Lesson 2 : Kalman Filter

1 Background

Kalman Filter was found by Dr. Rudolf Emil Kálmán. This algorithm is a powerful filtering algorithm that has been used in many applications most notably in signal processing, control, optimization, sensor fusion, system identification -etc , and it is able to be implemented online.

2 Kalman Filter (Linear System)

Consider a linear discrete time system as following:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v_k \\ y_{k+1} &= Cx_{k+1} + Du_{k+1} + w_{k+1} \end{aligned} \tag{1}$$

Where:

- x_k is voltage at terminal conductor of motor
- u_k is back emf constant
- y_k is angular velocity of motor
- A is angular velocity of motor
- C is angular velocity of motor
- B is angular velocity of motor
- D is angular velocity of motor
- v_k is angular velocity of motor
- w_k is angular velocity of motor

Apply Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\ P_{k+1|k} &= AP_{k|k}A^T + Q \end{aligned} \tag{2}$$

Measurement Update

$$
\begin{aligned}
\hat{y}_{k+1|k} &= C\hat{x}_{k+1|k} + Du_{k+1} \\
P_{xy,k+1|k} &= P_{k+1|k}C^T \\
P_{yy,k+1|k} &= CP_{k+1|k}C^T + R \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{xy,k+1|k}P^{-1}_{yy,k+1|k}(y_k - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - P_{xy,k+1|k}P^{-1}_{yy,k|k+1}P^T_{xy,k+1|k}
\end{aligned} \tag{3}
$$

In terms of Kalman Gain,

$$
\begin{aligned}
K_{k+1} &= P_{xy,k+1|k}P^{-1}_{yy,k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k} - Du_{k+1}) \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1}P_{yy,k+1|k}K^T_{k+1}
\end{aligned} \tag{4}
$$

3 Extended Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$
\begin{aligned}
x_{k+1} &= f_d(x_k + u_k) + v_k \\
y_{k+1} &= h_d(x_{k+1}, u_{k+1}) + w_{k+1}
\end{aligned} \tag{5}
$$

Where:

- x_k is voltage at terminal conductor of motor
- u_k is back emf constant
- y_k is angular velocity of motor
- f_d is angular velocity of motor
- h_d is angular velocity of motor
- v_k is angular velocity of motor
- w_k is angular velocity of motor

Apply Extended Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$
\begin{aligned}
\hat{x}_{k+1|k} &= f_d(\hat{x}_{k|k}, u_k) \\
P_{k+1|k} &= A_kP_{k|k}A^T_k + Q
\end{aligned} \tag{6}
$$

Measurement Update

$$\begin{aligned}
\hat{y}_{k+1|k} &= h_d(\hat{x}_{k+1|k}, u_{k+1}) \\
P_{xy,k+1|k} &= P_{k+1|k} C_{k+1}^T \\
P_{yy,k+1|k} &= C_{k+1} P_{k+1|k} C_{k+1}^T + R \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{xy,k+1|k} P_{yy,k+1|k}^{-1} (y_{k+1} - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - P_{xy,k+1|k} P_{yy,k|k+1}^{-1} P_{xy,k+1|k}^T
\end{aligned} \tag{7}$$

In terms of Kalman Gain,

$$\begin{aligned}
K_{k+1} &= P_{xy,k+1|k} P_{yy,k+1|k}^{-1} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - h_d(\hat{x}_{k+1|k}, u_{k+1})] \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{yy,k+1|k} K_{k+1}^T
\end{aligned} \tag{8}$$

Where from linearization of nonlinear function f_d and h_d using a Taylor series expansion, We get Jacobian matrix:

$$A_k = \frac{\partial f_d}{\partial x}|_{x=\hat{x}_{k|k}}$$

$$C_{k+1} = \frac{\partial h_d}{\partial x}|_{x=\hat{x}_{k+1|k}}$$

4 Unscented Kalman Filter (Nonlinear System)

Consider a nonlinear discrete time system as following:

$$\begin{aligned}
x_{k+1} &= f_d(x_k + u_k) + v_k \\
y_{k+1} &= h_d(x_{k+1}, u_{k+1}) + w_{k+1}
\end{aligned} \tag{9}$$

Apply Unscented Kalman Filter on the system

Initialize:
Select any

- $\hat{x}_{0|0}$ initial state estimate
- $P_{0|0}$ positive definite error covariance matrix

Time Update

$$\begin{aligned}
X_{k|k} &= [\hat{x}_{k|k} \quad ... \quad \hat{x}_{k|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k|k}} \quad -\sqrt{P_{k|k}}] \\
X_{k+1|k} &= f_d(X_{k|k}, u_k) \\
\hat{x}_{k+1|k} &= X_{k+1|k} w_m \\
P_{k+1|k} &= X_{k+1|k} W X_{k+1|k}^T + Q
\end{aligned} \tag{10}$$

Measurement Update

$$
\begin{aligned}
X^{(r)}_{k+1|k} &= [\hat{x}_{k+1|k} \quad ... \quad \hat{x}_{k+1|k}] + \sqrt{n_x + \lambda}[0 \quad \sqrt{P_{k+1|k}} \quad -\sqrt{P_{k+1|k}}] \\
Y_{k+1|k} &= h_d(X^{(r)}_{k+1|k}, u_{k+1}) \\
\hat{y}_{k+1|k} &= Y_{k+1|k} w_m \\
P_{xy,k+1|k} &= X^{(r)}_{k+1|k} W Y^T_{k+1|k} \\
P_{yy,k+1|k} &= Y_{k+1|k} W Y^T_{k+1|k} + R \\
K_{k+1} &= P_{xy,k+1|k} P^{-1}_{yy,k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - \hat{y}_{k+1|k}) \\
P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{yy,k+1|k} K^T_{k+1}
\end{aligned} \tag{11}
$$