Lesson 5 : DC Motor Control

1   Background

In application of DC Motor, we want to be able to control its position and angular velocity.

2   Velocity Control using PI Control (No Nonlinear) (2nd Order Differential Equation Design Workflow)

From DC motor Model, we have:

$$\dot{\omega}(t) = -a\omega(t) + bv_a(t) \tag{1}$$

We have to design a velocity controller, thus the feedback of the system is angular velocity of the dc motor.
We have our PI control:

$$v_a(t) = K_p(\omega_d(t) - \omega(t)) + K_i \int_0^t (\omega_d(t) - \omega(t))\, dt \tag{2}$$

From Equation 1,we get:

$$v_a(t) = \frac{1}{b}\dot{\omega}_d(t) + \frac{a}{b}\omega_d(t) \tag{3}$$

From Equation 2 and Equation 3, we have our controller design:

$$v_a(t) = \frac{1}{b}\dot{\omega}_d(t) + \frac{a}{b}\omega_d(t) + K_p(\omega_d(t) - \omega(t)) + K_i \int_0^t (\omega_d(t) - \omega(t))\, dt \tag{4}$$
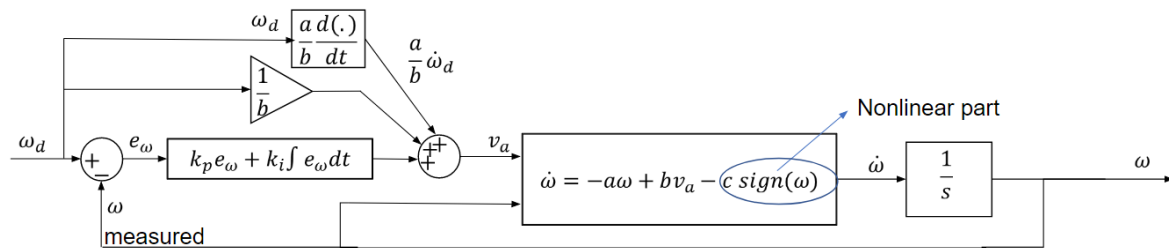

Figure 1: Velocity Control PI Controller

Substitute Equation 4 back to model in Equation 1, we get:

$$\dot{\omega}(t) = -a\omega(t) + b\left(\frac{1}{b}\dot{\omega}_d(t) + \frac{a}{b}\omega_d(t) + K_p(\omega_d(t) - \omega(t)) + K_i \int_0^t (\omega_d(t) - \omega(t))\, dt\right) \tag{5}$$

$$0 = -\dot{\omega}(t) - a\omega(t) + \dot{\omega}_d(t) + a\omega_d(t) + bK_p(\omega_d(t) - \omega(t)) + bK_i \int_0^t (\omega_d(t) - \omega(t))\, dt$$

$$0 = (\dot{\omega}_d(t) - \dot{\omega}(t)) + a(\omega_d(t) - \omega(t)) + bK_p(\omega_d(t) - \omega(t)) + bK_i \int_0^t (\omega_d(t) - \omega(t))\, dt$$

$$0 = (\dot{\omega}_d(t) - \dot{\omega}(t)) + (a + bK_p)(\omega_d(t) - \omega(t)) + bK_i \int_0^t (\omega_d(t) - \omega(t))\, dt$$

$$0 = \dot{e}_\omega + (a + bK_p)e_\omega + bK_i \int_0^t e_\omega\, dt$$

$$0 = \ddot{e}_\omega + (a + bK_p)\dot{e}_\omega + bK_i e_\omega \leftarrow \text{take derivative to cancel integral.}$$

Thus, we get:

$$\ddot{e}_\omega + (a + bK_p)\dot{e}_\omega + bK_i e_\omega = 0 \tag{6}$$

From 2nd Order differential equation standard form, we have:

$$\ddot{X} + 2\zeta\omega_n X + \omega_n^2 X = 0 \tag{7}$$

From Equation 6 and Equation 7, we get:

$$a + bK_p = 2\zeta\omega_n$$
$$bK_i = \omega_n^2$$
$$K_p = \frac{2\zeta\omega_n - a}{b}$$
$$K_i = \frac{\omega_n^2}{b}$$

From equation above, we want $K_p > 0$. Thus, $2\zeta\omega_n > a$, then $\zeta\omega_n > \frac{a}{2}$ to ensure stability.

3 Velocity Control using PID Control (No Nonlinear) (2nd Order Differential Equation Design Workflow)

We have to design a velocity controller, thus the feedback of the system is angular velocity of the dc motor.
We have our PID control:

$$v_a(t) = K_p(\omega_d(t) - \omega(t)) + K_i \int_0^t (\omega_d(t) - \omega(t))\, dt + K_d \frac{d}{dt}(\omega_d(t) - \omega(t)) \tag{8}$$

From Equation 3 and Equation 8, we have our controller design:

$$v_a(t) = \frac{1}{b}\dot{\omega}_d(t) + \frac{a}{b}\omega_d(t) + K_p(\omega_d(t) - \omega(t)) + K_i \int_0^t (\omega_d(t) - \omega(t))\, dt + K_d \frac{d}{dt}(\omega_d(t) - \omega(t)) \tag{9}$$

Substitute Equation 9 back to model in Equation 1, we get:

$$\dot{\omega} = -a\omega + b\left(\frac{1}{b}\dot{\omega}_d + \frac{a}{b}\omega_d + K_p(\omega_d - \omega) + K_i \int_0^t (\omega_d - \omega)\, dt + K_d \frac{d}{dt}(\omega_d - \omega) \right) \tag{10}$$

$$0 = -\dot{\omega} - a\omega + b(\frac{1}{b}\dot{\omega}_d + \frac{a}{b}\omega_d + K_p(\omega_d - \omega) + K_i \int_0^t (\omega_d - \omega)\, dt + K_d \frac{d}{dt}(\omega_d - \omega))$$

$$0 = -\dot{\omega} - a\omega + \dot{\omega}_d + a\omega_d + bK_p(\omega_d - \omega) + bK_i \int_0^t (\omega_d - \omega)\, dt + bK_d \frac{d}{dt}(\omega_d - \omega)$$

$$0 = (\dot{\omega}_d - \dot{\omega}) + (a + bK_p)(\omega_d - \omega) + bK_i \int_0^t (\omega_d - \omega)\, dt + bK_d \frac{d}{dt}(\omega_d - \omega)$$

$$0 = (1 + bK_d)(\dot{\omega}_d - \dot{\omega}) + (a + bK_p)(\omega_d - \omega) + bK_i \int_0^t (\omega_d - \omega)\, dt$$

$$0 = (1 + bK_d)\dot{e}_\omega + (a + bK_p)e_\omega + bK_i \int_0^t e_\omega\, dt$$

$$0 = (1 + bK_d)\ddot{e}_\omega + (a + bK_p)\dot{e}_\omega + bK_i e_\omega$$

Thus, we get:

$$(1 + bK_d)\ddot{e}_\omega + (a + bK_p)\dot{e}_\omega + bK_i e_\omega = 0$$
$$\ddot{e}_\omega + \frac{(a + bK_p)}{(1 + bK_d)}\dot{e}_\omega + \frac{bK_i}{(1 + bK_d)} = 0 \tag{11}$$

From 2nd Order differential equation standard form Equation 7, we have:

$$\frac{(a + bK_p)}{(1 + bK_d)} = 2\zeta\omega_n$$
$$\frac{bK_i}{(1 + bK_d)} = \omega_n^2$$

We have more freedom to choose Kp Ki Kd.

4 Position Control using PID Control (No Nonlinear) (2/3rd Order Differential Equation Design Workflow)

We have to design a position controller, thus the feedback of the system is shaft position of the dc motor.
We have our PID control:

$$v_a(t) = K_p(\theta_d(t) - \theta(t)) + K_i \int_0^t (\theta_d(t) - \theta(t))\, dt + K_d \frac{d}{dt}(\theta_d(t) - \theta(t)) \tag{12}$$

From Equation 1, it can be written in form of position as:

$$\ddot{\theta}(t) = -a\dot{\theta}(t) + bv_a(t) \tag{13}$$

Thus, we get:

$$v_a(t) = \frac{1}{b}\ddot{\theta}_d(t) + \frac{a}{b}\dot{\theta}_d(t) \tag{14}$$

From Equation 13 and Equation 14, we have our controller design:

$$v_a(t) = \frac{1}{b}\ddot{\theta}_d(t) + \frac{a}{b}\dot{\theta}_d(t) + K_p(\theta_d(t) - \theta(t)) + K_i \int_0^t (\theta_d(t) - \theta(t))\, dt + K_d \frac{d}{dt}(\theta_d(t) - \theta(t)) \tag{15}$$

Substitute Equation 15 to Equation 13, we get:

$$\ddot{\theta} = -a\dot{\theta} + b\left(\frac{1}{b}\ddot{\theta}_d + \frac{a}{b}\dot{\theta}_d + K_p(\theta_d - \theta) + K_i\int_0^t (\theta_d - \theta)\,dt + K_d\frac{d}{dt}(\theta_d - \theta)\right) \qquad (16)$$

$$\ddot{\theta} = -a\dot{\theta} + \ddot{\theta}_d + a\dot{\theta}_d + bK_p(\theta_d - \theta) + bK_i\int_0^t (\theta_d - \theta)\,dt + bK_d\frac{d}{dt}(\theta_d - \theta)$$

$$0 = -\ddot{\theta} - a\dot{\theta} + \ddot{\theta}_d + a\dot{\theta}_d + bK_p(\theta_d - \theta) + bK_i\int_0^t (\theta_d - \theta)\,dt + bK_d\frac{d}{dt}(\theta_d - \theta)$$

$$0 = (\ddot{\theta}_d - \ddot{\theta}) + a(\dot{\theta}_d - \dot{\theta}) + bK_p(\theta_d - \theta) + bK_i\int_0^t (\theta_d - \theta)\,dt + bK_d(\dot{\theta}_d - \dot{\theta})$$

$$0 = (\ddot{\theta}_d - \ddot{\theta}) + (a + bK_d)(\dot{\theta}_d - \dot{\theta}) + bK_p(\theta_d - \theta) + bK_i\int_0^t (\theta_d - \theta)\,dt$$

$$0 = \ddot{e}_\theta + (a + bK_d)\dot{e}_\theta + bK_p e_\theta + bK_i\int_0^t e_\theta\,dt$$

$$0 = \dddot{e}_\theta + (a + bK_d)\ddot{e}_\theta + bK_p\dot{e}_\theta + bK_i e_\theta$$

Thus, we get:

$$\dddot{e}_\theta + (a + bK_d)\ddot{e}_\theta + bK_p\dot{e}_\theta + bK_i e_\theta = 0 \qquad (17)$$

is the 3rd order differential equation with the characteristic form of:

$$\lambda^3 + (a + bK_d)\lambda^2 + bK_p\lambda + bK_i = 0 \qquad (18)$$

From Equation 18, we know that in 3rd order differential equation characteristic polynomial, there exist 3 roots and at least 1 root is real root (denoted by λ_1).Thus, we can write:

$$(\lambda + \lambda_1)(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0$$
$$\lambda^3 + 2\zeta\omega_n\lambda^2 + \omega_n^2\lambda + \lambda_1\lambda^2 + 2\zeta\omega_n\lambda\lambda_1 + \omega_n^2\lambda_1 = 0 \qquad (19)$$
$$\lambda^3 + (2\zeta\omega_n + \lambda_1)\lambda^2 + (2\zeta\omega_n\lambda_1 + \omega_n^2)\lambda + \omega_n^2\lambda_1 = 0$$

From Equation 18 and Equation 19, we have:

$$a + bK_d = 2\zeta\omega_n + \lambda_1$$
$$bK_p = 2\zeta\omega_n\lambda_1 + \omega_n^2$$
$$bK_i = \omega_n^2\lambda_1$$
$$K_d = \frac{2\zeta\omega_n + \lambda_1 - a}{b}$$
$$K_p = \frac{2\zeta\omega_n\lambda_1 + \omega_n^2}{b}$$
$$K_i = \frac{\omega_n^2\lambda_1}{b}$$