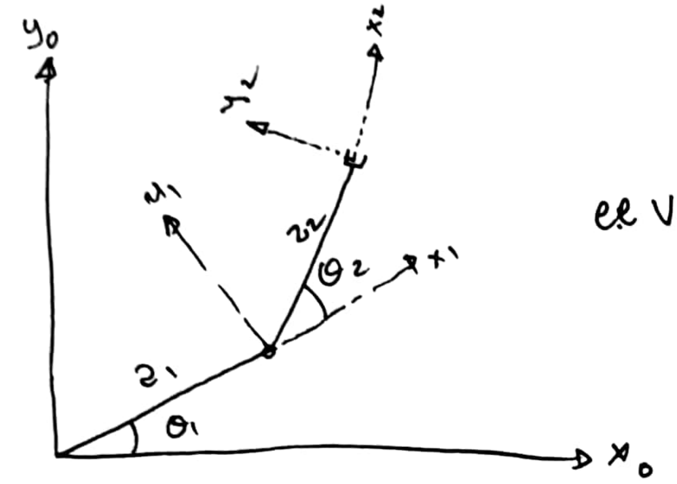
Forward Kinematic



$$\begin{cases} x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \end{cases}$$

Direction cosine

$x_2 \cdot x_0 = \cos(\theta_1+\theta_2)$

$x_2 \cdot y_0 = \sin(\theta_1+\theta_2)$

$y_2 \cdot x_0 = -\sin(\theta_1+\theta_2)$

$y_2 \cdot y_0 = \cos(\theta_1+\theta_2)$

Get rotation matrix

$$\begin{bmatrix} x_2\cdot x_0 & y_2\cdot x_0 \\ x_2\cdot y_0 & y_2\cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Inverse Kinematic

choose sign for elbow up/down

- $\theta_2 = \tan^{-1}\dfrac{\pm\sqrt{1-D^2}}{D}$
- $\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\dfrac{a_2\sin\theta_2}{a_1+a_2\cos\theta_2}\right)$

Velocity Kinematic

Relationship of tool velo. and joint velo. $\theta_1(t)$, $\theta_2(t)$

$\dot{x} = -a\sin\theta_1\dot{\theta}_1 - a_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$

$\dot{y} = +a_2\cos\theta_1\dot{\theta}_1 + a_2\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$

$x = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} -a\sin\theta_1 - a_2\sin(\theta_1+\theta_2) & -a_2\sin(\theta_1+\theta_2) \\ a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) \end{bmatrix}\dot{\theta}$$

$\dot{x} = J\dot{\theta}$ — ee velo, Jacobian, joint velo.

Because relationship of velo. is linear → determine joint velo. $\dot{\theta}$ from $\dot{x}$ is conceptually simple

Inverse Jacobian

$$\dot{\theta} = J^{-1}\dot{x}$$

$$J^{-1} = \frac{1}{a_1a_2\sin\theta_2}\begin{bmatrix} a_2\cos(\theta_1+\theta_2) & a_2 s(\theta_1+\theta_2) \\ -a_1 c\theta_1 - a_2 c(\theta_1+\theta_2) & -a s\theta_1 - a_2 s(\theta_1+\theta_2) \end{bmatrix}$$

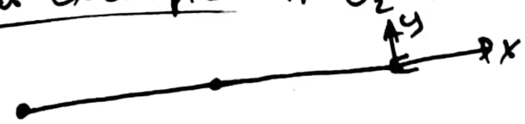
$$\boxed{J^{-1} = \frac{1}{\det(J)}\operatorname{adj}[J]}$$

but have a look at $\dfrac{1}{a_1a_2\sin\theta_2}$

we see that if $\sin\theta_2 = 0 \Rightarrow \theta_2 = 0$ or $\Rightarrow \theta_2 = \pi \Rightarrow J$ has no inverse

which is said to be Singularity.

For example if $\theta_2 = 0$

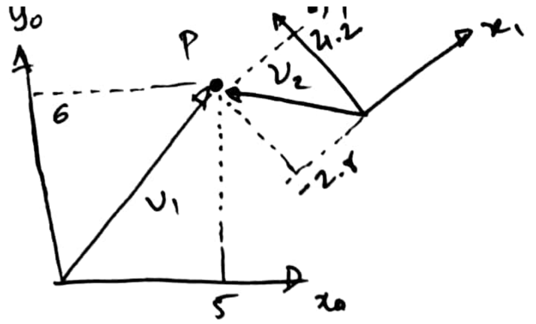


* the robot can not move to direction of $-x$ because it is block by arm link.

⇒ We want to avoid when do planning

Dynamic

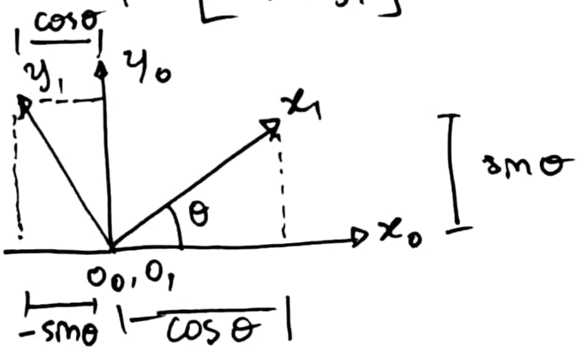
+ Lagrangian dynamic
+ Recursive Newton-Euler



$$P^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad P^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

Rotation in plane

$$R_1^0 = [x_1^0 \mid y_1^0]$$



$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

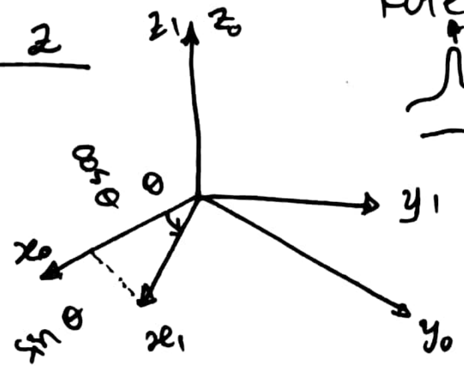
$$R_0^1 = (R_1^0)^T$$

$$(R_1^0)^T = (R_1^0)^{-1}$$

Rotation in 3D

Right hand Rule

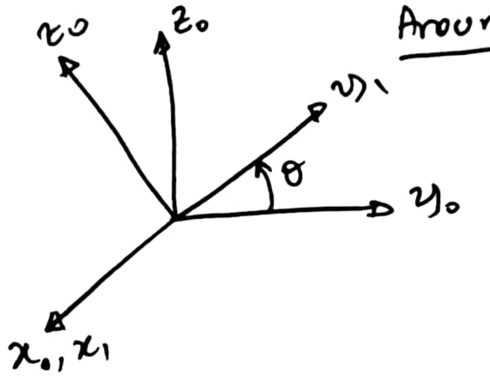
Around z



$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$$

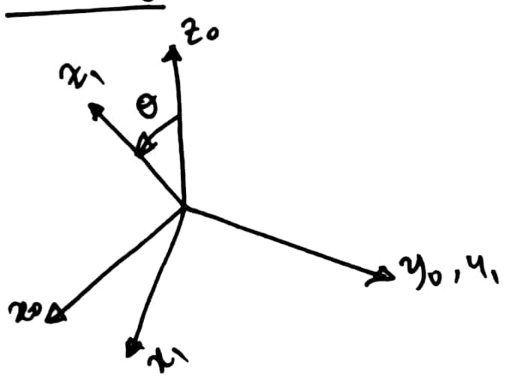
Let use instead: $R_{z,\theta}$

Around x



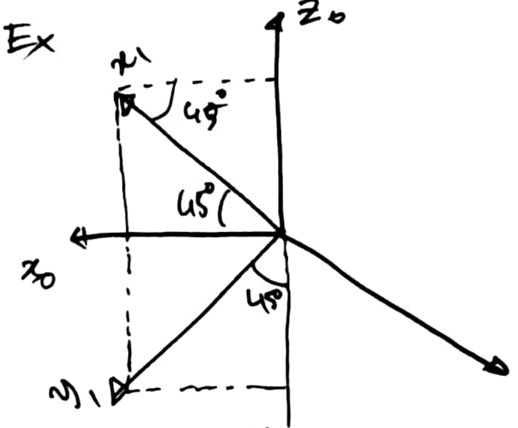
$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Around y



$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Ex



$$x_1 = \begin{bmatrix} \text{proj } x_1 \text{ on } x_0 \\ \text{proj } x_1 \text{ on } y_0 \\ \text{proj } x_1 \text{ on } z_0 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ \\ 0 \\ \sin 45^\circ \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$y_1 = \begin{bmatrix} \text{proj } y_1 \text{ on } x_0 \\ \text{proj } y_1 \text{ on } y_0 \\ \text{proj } y_1 \text{ on } z_0 \end{bmatrix} = \begin{bmatrix} \sin 45^\circ \\ 0 \\ \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$z_1 = \begin{bmatrix} \text{proj } z_1 \text{ on } x_0 \\ \text{proj } z_1 \text{ on } y_0 \\ \text{proj } z_1 \text{ on } z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow R_1^0 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

Rotational Transformation

$$p^0 = R_1^0 p^1$$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^1$ position of point p in coordinate 1

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^0$

Rotation Matrix

position of point p in coordinate 0

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^0 = [R] \begin{bmatrix} x \\ y \\ z \end{bmatrix}^1$$

☆ Role of Rotation matrices

1. Represent coordinate transformation of (point) in two diff. coord. sys.
2. Represent orientation of a transform coord. frame to a fixed frame
3. Work as an operator, take a vector → rotate it → new vector, all in a single frame

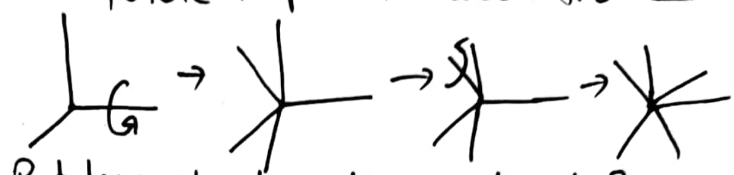
Rotation Respect to Current frame

$$\begin{cases} p^0 = R_1^0 p^1 \\ p^1 = R_2^1 p^2 \\ p^0 = R_2^0 p^2 \end{cases}$$

$$p^0 = R_1^0 R_2^1 p^2 \Rightarrow \boxed{R_2^0 = R_1^0 R_2^1}$$

In this rotation mode, we consecutive rotation from one frame after another rotated frame.

Rotate respect to local frame



Rotation about y than about z

$$\boxed{R = R_{y,\phi} R_{z,\theta}}$$

Rotate respect to fixed frame

$$R_2^0 = R_1^0 [(R_1^0)^{-1} R R_1^0] = R R_1^0$$

Rotation about y than about z

$$\boxed{R = R_{z,\theta} . R_{y,\phi}}$$

⊖ We have the rule:

- If rotate concurrent axis (Post multiplied)
- If rotate about fixed axis <Pre multiplied>

Euler angle

We can specify rotation of frame 1 respect to frame 0 by 3 angle (ϕ, θ, ψ). EX:

- First rotate about z by ϕ
- 2nd ——— y by θ
- 3rd ——— ~~z~~ by ψ

$$\Rightarrow R_{zyz} = R_{z,\psi} R_{y,\theta} R_{z,\phi}$$

We notice that the rotation in euler angle is in rotation about fixed frame ⇒ we premultiplied

→ It is ez to find R when we know sequence of rotation (ϕ, θ, ψ)

→ It is hard to find sequence of rotation from R

Roll, Pitch, Yaw

(ϕ, θ, ψ)

$z_0 \quad y_0 \quad x_0$

first rotate about x_0 by ψ

2nd — " — y_0 by θ

3rd — " — z_0 by ϕ

Fixed frame rotation

=> pre multiplied

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

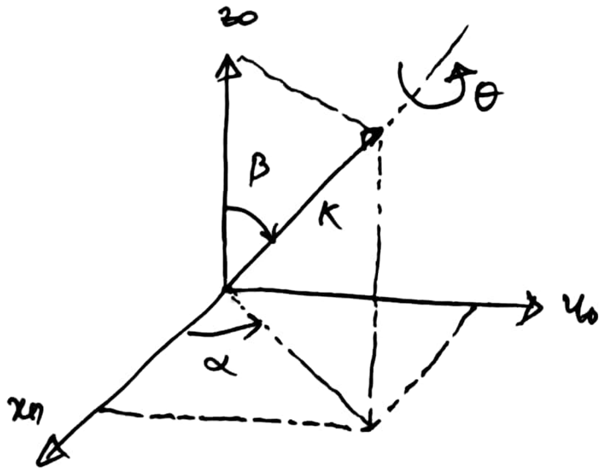
-> yaw, pitch, roll

Rotation about Abitrary Axis

Given an axis in 3D, rotate about it by θ

let $K = (K_x, K_y, K_z)$ unit vector that define axis in frame 0

find $R_{K,\theta}$.



There are multipleway:
Let rotate about z_0 axis then by y axis Using concurrent axis rotation type => than the axis z is align with K vector

using similarity transformation

$$R_{K,\theta} = R R_{z,\theta} R^{-1}$$

$$= R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,\beta} R_{z,-\alpha}$$

from fig. $\sin\alpha = \dfrac{k_y}{\sqrt{k_x^2 + k_y^2}}$

$$\cos\alpha = \frac{K_x}{\sqrt{K_x^2 + K_y^2}}$$

$\sin\beta = \sqrt{K_x^2 + K_y^2}$, $\cos\beta = K_z$

$R = R_{K,\theta}$ ← angle of rot.

arbi: axis

If given $R = [3\times3]$ rot. mat.
we can cal. $R_{K,\theta}$ by.

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$K = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$R_{K,\theta} = R_{-K,\theta}$$

Exponential Coordinate

- component of vector $K\theta \in \mathbb{R}^3$ are called exp. coord. of R.
- Skew-sym. (3×3)

$$S^T + S = 0$$

$S(K)$; $K = (K_x, K_y, K_z)$

$$\Rightarrow S(K) = \begin{bmatrix} 0 & -K_z & K_y \\ K_z & 0 & -K_x \\ -K_y & K_x & 0 \end{bmatrix}$$

Let $e^{S(K)\theta}$ be the matrix expo.

$$e^{S(K)\theta} = I + S(K)\theta + \frac{1}{2}S^2(K)\theta^2 + \frac{1}{3!}S^3(K)\theta^3 + \dots$$

Rigid Motion

A rigid motion is an ordered pair (d, R) where $d \in \mathbb{R}^3$ & $R \in SO(3)$. It is $\in SE(3) = \mathbb{R}^3 \times SO(3)$

$$p^0 = R_1^0 p^1 + d^0$$

distance of origin from frame 0 & 1

pose in frame 0

pose in frame 1

Rot. mat. of frame 1 respect to frame 0

---

3. coord. frame

$$p^1 = R_2^1 p^2 + d_2^1$$

&

$$p^0 = R_1^0 p^1 + d_1^0$$

$$\Rightarrow p^0 = R_1^0 (R_2^1 p^2 + d_2^1) + d_1^0$$

$$\Rightarrow p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

since p^0 & p^2 relationship is a rigid motion

$$p^0 = R_2^0 p^2 + d_2^0$$

$$\boxed{\begin{aligned} &\Rightarrow R_2^0 = R_1^0 R_2^1 \\ &\Rightarrow d_2^0 = d_1^0 + R_1^0 d_2^1 \end{aligned}}$$

Homogeneous Transformation

- Let $H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$ (4×4)

$$= \begin{bmatrix} 3\times3 & 3\times1 \\ 0\,0\,0 & 1 \end{bmatrix}$$ homogeneous transformation matrix

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

$$\boxed{P^0 = H_1^0 P^1}$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

Some basic H matrix

$$\mathrm{Trans}_{\begin{pmatrix} x\\y\\z \end{pmatrix}, \begin{pmatrix} a\\b\\c \end{pmatrix}} = \begin{bmatrix} 1&0&0&a \\ 0&1&0&b \\ 0&0&1&c \\ 0&0&0&1 \end{bmatrix}$$

$$\mathrm{Rot}_{\begin{pmatrix} x\\y\\z \end{pmatrix}, \begin{pmatrix} \alpha\\\beta\\\gamma \end{pmatrix}} = \begin{bmatrix} R_{\alpha,\beta,\gamma} & \begin{matrix}0\\0\\0\end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

distance from origin 0 to origin 1 define on frame 0

proj. x_1 on x_0 y_0 z_0; proj. y_1 x_0 y_0 z_0; proj. z_1 x_0 y_0 z_0

- Concurrent axis (post mul.)

$$H_2^0 = H_1^0 H$$

- Fixed axis (pre-mul.)

$$H_2^0 = H H_1^0$$

Quaternion

$Q = q_0 + i q_1 + j q_2 + k q_3$; $q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$

rotation by θ about unit vector $n = (n_x, n_y, n_z)$ is $Q = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$

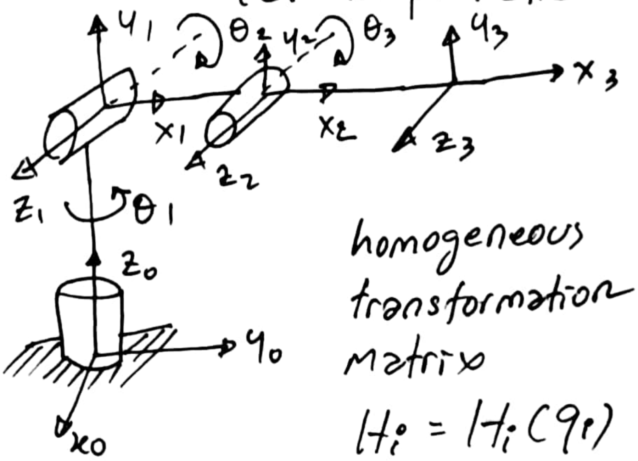
Forward Kinematics

A robot manipulator with n joint with have n+1 link

⊕ link 1, 2, ... n

⊕ joint (i) connect link (i−1) to link (i)

$$q_i = \begin{cases} \theta_i & \text{for revolute} \\ d_i & \text{for prismatic} \end{cases}$$



homogeneous transformation matrix

$$H_i = H_i(q_i)$$

revolute → θ_i d_i ← prismatic

$$H = \begin{bmatrix} R_n^0 & O_n^0 \\ 0 & 1 \end{bmatrix}$$

position and orientation of ee. in inertial frame given by

$$H = T_n^0 = \underbrace{H_1(q_1) \cdots H_n(q_n)}_{\text{each } H_i \text{ is}}$$

$$\begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix}$$

$$R_j^i = R_{i+1}^i \cdots R_j^{j-1}$$

$$O_j^i = O_{j-1}^i + R_{j-1}^i O_j^{j-1}$$

Denavit Hartenberg

Transformation matrix A_i as 4 basic tran

$$A_i = Rot_{z,\theta_i} \, Trans_{z,d_i} \, Trans_{x,a_i} \, Rot_{x,\alpha_i}$$

$$= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_i & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s\theta_i & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- a_i = link length
- α_i = link twist
- d_i = link offset
- θ_i = joint angle

Ex: 2 link planner

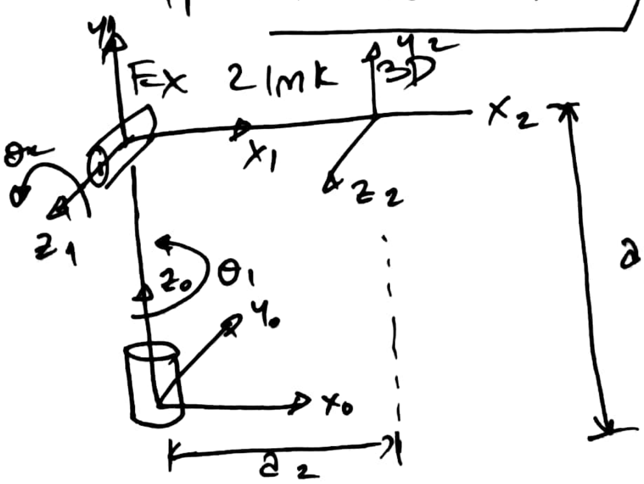
link	a	α	d	θ
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2

$$\Rightarrow A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_1^0 = A_1 \Rightarrow T_2^0 = A_1 A_2$$

Ex 2 link 3D



link	θ	α	r	d
1	θ_1	90°	0	a_1
2	θ_2	0	a_2	0

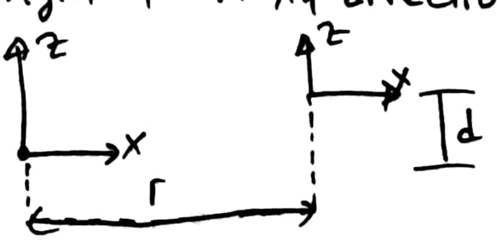
- θ amount of rotation of frame 0 along z-axis(0) to match frame 1 (pay attention to x axis, we want both x axis to match) and account for angle variable of joint as well.

[In our case the x_0 & x_1 is match so we don't have to rotate any thing, all it is left is joint variable θ, so ⇒ put θ in table]

- α amount of rotation of frame 0 along x-axis (1) to match frame 1 (pay att. to z axis, we want both z axis to match)

[In our case, we can rotate 90° of frame 0, along x_1 so f1 & f0 are all match. for f2,1, since all 3 axis are all match, we have to rotate 0°]

- r diplace-ment of origin 0 and origin 1 in x_1 direction.

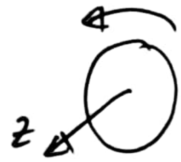


[In our case, f1 is on top of frame 0, so r=0, f_2 is at a_2 length origin from length 1 ⇒ $r = a_2$]

- d displacement of origin 0 and origin 1 in z_1 direction

[In our case, f_1 is at a_1 length in z direction from f_0, and f_2 is 0 length z axis from f_1]

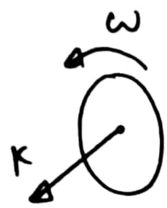
In DH: z axis always point at the direction of actuation



It is not strict rule, but it will make our live ezier.

Velocity Kinematic

Angular velo. Fixed Axis

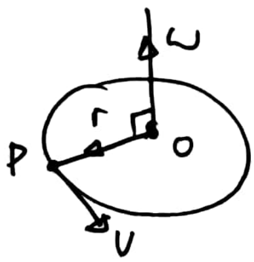


$\omega = \dot{\theta} K$ ← unit vector in diretion of axis of rotation

cross product

$v = \omega \times r$

linear velo — ang velo — vector from origin (assumed lie on the axis of rotation)



Skew-Sym Matrix

$S^T + S = 0$
$s_{ij} + s_{ji} = 0$
$s_{ii} = 0$

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

If $a = (a_x, a_y, a_z)$

$$\Rightarrow S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

axis of basis

ex: $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, S(j) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$S(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Prop. of skew. sym.

1). Linear

$$S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$$

2). $S(a)\, p = a \times p$

3). $R\, S(a)\, R^T = S(Ra)$

R is orthogonal $\in SO(3)$

Derivative of rotation matrix

- $R(\theta)$ = rotation matrix

$R \in SO(3) \Rightarrow$ orthogonal

$$R(\theta) R(\theta)^T = I$$

$$\left[\frac{\partial}{\partial\theta} R\right] R(\theta)^T + R(\theta) \left[\frac{\partial}{\partial\theta} R^T\right] = 0$$

Let

$$S = \left[\frac{\partial}{\partial\theta} R\right] R(\theta)^T \quad (*)$$

$$\Rightarrow S^T = \left(\left[\frac{\partial}{\partial\theta} R\right] R(\theta)^T\right)^T = R(\theta) \left[\frac{\partial}{\partial\theta} R^T\right]$$

$$S + S^T = 0$$

multiply $(*)$ both side by R

$$\boxed{SR = \left[\frac{\partial}{\partial\theta} R\right] \overbrace{R^T R}^{=I} = \frac{\partial}{\partial\theta} R}$$

Important

It say that the derivative of rotation mat. R is $S \cdot R$

skew. sym.

where $S = \left[\frac{\partial}{\partial\theta} R\right] R^T$

Ex: $R = R_{x,\theta}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow S = \left[\frac{\partial}{\partial\theta} R\right] R^T$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i) \text{ where } (i) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow x$$

thus $\partial / \partial\theta\, R_{x,\theta} = S(i) R_{x,\theta}$

then we shown that

$$\frac{\partial}{\partial\theta} R_{x,\theta} = S(i) R_{x,\theta}$$

$$\frac{\partial}{\partial\theta} R_{y,\theta} = S(j) R_{y,\theta}$$

$$\frac{\partial}{\partial\theta} R_{z,\theta} = S(k) R_{z,\theta}$$

$$\Downarrow$$

$$\frac{\partial}{\partial\theta} R_{k,\theta} = S(k) R_{k,\theta}$$

where $k = (k_x, k_y, k_z)$

Angular velo. Regular case

Consider general case of ang. velo about arbit. + posibly moving axis. suppose R is time varying $R = R(t) \in SO(3)$

$$\dot{R}(t) = \underbrace{S(\omega(t))}_{\text{skew. sym.}} R(t)$$

ω = ang. velo. of rotating frame with respect to fixed frame at time t.

We have:

$$p^0 = R_1^0 p^1$$

$$\Rightarrow \frac{\partial}{\partial t} p^0 = \dot{R}_1^0 p^1$$
$$= S(\omega) R_1^0 p^1$$
$$= \omega \times R_1^0 p^1$$
$$= \omega \times p^0$$

Ex $R(t) = R_{x,\theta(t)}$

$$\dot{R}(t) = \frac{\partial R}{\partial t} = \frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial t}$$
$$= \dot{\theta} S(t) R(t)$$
$$= S(\omega(t)) R(t)$$

where $\omega = i\dot{\theta}$, $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Addition of ang. velo

$$R_2^0(t) = R_1^0(t) R_2^1(t)$$

$$\Rightarrow \dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

$$\vdots$$

$$\omega_2^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

Given $R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$

$$\Rightarrow \dot{R}_n^0 = S(\omega_{0,n}^0) R_n^0$$

where

$$\boxed{\omega_{0,n}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^0 \cdots}$$

resultant of ang. velo due to rotation of several coord. frame.

Linear Velocity of point attached to moving frame

consider linear velo. of point that is rigidly attached to a moving frame. suppose p is rigid attached to frame 1, and frame 1 rotate about frame 0.

then coord of p is

$$p^0 = R_1^0(t) p^1$$

$$\Rightarrow \dot{p}^0 = (R_1^0(t) p^1)'$$

$$\Rightarrow \dot{p}^0 = \dot{R}_1^0(t) p^1 + R_1^0(t) \dot{p}^1$$

$\dot{p}^1 = 0$ p is rigid attach to fr. 1

$$= S(\omega^0) R_1^0(t) p^1$$
$$= S(\omega^0) p^0$$
$$= \omega^0 \times p^0$$

suppose the motion of f1 & f0 is more general.

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

$$p^0 = R_1^0 p^1 + o_1^0$$

for simplicity, omit (t), sup & sup. script.

$$\dot{p}^0 = \dot{R} p^1 + \dot{o}$$
$$= S(\omega) R p^1 + \dot{o}$$
$$= \omega \times r + v$$

- where $r = Rp^1$ is the vector from o_1 to p expressed is orientation of f0 & v is rate at origin o_1 is moving

- If the point p is moving in frame 1, then we must add term v is the rate of change of coord. p^1 expressed in f.0.

Derivation of Jacobian

- Consider n-link manip. with joint variable $q_1 \ldots q_n$. Let:

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & O_n^0(q) \\ 0 & 1 \end{bmatrix}$$

be tf. from ee. to base.

- As the robot move q & O_n^0 & R_n^0 are func. of time.
- We want to relate linear & ang velo of ee to joint velo $\dot{q}(t)$.

* Let

+ $S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$

be ang. velo. vector ω_n^0 of e.e

+ $v_n^0 = \dot{O}_n^0$

be lin. velo. of e.e

And we want to find expression in form:

$$v_n^0 = J_v \dot{q}$$
$$\omega_0^0 = J_\omega \dot{q}$$

- $J_v = 3 \times n$ mat.
- $J_\omega = 3 \times n$ mat.

we can write it as one

$$\xi = J\dot{q} \Leftrightarrow \xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix}, \quad J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$J = 6 \times n$ mat where n is number of link

body velo.

Ang. velo

We recall that ang. velo can be add as free vector if they express relative to a common fra.

$\Rightarrow$ we can det. ang. velo. of ee relative to base by express the ang. velo contributed by each joint in orientation of base and then sum it up.

- say i^{th} joint is revolute $\Rightarrow q_i = \theta_i$ & axis of rotation is z_{i-1}
- let ω_i^{i-1} represent ang. velo. of link i that created by rota. of joint i, express relative to frame $(i-1)$

$$\omega_i^{i-1} = \dot{q} z_{i-1}^{i-1} = \dot{q}_i k$$

where z is basis of axis $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = k$ unit vector

- If i^{th} joint is prismatic

$$\omega_i^{i-1} = 0$$

which correct since ang. velo of ee does not depend on θ_i but d_i instead.

$\Rightarrow$ Then the overall ang. velo. of ee ω_n^0 in base is

$$\Rightarrow \omega_n^0 = \rho_1 \dot{q}_1 k + \rho_2 \dot{q}_2 R_1^0 k + \cdots + \rho_n \dot{q}_n R_{n-1}^0 k$$
$$= \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0$$

where $\rho_i \begin{cases} = 1 & \text{for revolute} \\ = 0 & \text{for pris.} \end{cases}$

& $z_{i-1}^0 = R_{i-1}^0 k$

- $z_0^0 = k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow J_\omega = [\rho_1 z_0 \ldots \rho_n z_{n-1}]$

Linear. velo.

Lin. velo. of. ee is $\dot{O}_n^0$, by chain rule

$$\dot{O}_n^0 = \sum_{i=1}^{n} \frac{\partial O_n^0}{\partial q_i} \dot{q}_i$$

at ith column of J_v; J_w

$$J_{v_i} = \frac{\partial O_n^0}{\partial q_i}$$

In Intuition as if we not actuated any joint but ith and actuated ith joint at unit velocity

* Case 1: prismatic
  - direct pure translation of ee
  - direction of trans. is parrlel to axis z_{i-1}
  - magnitude is $\dot{d}_i$

  d_i is D-H joint varrable

$$\dot{O}_n^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i z_{i-1}^0$$

$$\Rightarrow J_{v_i} = z_{i-1}$$

* Case 2: Revolute
  - $q_i = \theta_i$

  $\Rightarrow$ linear velocity of ee is $\omega \times r$

  where
  - $\omega = \dot{\theta}_i z_{i-1}$
  - $r = O_n - O_{i-1}$

$$\Rightarrow J_{v_i} = z_{i-1} \overset{\text{cross}}{\times} (O_n - O_{i-1})$$

Combine linear & Ang. Velo Jac.

$$J_v = [J_{v_1} \; J_{v_2} \cdots J_{v_n}]$$

Where at ith column

$$J_{v_i} = \begin{cases} z_{i-1} \times (O_n - O_{i-1}) & \text{revol} \\ z_{i-1} & \text{pris} \end{cases}$$

$$J_\omega = [J_{\omega_1} \; J_{\omega_2} \cdots J_{\omega_n}]$$

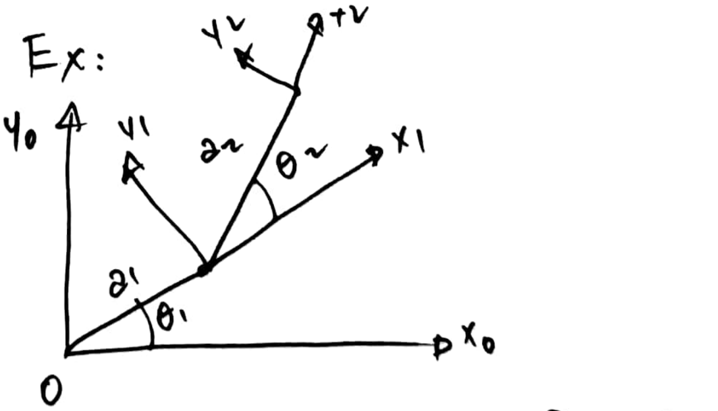
where at ith column

$$J_\omega = \begin{cases} z_{i-1} & \text{revol} \\ 0 & \text{pris} \end{cases}$$

from above formula $\rightarrow$ det of Jac is simple if forward kine are all worked out. To find Jac we need unit vector z_i & coord. of origin $O_1, \ldots O_n$.

- coord. for z_i with respect to base are given by first three element in <u>3rd</u> column of T_i^0,
- O_i is given by first three element of 4th column of T_i^0.

Ex:



+ we have 2 joint $\rightarrow n = 2 \Rightarrow$ Jac $= 6 \times 2$
+ joint revolute

$$J_{v_i} = z_{i-1} \times (O_n - O_{i-1})$$
$$J_{\omega_i} = z_{i-1}$$

• joint 1

$$\begin{cases} J_{v_1} = z_0 \times (O_2 - O_0) \\ J_{\omega_1} = z_0 \end{cases}$$

• joint 2

$$\begin{cases} J_{v_2} = z_1 \times (O_2 - O_1) \\ J_{\omega_1} = z_1 \end{cases}$$

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad O_1 = \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \\ 0 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

• cross product $u = \mathbb{R}^3$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + b_1 a_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

• $$[O_2 - O_0] = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix} = O_2$$

• $$[O_2 - O_1] = \begin{bmatrix} a_1\cos\theta_1 + a_2\cos\theta_1+\theta_2 \\ a_1\sin\theta_1 + a_2\sin\theta_1+\theta_2 \\ 0 \end{bmatrix} - \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2\cos\theta_1+\theta_2 \\ a_2\sin\theta_1+\theta_2 \\ 0 \end{bmatrix}$$

• $$[z_0 \times (O_2 - O_0)] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_1+\theta_2 \\ a_1 s\theta_1 + a_2 s\theta_1+\theta_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1+\theta_2 \\ a_1 c\theta_1 + a_2 c\theta_1+\theta_2 \\ 0 \end{bmatrix}$$

• $$[z_1 \times (O_2 - O_1)] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2\cos\theta_1+\theta_2 \\ a_2\sin\theta_1+\theta_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2\sin\theta_1+\theta_2 \\ a_2\cos\theta_1+\theta_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} J_{v_1} & J_{v_2} \\ J_{\omega_1} & J_{\omega_2} \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1+\theta_2 & -a_2 s\theta_1+\theta_2 \\ a_1 c\theta_1 + a_2 c\theta_1+\theta_2 & a_2 c\theta_1+\theta_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

From DH: $T_2^0 =$

$$\begin{array}{cccc} & & z_i & O_i \\ \end{array}$$
$$\begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since this is for frame 2 $\Rightarrow i = 2$

$$\Rightarrow z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad O_2 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

$$\begin{array}{cccc} & & z_1 & O_1 \\ \end{array}$$
$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Do example 4.6, 4.7, 4.8

The tool velocity

Many task need tool, tool is rigidly attached to ee. Relationship bet/ tool & ee frame is

$$T^6_{tool} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

• Assume ee velo is given & express in coord. relative to ee frame ξ^6_6. we want to find ξ^{tool}_{tool}.

• since frame is rigidly attached, $\Rightarrow$ ang. velo of tool = e.e.

we have

$$R^0_{tool} = R^0_6 R$$
$$\dot{R}^0_{tool} = \dot{R}^0_6 R$$

$$\Rightarrow S(\omega^0_{tool}) R^0_{tool} = S(\omega^0_6) R^0_6 R$$

$$S(\omega^0_{tool}) = S(\omega^0_6)$$

$$\Rightarrow \omega_{tool} = \omega_6$$

$$\Rightarrow \omega^{tool}_{tool} = \omega^{tool}_6 = R^T \omega^6_6$$

- If the ee frame is moving with body velo. $\xi = (v_6, \omega_6)$

$\Rightarrow$ lin. velo of tool origin

$$v_{tool} = v_6 + \omega_6 \times r$$

- r vector from origin of ee to origin of tool frame

$$\vdots$$

$$v^{tool}_6 = R^T v^6_6$$

$$\Rightarrow v^{tool}_{tool} = R^T v^6_6 - R^T S(d) \omega^6_6$$

$$\omega^{tool}_{tool} = R^T \omega^6_6$$

$$\Rightarrow \xi^{tool}_{tool} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times3} & R^T \end{bmatrix} \xi^6_6$$

In many cases, we want inverse problem; compute required ee velo. to get desired tool. velo. since

$$\begin{bmatrix} R & S(d)R \\ 0_{3\times3} & R \end{bmatrix} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times3} & R^T \end{bmatrix}^{-1}$$

$$\Rightarrow \xi^6_6 = \begin{bmatrix} R & S(d)R \\ 0_{3\times3} & R \end{bmatrix} \xi^{tool}_{tool}$$

General expression for trans. velo bet/ 2 rigidly attached moving frame:

$$\xi^A_A = \begin{bmatrix} R^A_B & S(d^A_B) R^A_B \\ 0_{3\times3} & R^A_B \end{bmatrix} \xi^B_B$$

Analytical Jac

Let $J_a(q)$ be analytical Jac. Based on a minimal represent for orientation of ee frame

Let $X = \begin{bmatrix} d(q) \\ \alpha(q) \end{bmatrix}$ = ee pose

- $d(q)$: usual vector from origin base to origin ee
- $\alpha(q)$: minimal representation for orien. of ee frame to base frame

ex: use $\alpha = (\phi, \theta, \psi)$ euler ang.

$$\dot{X} = \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = J_a(q)\dot{q}$$

$R = R_{z,\phi} R_{y,\theta} R_{z,\psi}$ euler ang

then

$$\dot{R} = S(\omega) R$$

where $\omega = \begin{bmatrix} c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta} \\ s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta} \\ \dot{\psi} + c_\theta \dot{\phi} \end{bmatrix}$

$$= \begin{bmatrix} c_\psi s_\theta & -s_\psi & 0 \\ s_\psi s_\theta & c_\psi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = B(\alpha)\dot{\alpha}$$

(proof this by ur own)

component of ω is nutation, spin, precession.

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ \omega \end{bmatrix} = J(q)\dot{q}$$

$$\Rightarrow J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} J_a(q)\dot{q}$$

then we can compute

$$J_a(q) = \begin{bmatrix} I & 0 \\ 0 & B^{-1}(\alpha) \end{bmatrix} J(q)$$

If det $B(\alpha) \neq 0$

Singularities

$J(q) = 6 \times n$ define mapping

$$\xi = J(q)\dot{q}$$

It is implied that all possible ee velo are linear combination of column of Jac.

$$\xi = J_1\dot{q}_1 + J_2\dot{q}_2 + \dots + J_n\dot{q}_n$$

when rank $(J) = 6$; ee can execute any arbitary velo.

It is always the case that rank $(J) \leq \min(6, n)$. J matrix rank is not always constant.

If rank (J) < config value called singularities.

What happen in singularities?

- Sing. represent config. from certain direction of motion may be unattainable.
- bounded ee velo may correspond to unbounded joint velo.
- bounded joint torque may correspond to unbounded ee force & torque
- correspond to point on the boundary of workspace = max reach

Many method can used to det. singularities. Let use one that If det $(J) = 0$ = sing. But it is hard to solved. Let use a technique of decoupling sing.

Sing. decoupling

+ arm sing. = sing. from motion of arm (normally first 3 joint)

+ wrist sing. = sing. from motion of spherical wrist

- Consider $n = 6$ joint case:

$\Rightarrow J = 6 \times 6$

$\Rightarrow$ sing. when $\det(J) = 0$

$$J = [J_P \mid J_O] = \left[\begin{array}{c|c} J_{11} & J_{12} \\ \hline J_{21} & J_{22} \end{array}\right]$$

+ final 3 joint. revolute

$$\Rightarrow J_O = \begin{bmatrix} z_3 \times (O_6 - O_3) & z_4 \times (O_6 - O_4) & z_5 \times (O_6 - O_5) \\ z_3 & z_4 & z_5 \end{bmatrix}$$

since wrist axes intersect at a common point O if we choose coord. frame so that $O_3 = O_4 = O_5 = O$ then

$$J_O = \begin{bmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

$$\Rightarrow \det J = \underset{(3\times3)}{\det J_{11}} \; \underset{(3\times3)}{\det J_{22}}$$

- J_{11} ith column $\begin{cases} z_{i-1} \times (O - O_{i-1}) & \text{revo.} \\ z_{i-1} & \text{pris} \end{cases}$

- $J_{22} = [z_3 \;\; z_4 \;\; z_5]$

$\Rightarrow$ Sing. config = when $\begin{cases} \det J_{11} = 0 \\ \det J_{22} = 0 \end{cases}$

Static Force / Torque Relation

Interect with env. produce force and moment at ee or tool.

=> produce torque at joint.

we want to use Jac. as relation bett ee force and joint torque

- Let $F = \begin{pmatrix} F_x \\ F_y \\ F_z \\ n_x \\ n_y \\ n_z \end{pmatrix}$, F = force vector, n = moment vector at ee.
- Let τ be corresponding vector of joint torque.

$$\Rightarrow \tau = J^T(q) F$$

this relation is obtain from principal of virtual work. Let δx & δq be infinitesimal displacement in task space & joint space. we call it virtual displacement if they are consistent with any contraint imposed on sys.

$$\delta x = J(q)\,\delta q$$

The virtual work of sys is

$$\delta w = F^T \delta x - \tau^T \delta q$$

$$\Rightarrow \delta w = (F^T J - \tau^T)\delta q$$

from this, it say that the equ. is = 0 when manipulator is in equil. which give the poof of

$$\tau = J^T(q) F$$

Ex : 2D 2 joint

we have $J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow \tau = J^T F$$

$$\Rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_1 c_1 + a_2 c_{12} & 0 & 0 & 0 & 1 \\ -a_2 s_{12} & a_2 c_{12} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

Inverse Velo. and Accerleration

Det. joint velo. $\dot{q}$ that produce desired ee velo.

when J is square & non sing.

$$\Rightarrow \dot{q} = J^{-1} \xi$$

If mani. joint are not = 6

=> J cannot be inverted

=> solution of $\xi = J\dot{q}$ exist iff ξ lie in range space of Jac.

- Can be det by simple test.

  rank(J) = rank$[J \mid \xi]$ ← aug. matrix

  => ξ belong to range of J

- For $n > 6$, solve $\dot{q}$ with right pseudo inverse J

$$\dot{q} = J^{+}\xi + (I - J^{+}J)\,b$$

b = arbitary vector

Inverse [redacted] Acceleration

$\dot{X} = J_a(q)\dot{q}$

$\Rightarrow \ddot{X} = J_a(q)\ddot{q} + \left(\frac{d}{dt} J_a(q)\right)\dot{q}$

If e.e acc. $\ddot{X}$ is given

$\Rightarrow$ Instantaneous joint acc $\ddot{q}$ is

$J_a(q)\ddot{q} = \ddot{X} - \left(\frac{d}{dt} J_a(q)\right)\dot{q}$

for 6 DoF :

- $\dot{q} = J_a(q)^{-1}\dot{X}$
- $\ddot{q} = J_a(q)^{-1}\left[\ddot{X} - \left(\frac{d}{dt} J_a(q)\right)\dot{q}\right]$

if $\det J_a(q) \neq 0$

Manipulability

$\xi = J\dot{q}$

think of J as scaling input $\dot{q}$ to make ξ. Consider the set of all joint velo $\dot{q}$

$\|\dot{q}\|^2 = \dot{q}_1^2 + \dot{q}_2^2 + \dots + \dot{q}_n^2 \leq 1$

if we use minimum norm solution

$\dot{q} = J^+\xi$

$\Rightarrow \|\dot{q}\|^2 = \dot{q}^T\dot{q} = (J^+\xi)^T J^+\xi$
$= \xi^T(JJ^T)^{-1}\xi$

This give us as a quantitative characterization of scaling effected by Jac.

If J is full rank, above equa. define an m-dim ellipsoid know as manipulability ellipsoid.

If input $\dot{q}$ vector has unit norm. then output ξ will lie in ellip given by equa.

It can be [redacted] seen that the equa. define ellipsoid by replace Jac by SVD ; $J = U\Sigma V^T$

$$\xi^T(JJ^T)^{-1}\xi = (U^T\xi)^T \Sigma_m^{-2}(U^T\xi)$$

- $\Sigma_m^{-2} = \begin{bmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \ddots & \\ & & & \sigma_m^{-2} \end{bmatrix}$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$

- let $v = U^T\xi$

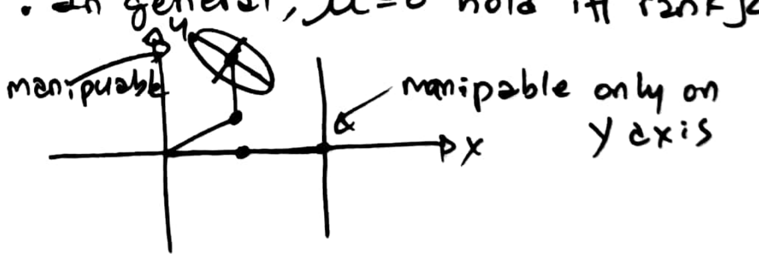
$\Rightarrow v^T \Sigma_m^{-2} v = \sum \frac{v_i^2}{\sigma_i^2} \leq 1$

- In [redacted] origal coord. sys.
- axis of ellip is vector $\sigma_i u_i$.
- volume of ellip $V = K\sigma_1\sigma_2 \dots \sigma_m$

 K = cte depend on dim m of ellip.
- Manipulability measure by

 $\mu = \sigma_1\sigma_2 \dots \sigma_m$

- In general, $\mu = 0$ hold iff rank $J <$ m



ex: 2 link 2 joint good old 2D

$J = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$

$\Rightarrow \mu = |\det J| = a_1 a_2 |s_2|$

$\Rightarrow$ Use manip.ty to det. optimal config to perform task. ex: 2 joint is good to perform at $\theta_2 = \pm\pi/2$

$\Rightarrow$ help design mechanical. ex: maximize $\mu = a_1 a_2 |s_2|$ to get a_1, a_2 best length.

Inverse kinematic

General IK

Given homog. trans.

$$H = \begin{bmatrix} R & O \\ 0 & 1 \end{bmatrix}$$

find solutions of equa.

$$T_n^0(q_1 \dots q_n) = H$$

where

- $T_n^0(q_1, \dots q_n) = A_1(q_1) \dots A_n(q_n)$
- H represent desired pos & orien. of e.e and our task is to find $(q_1 \dots q_n) \to T_n^0 = H$

<u>ex</u>: stanford manip.

from DH, we have $T_6^0 = A_1 \dots A_6 = \begin{bmatrix} r & r & r & d \\ r & r & r & d \\ r & r & r & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$

wher each r. element ~~is~~ trig. equa.

Let $H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be desired

pos & orien. of ee

$\Leftrightarrow H = T_6^0 \Leftrightarrow \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} = \begin{bmatrix} r & r & r & d \\ r & r & r & d \\ r & r & r & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ solve for $q_1 \dots q_n$ from 16 equa.

one possible solution is :

$$\begin{cases} \theta_1 = \pi/2, \theta_2 = \pi/2, d_3 = 0.5, \theta_4 = \pi/2, \theta_5 = 0 \\ \theta_6 = \pi/2 \end{cases}$$

$\Rightarrow$ difficult to solve

$\Rightarrow$ may or may not exist

$\Rightarrow$ not unique

$\Rightarrow$ must consider physical constraint

Kinematic decoupling

$\Rightarrow$ Decoupling into 2 problem

- inverse position kinematic.
- inverse orientation kinematic.

- for ex: 6 DOF = find ik position of centers of wrist sphere then find ik orien. of sphere.

Read more: 5.3, (5.3.1, .2), 5.4 do example. This section are geometrical approach which are some trig stuff

Numerical IK

- Let $x^d \in \mathbb{R}^m$ vector of Cartesian coord. ex: $x^d = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $x^d = \begin{pmatrix} x \\ y \\ z \\ \gamma \\ \delta \\ \phi \end{pmatrix}$

If we set

$$G(q) = x^d - f(q)$$

solution of $G(q)$ is q^d that sat. $G(q^d) = x^d - f(q^d) = 0$

- popular method

  + J inverse = newton-raphson of root finding.

  + J^T = gradient search.

Jac inv. method

- with x^d, expand forward kinematic func. $f(q)$ in tyler serie at q^d.

$$f(q) = f(q^d) + J(q^d)(q - q^d) + hot$$

Let take (J_a) to be our J

$$\Rightarrow q^d - q = J^{-1}(q)(x^d - f(q))$$

assume J is square & invertable.
to find q^d, we begin by guess q_0 then find seq. of update as:

$$\boxed{q_k = q_{k-1} + \alpha_k J^{-1}(q_{k-1})\overbrace{(x^d - f(q_{k-1}))}^{err}}$$

where: α is step size $\alpha > 0$.
choose to aid convergence,
- can be scalar or diag. mat.
- can be cte or func of k

⊕ if J is not square or invertable.
use $J^+ = J_a^+$ instead of J_a^{-1}

$$J^+ = J^T(JJ^T)^{-1}$$

$$q_k = q_{k-1} + \alpha_k J^+(q_{k-1})\overbrace{(f(q^d) - f(q_{k-1}))}^{err}$$

Jac. Transpose

- Define an optimization problem

$$\min_q F(q) = \min_q \frac{1}{2}(f(q) - x^d)^T (f(q) - x^d)$$

where: x^d = desired config
$f(q)$ = forward kine.

The gradient of cost func $F(q)$ is

$$\nabla F(q) = J^T(q)(f(q) - x^d)$$

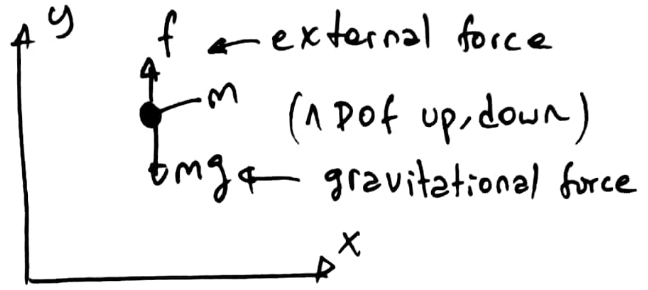
- A gradient decent alg. to minimize $F(q)$ is

$$q_k = q_{k-1} - \alpha_k \nabla F(q_{k-1})$$

$$q_k = q_{k-1} - \alpha_k J^T(q_{k-1})\overbrace{(F(q_{k-1}) - x^d)}^{err}$$

Dynamic

- Euler-Lagrange
- holonomic constraint
- principle of virtual work
- Newton-euler formulation for recursive numerical calculation



Euler-Lagrange

by newton 2nd:

$$m\ddot{y} = f - mg$$

- $m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right)$

$$= \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

$\Rightarrow K = \frac{1}{2}m\dot{y}^2$ = kinetic energy

- $mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial P}{\partial y}$

$\Rightarrow P = mgy$ = potential energy due to gravity

Define $\mathcal{L} = K - P = \frac{1}{2}m\dot{y}^2 - mgy$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial K}{\partial \dot{y}} \quad \& \quad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial P}{\partial y}$$

sub. back to original equa.

$$\Rightarrow \frac{d}{dt}\frac{\partial K}{\partial \dot{y}} = f - \left(+\frac{\partial P}{\partial y}\right)$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} = f - \left(-\frac{\partial \mathcal{L}}{\partial y}\right)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f}$$

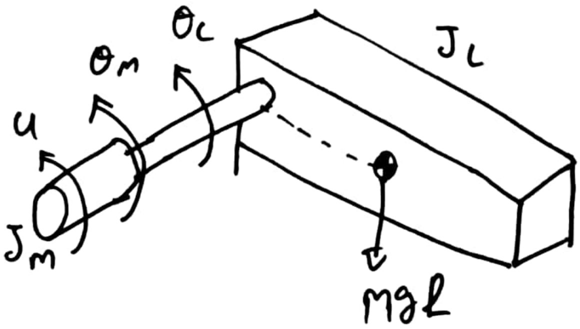
⊕ $\mathcal{L}$ is Lagrangian
⊕ The equa. is Euler-Lagrange eq.

⊛ n-DOF sys.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k \; ; k = 1, \dots n$$

+ τ_k = (generalized) force associated with q_k.

Ex : single link manip.



let θ_ℓ angle of link

θ_m motor shaft

$\Rightarrow \theta_m = r\theta_L$: r = gear ratio.

$\Rightarrow$ Since the sys. has 1 DOF we can generalized coord. [redacted]

θ_m or θ_L

Let choose $q = \theta_\ell$ as gen. coord.

we have kinetic energy

$$⊛ \; K = \frac{1}{2} J_m \underbrace{\dot{\theta}_m^2}_{\propto r^2 \dot{\theta}_\ell^2} + \frac{1}{2} J_\ell \dot{\theta}_\ell^2$$

$$= \frac{1}{2}(r^2 J_m + J_\ell)\dot{q}^2$$

where:

J_m = rot. inertia of motor

J_ℓ = —— " —— link

we have potential energy

$$⊛ \; P = Mg\ell(1 - \cos q)$$

where:

M : total mass of link

ℓ = distance from joint axis to link center of mass

Define $I = r^2 J_m + J_\ell$

$$\Rightarrow \mathcal{L} = \frac{1}{2} I \dot{q}^2 - Mg\ell(1 - \cos q)$$

$q = \theta_\ell$

$$\Rightarrow \mathcal{L} = \frac{1}{2} I \dot{\theta}_\ell^2 - Mg\ell(1 - \cos\theta_\ell)$$

Lagrange :

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_\ell} - \frac{\partial \mathcal{L}}{\partial \theta_\ell} = \tau_k$$

- $\frac{\partial \mathcal{L}}{\partial \dot{\theta}_\ell} = I\dot{\theta}_\ell$
- $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_L}\right) = I\ddot{\theta}_\ell$
- $\frac{\partial \mathcal{L}}{\partial \theta_\ell} = Mg\ell \sin\theta_\ell$

$$\Leftrightarrow \; I\ddot{\theta}_\ell + Mg\ell\sin\theta_\ell = \tau_\ell$$

The generalized force τ_ℓ represents those ext. forces & torques that are not derived from a potential func.

For this example : τ_ℓ consists of

- input motor torque $u = r\tau_m$, reflected to link
- damping torques $B_m\dot{\theta}_m$ & $B_\ell\dot{\theta}_\ell$

Reflecting the motor damping to link yield $\tau_L = u - B\dot{q}$

$B = rB_m + B_\ell$.

Then the complete dynamic eq. is

$$\boxed{I\ddot{\theta}_\ell + B\dot{\theta}_\ell + Mg\ell\sin\theta_\ell = u}$$

<u>Do example 6.2</u>