Lesson 0 : Transfer Function

1 Background

Transfer Function is the ratio of Laplace Transform of Output of the system to the Laplace Transform of Input of the system, when all the initial condition are assumed to be zero. (Very important that if it is not zero then the system is not Linear Time Invariant)(We can not take a Laplace Transform of a nonlinear system).

2 Single Input Single Output (SISO)

Let:

- $x(t)$ is Input of the system
- $y(t)$ is Output of the system
- $h(t)$ is the system

We have:

$$y(t) = x(t) * h(t) \tag{1}$$

Taking Laplace Transform of Equation 1, We get:

$$Y(s) = X(s) * H(s) \tag{2}$$

By convolution property:

$$\boxed{H(s) = \frac{Y(s)}{X(s)}} \tag{3}$$

3 Example of Determine a Transfer Function

Determine a Transfer Function of a system below:
System 1

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Solution
We have:

- $x(t)$ is Input of the system
- $y(t)$ is Output of the system

Taking Laplace Transform of the system:

$$\mathcal{L}[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)] = \mathcal{L}[x(t)]$$

We get:

$$\mathcal{L}[\frac{d^2y(t)}{dt^2}] = s^2Y(s) - sY(0^-) - y'(0^-)$$

$$\mathcal{L}[3\frac{dy(t)}{dt}] = 3[sY(s) - y(0)]$$
$$\mathcal{L}[2y(t)] = 2Y(s)$$
$$\mathcal{L}[X(t)] = X(s)$$
$$s^2Y(s) - sY(0^-) - y'(0^-) + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$

Put Initial Condition to zero, we get:

$$s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$$

$$Y(s)[s^2 + 3s + 2] = X(s)$$
$$\rightarrow \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2}$$
$$\rightarrow H(s) = \frac{1}{s^2 + 3s + 2}$$
$$\rightarrow \boxed{H(s) = \frac{1}{(s+1)(s+2)}}$$

System 2

$$\dot{\phi}(t) = k(\phi_{ref}(t) - \phi(t))$$

Solution

$$\frac{1}{k}\dot{\phi}(t) = \phi_{ref}(t) - \phi(t)$$
$$\frac{1}{k}\dot{\phi}(t) + \phi(t) = \phi_{ref}(t)$$

Taking Laplace Transform of the system:

$$\mathcal{L}[\frac{1}{k}\dot{\phi}(t) + \phi(t)] = \mathcal{L}[\phi_{ref}(t)]$$

We have:

- $\phi_{ref}(t)$ is Input of the system
- $\phi(t)$ is Output of the system

$$\frac{1}{k}sY(s) + Y(s) = X(s)$$
$$Y(s)[\frac{1}{k}s + 1] = X(s)$$
$$\rightarrow \frac{Y(s)}{X(s)} = \frac{1}{\frac{1}{k}s + 1}$$
$$\rightarrow \boxed{H(s) = \frac{1}{\frac{1}{k}s + 1}}$$

System 3

$$\ddot{\phi}(t) = k(\phi_{ref}(t) - \phi(t))$$

Solution

$$\frac{1}{k}\ddot{y}(t) + y(t) = x(t)$$

Taking Laplace Transform of the system:

$$\mathcal{L}[\frac{1}{k}\ddot{y}(t) + y(t)] = \mathcal{L}[x(t)]$$

$$\frac{1}{k}s^2Y(s) + Y(s) = X(s)$$

$$Y(s)[\frac{1}{k}s^2 + 1] = X(s)$$

$$\rightarrow \boxed{H(s) = \frac{1}{\frac{1}{k}s^2 + 1}}$$

System 4

$$\ddot{y}(t) + ky(t) = kx(t)$$

Solution

Taking Laplace Transform of the system:

$$s^2Y(s) + kY(s) = kX(s)$$

$$Y(s)[s^2 + k] = kX(s)$$

$$\rightarrow \boxed{H(s) = \frac{k}{s^2 + k}}$$

4 Example of Determine System from TF

Below is a system transfer function that transfer wheel position θ to wheel velocity $\dot{\theta}$. Determine the system function and discretize it. We have a TF:

$$\frac{Y(s)}{X(s)} = \frac{s}{as + 1}$$

$$(as + 1)Y(s) = sX(s)$$

$$asY(s) + Y(s) = sX(s)$$

Taking a Reverse Laplace Transform:

$$a\dot{y}(t) + y(t) = \dot{x}(t)$$

$$\rightarrow \boxed{\dot{y}(t) = -\frac{1}{a}y(t) + \frac{1}{a}\dot{x}(t)}$$

Discretize the model:

$$\frac{y_{k+1} - y_k}{T_s} = -\frac{1}{a}y_k + \frac{1}{a}\frac{x_{k+1} - x_k}{T_s}$$

$$y_{k+1} - y_k = -\frac{T_s}{a}y_k + \frac{1}{a}(x_{k+1} - x_k)$$

$$y_{k+1} = y_k - \frac{T_s}{a}y_k + \frac{1}{a}(x_{k+1} - x_k)$$

$$y_{k+1} = (1 - \frac{T_s}{a})y_k + \frac{1}{a}(x_{k+1} - x_k)$$

$$\rightarrow \boxed{\dot{\theta}_{k+1} = (1 - \frac{T_s}{a})\dot{\theta}_k + \frac{1}{a}(\theta_{k+1} - \theta_k)}$$