Foundation - Lesson 5 : Partial Fraction Decomposition

1 Partial Fraction Decomposition

Usually we have a Function:

$$F(s) = \frac{A(s)}{B(s)}$$

Where:

- $A(s)$ is a polynomial which order is smaller than $B(s)$
- $B(s)$ is a polynomial which order is greater than $A(s)$

To perform the Partial Fraction Decomposition, first we have to get $F(s)$ into the ZPK (Zero, Pole, Gain) format which is:

$$F(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)...(s+z_n)}{(s+p_1)(s+p_2)(s+p_3)...(s+p_n)}$$

Where:

- z is roots of $A(s)$ that is the zeros of $F(s)$
- p is roots of $B(s)$ that is the poles of $F(s)$

Given denominator of $F(s)$, determine the pole of the polynomial $(s+p_1)...(s+p_n)$. From the result we can divide into 3 cases.

1.1 Case 1: Distinct Real Poles

In this case we can propose that the $F(s) = \frac{A(s)}{B(s)}$ can be written into:

$$F(s) = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + ... + \frac{a_n}{s+p_n}$$

Example :

$$F(s) = \frac{s^2+8s+15}{s^3+3s^2+2s}$$

We can see that Nominator order is greater than Denominator order. And the denominator s^3+3s^2+2s has the roots $s_1 = 0, s_2 = -2, s_3 = -1 \rightarrow p_1 = 0, p_2 = 2, p_3 = 1$. Thus we have:

$$F(s) = \frac{s^2+8s+15}{s^3+3s^2+2s} = \frac{s^2+8s+15}{(s+0)(s+2)(s+1)} = \frac{a_1}{s+0} + \frac{a_2}{s+2} + \frac{a_3}{s+1}$$

So we have to find a_1, a_2, a_3 to make it work. We can use 2 methods to do it.

- Method 1: Multiplication

$$\frac{s^2+8s+15}{s(s+2)(s+1)} = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{a_3}{s+1}$$

Multiply both side in terms of a_1 (s):

$$s(\frac{s^2+8s+15}{s(s+2)(s+1)}) = s\frac{a_1}{s} + s\frac{a_2}{s+2} + s\frac{a_3}{s+1}$$
$$\frac{s^2+8s+15}{(s+2)(s+1)} = a_1 + s\frac{a_2}{s+2} + s\frac{a_3}{s+1}$$

Substitute $s = 0$

$$\frac{0+0+15}{(0+2)(0+1)} = a_1 + 0 + 0$$
$$a_1 = \frac{15}{2}$$

Multiply both side in terms of a_2 $(s+2)$:

$$(s+2)(\frac{s^2+8s+15}{s(s+2)(s+1)}) = (s+2)\frac{a_1}{s} + (s+2)\frac{a_2}{s+2} + (s+2)\frac{a_3}{s+1}$$
$$\frac{s^2+8s+15}{(s)(s+1)} = (s+2)\frac{a_1}{s} + a_2 + (s+2)\frac{a_3}{s+1}$$

Substitute $s = -2$

$$\frac{(-2)^2+8(-2)+15}{(-2)(-2+1)} = (-2+2)\frac{a_1}{-2} + a_2 + (-2+2)\frac{a_3}{-2+1}$$
$$\frac{4-16+15}{2} = 0 + a_2 + 0$$
$$a_2 = \frac{3}{2}$$

Multiply both side in terms of a_3 $(s+1)$:

$$(s+1)(\frac{s^2+8s+15}{s(s+2)(s+1)}) = (s+1)\frac{a_1}{s} + (s+1)\frac{a_2}{s+2} + (s+1)\frac{a_3}{s+1}$$
$$\frac{s^2+8s+15}{(s)(s+2)} = (s+1)\frac{a_1}{s} + (s+1)\frac{a_2}{s+2} + a_3$$

Substitute $s = -1$

$$\frac{(-1)^2+8(-1)+15}{(-1)(-1+2)} = (-1+1)\frac{a_1}{-1} + (-1+1)\frac{a_2}{-1+2} + a_3$$
$$\frac{1+-8+15}{-1} = 0 + 0 + a_3$$
$$a_3 = -8$$

So we get:

$$\boxed{F(s) = \frac{s^2+8s+15}{s^3+3s^2+2s} = \frac{\frac{15}{2}}{s} + \frac{\frac{3}{2}}{s+2} + \frac{-8}{s+1}}$$

- Method 2: Coefficient

$$\frac{s^2+8s+15}{s(s+2)(s+1)} = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{a_3}{s+1}$$

Get the right-hand side denominator the same as left-hand side.

$$\begin{aligned}
\frac{s^2+8s+15}{s(s+2)(s+1)} &= \frac{(s+1)(s+2)a_1 + s(s+1)a_2 + s(s+2)a_3}{s(s+2)(s+1)} \\
s^2+8s+15 &= (s+1)(s+2)a_1 + s(s+1)a_2 + s(s+2)a_3 \\
&= (s^2+2s+s+2)a_1 + (s^2+s)a_2 + (s^2+2s)a_3 \\
&= (s^2+3s+2)a_1 + (s^2+s)a_2 + (s^2+2s)a_3 \\
&= s^2a_1 + 3sa_1 + 2a_1 + s^2a_2 + sa_2 + s^2a_3 + 2sa_3 \\
s^2+8s+15 &= s^2(a_1+a_2+a_3) + s(3a_1+a_2+2a_3) + (2a_1) \\
1 &= a_1+a_2+a_3 \\
8 &= 3a_1+a_2+2a_3 \\
15 &= 2a_1 \\
a_1 &= \frac{15}{2} \\
a_2 &= \frac{3}{2} \\
a_3 &= -8
\end{aligned}$$

So we get:

$$\boxed{F(s) = \frac{s^2+8s+15}{s^3+3s^2+2s} = \frac{\frac{15}{2}}{s} + \frac{\frac{3}{2}}{s+2} + \frac{-8}{s+1}}$$

1.2 Case 2: Repeated Real Poles

In this case we can propose that the $F(s) = \frac{A(s)}{B(s)}$ can be written into:

$$F(s) = \frac{a_1}{s+p} + \frac{a_2}{(s+p)^2} + ... + \frac{a_n}{(s+p)^n}$$

Example :

$$F(s) = \frac{s^2+2s+3}{(s+1)^3}$$

The denominator $(s+1)^3$ has a repeated real pole at $p=-1$. $F(s)$ can be written as:

$$F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}$$

- Method 1 : Coefficient

Determine a_1, a_2, a_3

$$\begin{aligned}
\frac{s^2+2s+3}{(s+1)^3} &= \frac{(s+1)^2a_1}{(s+1)^2(s+1)} + \frac{(s+1)a_2}{(s+1)(s+1)^2} + \frac{a_3}{(s+1)^3} \\
\frac{s^2+2s+3}{(s+1)^3} &= \frac{(s+1)^2a_1 + (s+1)a_2 + a_3}{(s+1)^3} \\
s^2+2s+3 &= (s+1)^2a_1 + (s+1)a_2 + a_3 \\
s^2+2s+3 &= s^2a_1 + 2sa_1 + a_1 + sa_2 + a_2 + a_3 \\
s^2+2s+3 &= s^2a_1 + s(2a_1+a_2) + (a_1+a_2+a_3)
\end{aligned}$$

$$\begin{aligned}
1 &= a_1 \\
2 &= 2a_1 + a_2 \\
3 &= a_1 + a_2 + a_3 \\
\rightarrow a_1 &= 1 \\
\rightarrow a_2 &= 0 \\
\rightarrow a_3 &= 2
\end{aligned}$$

Thus we get:

$$\boxed{F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{1}{s+1} + \frac{0}{(s+1)^2} + \frac{2}{(s+1)^3} = \frac{1}{s+1} + \frac{2}{(s+1)^3}}$$

- Method 2 : Derivative

From finding the common denominator above:

$$s^2+2s+3 = (s+1)^2 a_1 + (s+1)a_2 + a_3$$

Substitute $s=-1$

$$\begin{aligned}
(-1)^2 + 2(-1) + 3 &= (-1+1)^2 a_1 + (-1+1)a_2 + a_3 \\
(-1)^2 + 2(-1) + 3 &= 0 + 0 + a_3 \\
a_3 &= 2
\end{aligned}$$

Take derivative of $s^2+2s+3 = (s+1)^2 a_1 + (s+1)a_2 + a_3$ both side, we get:

$$2s+2 = 2(s+1)a_1 + a_2$$

Substitute $s=-1$

$$\begin{aligned}
2(-1) + 2 &= 2(-1+1)a_1 + a_2 \\
2(-1) + 2 &= 0 + a_2 \\
a_2 &= 0
\end{aligned}$$

Take derivative of $2s+2 = 2(s+1)a_1 + a_2$ both side, we get:

$$\begin{aligned}
2 &= 2a_1 \\
a_1 &= 1
\end{aligned}$$

Thus we get:

$$\boxed{F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{1}{s+1} + \frac{0}{(s+1)^2} + \frac{2}{(s+1)^3} = \frac{1}{s+1} + \frac{2}{(s+1)^3}}$$

1.3 Case 3: Complex Conjugate Poles

In this case we can propose that the $F(s) = \frac{A(s)}{B(s)}$ can be written into:

$$F(s) = \frac{A(s)}{(s+\alpha)^2 + \omega^2}$$

Where from general denominator:

$$s^2 + ds + e = 0$$

$$\alpha = \frac{d}{2}$$

$$\omega = \frac{\sqrt{4e - d^2}}{2}$$

Example :

$$F(s) = \frac{s-1}{s^2 + 2s + 2}$$

From denominator $s^2 + 2s + 2$ in general form $d = 2, e = 2$, we get:

$$\alpha = 1$$

$$\omega = 1$$

Thus:

$$\boxed{F(s) = \frac{s-1}{s^2 + 2s + 2} = \frac{s-1}{(s+1)^2 + 1^2}}$$

2 Improper Complex Function

We have a transfer function:

$$F(s) = \frac{A(s)}{B(s)}$$

Where:

- $A(s)$ is a polynomial which order is greater than $B(s)$
- $B(s)$ is a polynomial which order is smaller than $A(s)$

We can use synthetic division to make the $A(s)$ smaller than $B(s)$ (Polynomial Equation division).