega_n^2}}$$
$$r_1, r_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

2.2 Critically damped Case ($\zeta = 1$)

We have the root : $r_1, r_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$
By substitute $\zeta = 1$, we get the root :

$$r_1, r_2 = -\omega_n$$

Above equation can be written as:

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{a_1}{s+\omega_n} + \frac{a_2}{(s+\omega_n)^2}$$

After using Partial Fraction Decomposition, we get:

$$a_1 = x_0$$

$$a_2 = \dot{x}_0 + x_0\omega_n$$

Thus the solution of differential equation $\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$ where $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ is :

$$x(t) = x_0 e^{-\omega_n t} + t e^{-\omega_n t}(\dot{x}_0 + x_0\omega_n)$$

Where:

$$a_1 = x_0$$
$$a_2 = \dot{x}_0 + x_0\omega_n$$

2.3 Under damped Case ($\zeta < 1$)

We have the root : $r_1, r_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
By modify the square root part, we get :

$$\begin{aligned} r_1, r_2 &= -\zeta\omega_n \pm \omega_n\sqrt{-1(1-\zeta^2)} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{(1-\zeta^2)}\sqrt{-1} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{(1-\zeta^2)}i \end{aligned}$$

Let:

$$\begin{aligned} \sigma &= \zeta\omega_n \\ \omega_d &= \omega_n\sqrt{(1-\zeta^2)} \end{aligned}$$

We can write the root as :

$$r_1, r_2 = -\sigma \pm \omega_d i$$

Rewrite the root in form of :

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s+\alpha)^2 + w_n^2$$

We get :

$$\begin{aligned} \alpha &= \zeta\omega_n \\ \omega &= \sqrt{(1-\zeta^2)\omega_n^2} \end{aligned}$$

Above equation can be written as:

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{(s+\alpha)^2 + w_n^2}$$

After using Partial Fraction Decomposition, The solution of differential equation $\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$ where $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ is :

$$x(t) = acos(\omega t) + bsin(\omega t)$$

Where:

$$\begin{aligned} a &= x_0 e^{-\alpha t} \\ b &= \frac{\dot{x}_0 + x_0\zeta\omega_n}{\omega} e^{-\alpha t} \\ \alpha &= \zeta\omega_n \\ \omega &= \sqrt{(1-\zeta^2)\omega_n^2} \end{aligned}$$

Or in simple form of :

$$x(t) = Ae^{-\zeta\omega_n t}cos(\omega_d t - \phi)$$

Where:

$$\begin{aligned} A &= \sqrt{x_0^2 + \frac{(\dot{x}_0 + x_0\zeta\omega_n)^2}{(1-\zeta^2)\omega_n^2}} \\ \omega_d &= \omega_n\sqrt{(1-\zeta^2)} \\ \phi &= atan2(\frac{\dot{x}_0 + x_0\zeta\omega_n}{\omega_n\sqrt{(1-\zeta^2)}}, x_0) \end{aligned}$$