Foundation - Lesson 8 : Linear Approximation with Taylor Series

1 What is a Linear System ?

Linear System is a system that comply to 2 rules.

- Superposition (Addition).
- Homogeneous (Multiplication).

Superposition

Given that we have a function $y = f(x)$.

- If we have a value x_1 substitute to the function we get y_1 : $y_1 = f(x_1)$
- If we have a value x_2 substitute to the function we get y_1 : $y_2 = f(x_2)$.
- If we have a value x_1+x_2 substitute to the function we should get y_1+y_2 : $y_1+y_2 = f(x_1 + x_2)$

Homogeneous

Given that we have a function $y = f(x)$.

- If we have a value αx_1 substitute to the function we get y_1 : $y_1 = f(\alpha x_1)$
- If we have a value x_1 substitute to the function then multiply by α we should get $y_1 = \alpha f(x_1)$

1.1 Example

- Find out if the function is linear : $y = x$

Superposition test:
$y_1 = x_1$
$y_2 = x_2$
Add both result together $y_1 + y_2 = x_1 + x_2$
Substitute $x_1 + x_2$ to the function we get $y_1 + y_2$. Thus, $y_1 + y_2 = y_{12}$. TEST PASS.
Homogeneous test:
Substitute αx we get $y = \alpha x$
Substitute x and multiply by α we get $y = \alpha x$. Thus, $\alpha x = \alpha x$. TEST PASS.
Both test is passed and thus the system is linear.

- Find out if the function is linear : $y = x^2$

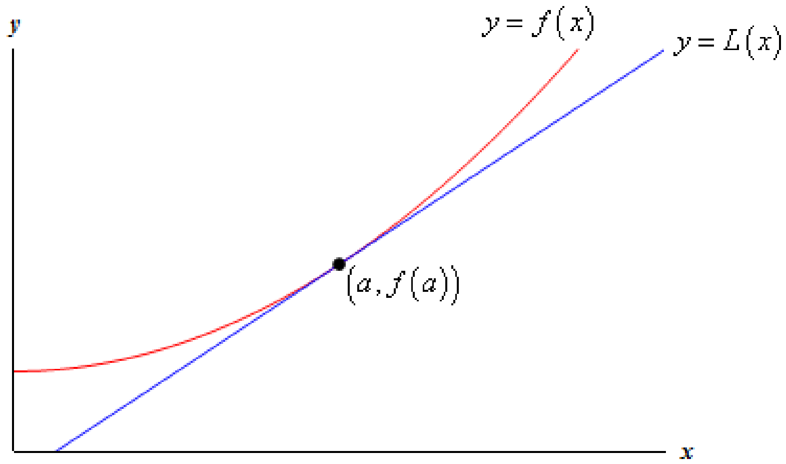
Superposition test:
$y_1 = x_1^2$
$y_2 = x_2^2$
Add both result together $y_1 + y_2 = x_1^2 + x_2^2$
Substitute $x_1 + x_2$ to the function we get $(x_1 + x_2)^2$. Thus, $y_1 + y_2 != y_{12}$. TEST FAIL.
The test is failed and thus the system is nonlinear.

2 Linearization Process

One of the Linearization method is by using Tyler Expansion Series within an operational range for stability.

$$y \approx y(x_0) + \left[\frac{dy}{dx}|_{x_0}\frac{(x-x_0)}{1!}\right] + \left[\frac{d^2y}{dx^2}|_{x_0}\frac{(x-x_0)^2}{2!}\right] + ...[HigherOrderTerm]$$

Let take a look at the plot:



$y = L(x)$ is the linear approximation of $y = f(x)$ and $a = x_0$ is an equilibrium point. We can see that we want to pick an operational range where the function is stable because the $y = L(x)$ is close to $y = f(x)$. As we move a way from the operational range, the approximation is starting to diverge from the real solution.

2.1 Example

- Linearize : $y = x^2$

We have:

$$y \approx y(x_0) + \left[\frac{dy}{dx}|_{x_0}\frac{(x-x_0)}{1!}\right] + \left[\frac{d^2y}{dx^2}|_{x_0}\frac{(x-x_0)^2}{2!}\right] + ...[HigherOrderTerm]$$

Only consider the first order term and eliminate HOT because in HOT the variable x is subject to power number that will make it nonlinear. We get:

$$y \approx y(x_0) + \left[\frac{dy}{dx}|_{x_0}\frac{(x-x_0)}{1!}\right]$$

We get:

$$\frac{dy}{dx}|_{x_0} = \frac{d(x^2)}{dx}|_{x_0} = 2x|_{x_0} = 2x_0$$

We get:

$$y \approx y(x_0) + \left[2x_0\frac{(x-x_0)}{1!}\right]$$

$$y \approx y(x_0) + [2x_0(x-x_0)]$$

$$\boxed{y \approx y(x_0) + 2x_0x - 2x_0^2}$$

Let pick an equilibrium point $x_0 = 2$

$$\begin{aligned} y &= 2^2 + 2 \times 2x - 2 \times 2^2 \\ y &= 4 + 4x - 8 \\ y &= 4x - 4 \end{aligned}$$

Now that we have a original function $y = x^2$ and approximation function at $x_0 = 2$ $y = 4x - 4$. Let compare:

$$\begin{aligned} x &= 2 \\ => y_{ori} &= 2^2 = 4 \\ => y_{lin} &= 4 \times 2 - 4 = 4 \end{aligned}$$

Both are equal to each other at equilibrium point.

$$\begin{aligned} x &= 3 \\ => y_{ori} &= 3^2 = 9 \\ => y_{lin} &= 4 \times 3 - 4 = 8 \end{aligned}$$

A way from the equilibrium point, it starts to diverge.