Foundation - Lesson 2 : 1st Order Differential Equation

1 Background

Differential equation is set an equation that its solution is a function and involve of its derivative. In engineering, these equations is usually used to govern a dynamics system model and the rate of change of state. In 1st Order Differential Equation is equation consist of first derivative of function in form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Where :

- $P(x)$ is function of x
- $Q(x)$ is function of x

2 Method of Solving 1st Order Homogeneous Differential Equation

2.1 Method of Separation of Variable

2.1.1 When to use the Method

All y, dy term and x, dx can explicitly move to different side of the equation.
For example:

$$\frac{dy}{dx} = 5xy$$
$$\rightarrow \frac{dy}{y} = 5xdx$$

2.1.2 How to use the Method

- Step 1 : Move all y, dy term and x, dx to different side of the equation.
- Step 2 : Integrate both side with respect to dx and dy respectively.
- Step 3 : Simplify the equation.

2.1.3 Example

Solve:

$$\frac{dy}{dx} = 5xy$$

- Step 1

$$\frac{dy}{y} = 5xdx$$

- Step 2

$$\int \frac{1}{y}\,dy = 5\int x\,dx$$
$$ln|y| = \frac{5}{2}x^2 + c$$

- Step 3

$$ln|y| = \frac{5}{2}x^2 + c$$
$$e^{ln|y|} = e^{\frac{5}{2}x^2+c}$$
$$y = e^{\frac{5}{2}x^2+c}$$
$$y = e^{\frac{5}{2}x^2}e^c$$
$$y = Ce^{\frac{5}{2}x^2} \leftarrow Solution$$

3 Method of Solving 1st Order Non-Homogeneous Differential Equation

3.1 Method of Variable Substitution

3.1.1 When to use the Method

Use in general form of 1st order linear differential equation of:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

3.1.2 How to use the Method

- Step 1 : Substitute $y = uv$ and $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ to equation.
- Step 2 : Factoring v out. example: $v(term_u, term_x)$.
- Step 3 : Put v term equal to zero and solve for u using separation of variable.
- Step 4 : Substitute u back to equation Step 2 where v term is zero and Solve for v.
- Step 5 : After getting u and v, substitute back into $y = uv$ for a solution of function.

3.1.3 Example

Solve:

$$\frac{dy}{dx} - \frac{y}{x} = 1$$

- Step 1

$$u\frac{dv}{dx} + v\frac{du}{dx} - \frac{uv}{x} = 1$$

- Step 2

$$u\frac{dv}{dx} + v(\frac{du}{dx} - \frac{u}{x}) = 1$$

- Step 3

$$
\begin{aligned}
(\frac{du}{dx} - \frac{u}{x}) &= 0 \\
\frac{du}{dx} &= \frac{u}{x} \\
\frac{du}{u} &= \frac{dx}{x} \\
\int \frac{du}{u} &= \int \frac{dx}{x} \\
ln|u| &= ln|x| + C \\
ln|u| &= ln|x| + ln|K| \leftarrow \text{let C = ln|k| make easier} \\
u &= Kx
\end{aligned}
$$

- Step 4

$$
\begin{aligned}
Kx\frac{dv}{dx} &= 1 \\
dv &= \frac{1}{Kx}dx \\
\int dv &= \frac{1}{K}\int \frac{1}{x}dx \\
v &= \frac{1}{K}(ln|x| + D) \\
v &= \frac{1}{K}ln|Lx|
\end{aligned}
$$

- Step 5

Our Solution is:

$$y = uv = Kx\frac{1}{K}ln|Lx| = xln|Lx|$$

3.2 Method of Integrating Factor

Use in general form of 1st order linear differential equation of:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

3.2.1 How to use the Method

- Step 1 : Calculate Integrating Factor $I(x) = e^{\int P(x)dx}$.
- Step 2 : Multiply both side of the equation by $I(x)$
- Step 3 : Form $\frac{d}{dx}(y.I(x)) = I(x)Q(x)$ and Integrate both side by dx.
- Step 4 : Solve for y and simplify.

3.2.2 Example

Solve:

$$cos(x)\frac{dy}{dx} + sin(x)y = 1$$

Then :

$$\frac{dy}{dx} + tan(x)y = \frac{1}{cos(x)}$$

We have $P(x) = tan(x)$ and $Q(x) = \frac{1}{cos(x)}$

- Step 1

$$I(x) = e^{\int P(x)dx} = e^{\int tan(x)dx} = e^{ln|sec(x)|} = sec(x)$$

- Step 2

$$sec(x)\frac{dy}{dx} + sec(x)tan(x)y = sec(x)\frac{1}{cos(x)}$$

$$sec(x)\frac{dy}{dx} + sec(x)tan(x)y = sec^2(x)$$

- Step 3

$$\frac{d}{dx}(y.sec(x)) = sec^2(x)$$

$$\int \frac{d}{dx}(y.sec(x))dx = \int sec^2(x)dx$$

$$y.sec(x) = \int sec^2(x)dx$$

$$y.sec(x) = tan(x) + C$$

- Step 4

$$y.sec(x) = tan(x) + C$$

$$y = \frac{tan(x) + C}{sec(x)}$$

$$y = sin(x) + Ccos(x)$$