Lesson 2 : Kalman Filter

1 Background

Kalman Filter was found by Dr. Rudolf Emil Kálmán. This algorithm is a powerful filtering algorithm that has been used in many applications most notably in signal processing, control, optimization, sensor fusion, system identification -etc , and it is able to be implemented online.

2 Kalman Filter (Linear System)

Consider a linear discrete time system as following:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v_k \\ y_{k+1} &= Cx_{k+1} + Du_{k+1} + w_{k+1} \end{aligned} \tag{1}$$

Where:

- x_k $\in \mathbb{R}^{n_x}$ state system
- u_k $\in \mathbb{R}^{n_u}$ input system
- y_k $\in \mathbb{R}^{n_z}$ measurement system
- A $\in \mathbb{R}^{n_x \times n_x}$ system matrix
- C $\in \mathbb{R}^{n_z \times n_x}$ observation matrix
- B $\in \mathbb{R}^{n_x \times n_u}$ some matrix
- D $\in \mathbb{R}^{n_z \times n_u}$ some matrix
- v_k $\in \mathbb{R}^{n_x}$ independent process noises
- w_k $\in \mathbb{R}^{n_z}$ independent measurement noises
- Q $\in \mathbb{R}^{n_x \times n_x}$ Gaussian covariance matrix of v
- R $\in \mathbb{R}^{n_z \times n_z}$ Gaussian covariance matrix of w

Apply Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$
\begin{aligned}
\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\
P_{k+1|k} &= AP_{k|k}A^T + Q
\end{aligned} \tag{2}
$$

Measurement Update

$$
\begin{aligned}
\hat{y}_{k+1|k} &= C\hat{x}_{k+1|k} + Du_{k+1} \\
P_{xy,k+1|k} &= P_{k+1|k}C^T \\
P_{yy,k+1|k} &= CP_{k+1|k}C^T + R \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{xy,k+1|k}P^{-1}_{yy,k+1|k}(y_k - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - P_{xy,k+1|k}P^{-1}_{yy,k|k+1}P^T_{xy,k+1|k}
\end{aligned} \tag{3}
$$

In terms of Kalman Gain,

$$
\begin{aligned}
K_{k+1} &= P_{xy,k+1|k}P^{-1}_{yy,k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k} - Du_{k+1}) \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1}P_{yy,k+1|k}K^T_{k+1}
\end{aligned} \tag{4}
$$

3 Extended Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$
\begin{aligned}
x_{k+1} &= f_d(x_k + u_k) + v_k \\
y_{k+1} &= h_d(x_{k+1}, u_{k+1}) + w_{k+1}
\end{aligned} \tag{5}
$$

Where:

- x_k $\in \mathbb{R}^{n_x}$ state system at discrete time
- u_k $\in \mathbb{R}^{n_u}$ input system
- y_k $\in \mathbb{R}^{n_z}$ measurement
- f_d some known function
- h_d some known function
- v_k $\in \mathbb{R}^{n_x}$ independent process noises
- w_k $\in \mathbb{R}^{n_z}$ independent measurement noises

Apply Extended Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$\begin{aligned}\hat{x}_{k+1|k} &= f_d(\hat{x}_{k|k}, u_k) \\ P_{k+1|k} &= A_k P_{k|k} A_k^T + Q\end{aligned} \tag{6}$$

Measurement Update

$$\begin{aligned}\hat{y}_{k+1|k} &= h_d(\hat{x}_{k+1|k}, u_{k+1}) \\ P_{xy,k+1|k} &= P_{k+1|k} C_{k+1}^T \\ P_{yy,k+1|k} &= C_{k+1} P_{k+1|k} C_{k+1}^T + R \\ \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{xy,k+1|k} P_{yy,k+1|k}^{-1} (y_{k+1} - \hat{y}_{k+1|k}) \\ P_{k+1|k+1} &= P_{k+1|k} - P_{xy,k+1|k} P_{yy,k|k+1}^{-1} P_{xy,k+1|k}^T\end{aligned} \tag{7}$$

In terms of Kalman Gain,

$$\begin{aligned}K_{k+1} &= P_{xy,k+1|k} P_{yy,k+1|k}^{-1} \\ \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - h_d(\hat{x}_{k+1|k}, u_{k+1})] \\ P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{yy,k+1|k} K_{k+1}^T\end{aligned} \tag{8}$$

Where from linearization of nonlinear function f_d and h_d using a Taylor series expansion, We get Jacobian matrix:

$$\begin{aligned}A_k &= \frac{\partial f_d}{\partial x}|_{x=\hat{x}_{k|k}} \\ C_{k+1} &= \frac{\partial h_d}{\partial x}|_{x=\hat{x}_{k+1|k}}\end{aligned}$$

4 Unscented Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$\begin{aligned}x_{k+1} &= f_d(x_k + u_k) + v_k \\ y_{k+1} &= h_d(x_{k+1}, u_{k+1}) + w_{k+1}\end{aligned} \tag{9}$$

Apply Unscented Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$\begin{aligned}X_{k|k} &= [\hat{x}_{k|k} \quad ... \quad \hat{x}_{k|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k|k}} \quad -\sqrt{P_{k|k}}] \\ X_{k+1|k} &= f_d(X_{k|k}, u_k) \\ \hat{x}_{k+1|k} &= X_{k+1|k} w_m \\ P_{k+1|k} &= X_{k+1|k} W X_{k+1|k}^T + Q\end{aligned} \tag{10}$$

Measurement Update

$$
\begin{aligned}
X^{(r)}_{k+1|k} &= [\hat{x}_{k+1|k} \quad ... \quad \hat{x}_{k+1|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k+1|k}} \quad -\sqrt{P_{k+1|k}}] \\
Y_{k+1|k} &= h_d(X^{(r)}_{k+1|k}, u_{k+1}) \\
\hat{y}_{k+1|k} &= Y_{k+1|k} w_m \\
P_{xy,k+1|k} &= X^{(r)}_{k+1|k} W Y^T_{k+1|k} \\
P_{yy,k+1|k} &= Y_{k+1|k} W Y^T_{k+1|k} + R \\
K_{k+1} &= P_{xy,k+1|k} P^{-1}_{yy,k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{yy,k+1|k} K^T_{k+1}
\end{aligned} \tag{11}
$$