Olympiad stuff you should know (and some stuff that could be nice to know)
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Contents

1 Algebra

1.1 Identities

- Binomial theorem and multinomial theorem

- Factorization of $a^n - b^n$ and $a^n + b^n$

- Brahmagupta's identity

- Euler's four-square identity

- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

1.2 Sequences, sums and products

- Telescoping series

- Difference equations

- Arithmetic progression/sum, geometric progression/sum, geometric series

- Fibonacci sequence

1.3 Functional equations

- Basic definitions: domain, codomain, injective, surjective, bijective, involution, additive, linear, cyclic, periodic.

- Cauchy's functional equation

- Jensen's functional equation

- Monotonocity and continuity

- Rewriting to recurrence relation

- Polynomial functional equations

    - Expanding to $\mathbb{C}$ and using fundamental theorem of algebra[1]
    - Polynomials are continuous

1.4 Inequalities

- Triangle inequality and $x^2 \geq 0$

- Jensen's inequality

- QM-AM-GM-HM

- Power Means – generalization of AM-GM

- Muirhead's inequality

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[1]great example where this is the key part: find all polynomials $p(x)$ such that $p(x^2) = p(x)p(x+1)$.

- Hölder's inequality

- Cauchy-Schwarz, Titus Lemma

- Rearrangement inequality and Chebyshev's inequality

- Karamata's inequality

- Schur's inequality

- Lagrange multipliers (xD)

- Substitutions like Ravi substitution.

1.5   Polynomials

- The fundamental theorem of algebra

- Polynomial division algorithm

- Vietas formulas

- Remainder factor theorem and Rational root theorem

- Roots of unity

- Lagrange interpolation

- Irreducibility

    - Gauss' lemma
    - Eisensteins (expanded) irreducibility criterion
    - Cohn's irreducibility criterion
    - Perron's irreducibility criterion

- Algebraic numbers and the minimal polynomial

- Calculus

    - [2] $(x-a)^k \mid P(x) \implies (x-a)^{k-1} \mid P'(x)$
    - Intermediate value theorem
    - Rolle's theorem
    - Taylor series

- Symmetric polynomials

- Rouché's theorem

- Chebyshev polynomials

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[2]A nice use of this theorem is to show that the Taylor polynomial of e^x around $x = 0$ has only simple zeros (in the complex plane)

2 Discrete mathematics

2.1 Combinatorics

- Sets

  - Basic definitions: elements, subsets, union, intersection, difference, complement, cartesian product, function, image, pre-image, injective, surjective, bijective, composition, cardinality.

- Permutations and combinations

- Bijections

- Double counting

- Pascal's triangle and binomial coefficients

  - Symmetry of Pascal's triangle
  - Row-, column- and diagonal-row sums
  - Pascal's identity
  - Hockey-stick identity
  - Vandermonde's identity

- Recursion

- Stars and bars

- Pidegeonhole principle

- PIE (Principle of Inclusion-Exclusion)

- Some abstract algebra stuff

  - Permutation groups
  - Group actions, especially Burnside's formula/lemma

- Hall's marriage theorem

- Dilworth's theorem

2.2   Graph theory

- Basic definitions: graph, vertice, edge, face, adjacent, incident, degree, path, cycle, length of path and cycle, walk, connect/disconnected, tree, forest, Hamiltoninan path/cycle, Eulerian circuit, complete, planar, bipartite, k-partite

- Handshake lemma

- Eulerian graphs, planar graphs, the complete graph K_n

- Euler's formula on planar graphs

- Kuratowski's theorem

- Dirac's theorem

- Ore's theorem

- Cayley's theorem

- Turan's lemma

- Four colour theorem

- Zarankiewicz's Lemma

- Ramsey theory

2.3   Other that doesn't fit into any category?

- Bayes' theorem

- Invariants

- Monovariants

- Generating functions

- Game theory

3   Geometry

3.1   Standard normal geometry

- Similar Triangles

- Cyclic Quads

- Circumcentre

- Orthocentre and its Configurations

- Incentre

- Centroid

- Tangents

- Incircles and Excircles

- Incentre/Excentre Lemma

- Power of a Point

- Radical Axis

3.2   Some Slightly Less Standard Geometry

- Homotheties

- Nine-Point Circle

- Basic Trig

- Ceva

- Menelaus

3.3   Configurations which often come up

- Incircle/Excircle Configurations

- Simson Lines

- Midpoints of Altitudes

- Isogonal/Isotomic Conjugates

- Symmedians

- Curvilinear and Mixtilinear Incircles

3.4   Bish Bosh Bashing

- Trig bashing/length chasing

- Cartesians

- Complex Bash

- Areal/Barycentric Bash

3.5 Special Points

- HM Point

- Isogonal Conjugate of above

- Fermat Point

- Isodynamic Point

- Bevan Point

3.6 Weird Geometry Stuff

3.6.1 Inversion

- Definition of Inversion
- Angle Conservation in Inversion
- Overlays
- Orthogonal Circles
- Inversion Distance Formula (never used this but i guess it is in egmo lol)

3.6.2 Projective

- Cross-Ratio
- Harmonic Bundles and Quadrilaterals
- Apollonian Circles
- Poles/Polars and Bruhcard's Theorem
- Pascal's Theorem
- Brianchon's Theorem
- Purely Projective problems
- Projective Transformations

3.6.3 Complete Quadrilaterals

- sorry i haven't covered this so literally just copying it from egmo contents
- Spiral Similarity
- Miquel's Theorem and Miquel Points
- Gauss-Bodenmiller Theorem
- Miquel Points of Cyclic Quadrilaterals

3.7 Random High-Tech Theorems and Tricks that are Good to Know

- Napoleon's theorem

- Cayley-Bacharach

- Sawayama-Thebault

- Circle Tangency through Homotehty Trick (see ISL 2018 G5 or IMO 2011/6)

- Moving Points

4 Number Theory

4.1 Divisibility

- Euclid's algorithm, division algorithm and gcd

- Bezout's identity

- Divisibility relations like Euclid's lemma

- Fundamental theorem of arithmetic

4.2 Modular arithmetic

- Basic congruence rules

- Linear congruences and Chinese remainder theorem

- Euler's theorem and Fermat's little theorem

- Wilson's theorem

- Order of an element

- Primitive roots

- Quadratic congruences

  - Quadratic residues
  - Legendre symbol
  - $x^2 \equiv -1 \pmod{p} \iff p \equiv 1 \pmod{4}$
  - Euler's criterion
  - Law of quadratic reciprocity

- Lucas' theorem

- Cyclotomic polynomials

- Lagrange's theorem for polynomials

4.3 Diophantine equations

- Linear diophantine equations

- Finding contradiction modulo n

- p-adic valuation

  - Basic theorems and definition
  - Legendre's formula
  - Lifting the exponent lemma

- Pell's equation

- Hensel's lemma

- Pythagoras' theorem and Fermat's Last Theorem

4.4 Abstract algebra stuff

- Basic definitions: rings, integral domains, fields, ideals and principal ideal domains (PID), Euclidean domains (ED), unique factorization domains (UFD)

- Eisenstein's criterion and Gauss' lemma

- Gaussian integers $\mathbb{Z}[i]$

- Field extensions of $\mathbb{Q}$ like $\mathbb{Q}(\sqrt{d})$ and the ring of integers $\mathbb{Z}[\omega]$ of $\mathbb{Q}(\sqrt{d})$.

4.5 Other

- Infinite descent

- Bertrand's postulate

- Zsigmondy's theorem

- Vieta root jumping

- Chicken McNugget theorem

- Pidgeonhole principle

- Sum of divisor function and number of divisors function

- Fibonacci Numbers, Fermat numbers, Mersenne primes and Euler-Euclid theorem (bijection between perfect numbers and Mersenne primes)