Lesson 2 : Kalman Filter

1   Background

Kalman Filter was found by Dr. Rudolf Emil Kálmán. This algorithm is a powerful filtering algorithm that has been used in many applications most notably in signal processing, control, optimization, sensor fusion, system identification -etc , and it is able to be implemented online.

2   Kalman Filter (Linear System)

Consider a linear discrete time system as following:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + v_k \\
y_{k+1} &= Cx_{k+1} + Du_{k+1} + w_{k+1}
\end{aligned}
\tag{1}
$$

Where:

- x_k $\in \mathbb{R}^{n_x}$ state system

- u_k $\in \mathbb{R}^{n_u}$ input system

- y_k $\in \mathbb{R}^{n_z}$ measurement system

- A $\in \mathbb{R}^{n_x \times n_x}$ system matrix

- C $\in \mathbb{R}^{n_z \times n_x}$ observation matrix

- B $\in \mathbb{R}^{n_x \times n_u}$ some matrix

- D $\in \mathbb{R}^{n_z \times n_u}$ some matrix

- v_k $\in \mathbb{R}^{n_x}$ independent process noises

- w_k $\in \mathbb{R}^{n_z}$ independent measurement noises

- Q $\in \mathbb{R}^{n_x \times n_x}$ Gaussian covariance matrix of v

- R $\in \mathbb{R}^{n_z \times n_z}$ Gaussian covariance matrix of w

Apply Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate

- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$P_{k+1|k} = AP_{k|k}A^T + Q \tag{2}$$

Measurement Update

$$\hat{y}_{k+1|k} = C\hat{x}_{k+1|k} + Du_{k+1}$$
$$P_{xy,k+1|k} = P_{k+1|k}C^T$$
$$P_{yy,k+1|k} = CP_{k+1|k}C^T + R \tag{3}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{xy,k+1|k}P_{yy,k+1|k}^{-1}(y_k - \hat{y}_{k+1|k})$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{xy,k+1|k}P_{yy,k|k+1}^{-1}P_{xy,k+1|k}^T$$

In terms of Kalman Gain,

$$K_{k+1} = P_{xy,k+1|k}P_{yy,k+1|k}^{-1}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k} - Du_{k+1}) \tag{4}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1}P_{yy,k+1|k}K_{k+1}^T$$

3 Extended Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$x_{k+1} = f_d(x_k + u_k) + v_k$$
$$y_{k+1} = h_d(x_{k+1}, u_{k+1}) + w_{k+1} \tag{5}$$

Where:

- x_k $\in \mathbb{R}^{n_x}$ state system at discrete time

- u_k $\in \mathbb{R}^{n_u}$ input system

- y_k $\in \mathbb{R}^{n_z}$ measurement

- f_d some known function

- h_d some known function

- v_k $\in \mathbb{R}^{n_x}$ independent process noises

- w_k $\in \mathbb{R}^{n_z}$ independent measurement noises

Apply Extended Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate

- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$
\begin{aligned}
\hat{x}_{k+1|k} &= f_d(\hat{x}_{k|k}, u_k) \\
P_{k+1|k} &= A_k P_{k|k} A_k^T + Q
\end{aligned}
\tag{6}
$$

Measurement Update

$$
\begin{aligned}
\hat{y}_{k+1|k} &= h_d(\hat{x}_{k+1|k}, u_{k+1}) \\
P_{xy,k+1|k} &= P_{k+1|k} C_{k+1}^T \\
P_{yy,k+1|k} &= C_{k+1} P_{k+1|k} C_{k+1}^T + R \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{xy,k+1|k} P_{yy,k+1|k}^{-1}(y_{k+1} - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - P_{xy,k+1|k} P_{yy,k|k+1}^{-1} P_{xy,k+1|k}^T
\end{aligned}
\tag{7}
$$

In terms of Kalman Gain,

$$
\begin{aligned}
K_{k+1} &= P_{xy,k+1|k} P_{yy,k+1|k}^{-1} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - h_d(\hat{x}_{k+1|k}, u_{k+1})] \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{yy,k+1|k} K_{k+1}^T
\end{aligned}
\tag{8}
$$

Where from linearization of nonlinear function f_d and h_d using a Taylor series expansion, We get Jacobian matrix:

$$
A_k = \frac{\partial f_d}{\partial x}\big|_{x=\hat{x}_{k|k}}
$$

$$
C_{k+1} = \frac{\partial h_d}{\partial x}\big|_{x=\hat{x}_{k+1|k}}
$$

4   Unscented Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$
\begin{aligned}
x_{k+1} &= f_d(x_k + u_k) + v_k \\
y_{k+1} &= h_d(x_{k+1}, u_{k+1}) + w_{k+1}
\end{aligned}
\tag{9}
$$

Apply Unscented Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate

- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$
\begin{aligned}
X_{k|k} &= [\hat{x}_{k|k} \quad ... \quad \hat{x}_{k|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k|k}} \quad -\sqrt{P_{k|k}}] \\
X_{k+1|k} &= f_d(X_{k|k}, u_k) \\
\hat{x}_{k+1|k} &= X_{k+1|k} w_m \\
P_{k+1|k} &= X_{k+1|k} W X_{k+1|k}^T + Q
\end{aligned}
\tag{10}
$$

Measurement Update

$$
\begin{aligned}
X^{(r)}_{k+1|k} &= [\hat{x}_{k+1|k} \quad ... \quad \hat{x}_{k+1|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k+1|k}} \quad -\sqrt{P_{k+1|k}}] \\
Y_{k+1|k} &= h_d(X^{(r)}_{k+1|k}, u_{k+1}) \\
\hat{y}_{k+1|k} &= Y_{k+1|k}w_m \\
P_{xy,k+1|k} &= X^{(r)}_{k+1|k}WY^{T}_{k+1|k} \\
P_{yy,k+1|k} &= Y_{k+1|k}WY^{T}_{k+1|k} + R \\
K_{k+1} &= P_{xy,k+1|k}P^{-1}_{yy,k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1}P_{yy,k+1|k}K^{T}_{k+1}
\end{aligned}
\tag{11}
$$