Supplementary information: Boundedness and exponential stabilization for time-space fractional parabolic-elliptic Keller-Segel model in higher dimensions

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Supplementary Note : Supplementary Explanation of Proof of Boundedness of Solutions

We will consider an initial boundary value problem for a time-space fractional parabolic-elliptic Keller-Segel (KS) model

$$\begin{cases} {}_0^C\mathrm{D}_t^\beta u = -(-\Delta)^{\alpha/2}(\rho(v)u), & (t,x)\in(0,T]\times\Omega, \\ (-\Delta)^{\alpha/2}v + v = u, & (t,x)\in(0,T]\times\Omega, \\ \partial_\nu u = \partial_\nu v = 0, & (t,x)\in(0,T]\times\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where $\beta\in(0,1),\ \alpha\in(1,2),\ \Omega\subset\mathbb{R}^n,\ n\geq 3$.

Lemma 4.3 Assume that $\rho(v)=v^{-k}, k>0$ and that $n\geq 1$. Let $(u(t,x),v(t,x))$ be a unique non-negative classical solution to (1) on $[0,T_{max}),\ T_{max}\in(0,\infty]$. Then, there exists a positive constant $C=C\left(n,\Omega,k,u_0,\alpha\right)$ satisfying

$$\|u(t,\cdot)\|_{L^\infty(\Omega)}\leqslant C, \quad \text{for all} \quad t\in(0,T_{max}), \tag{2}$$

$$\|v(t,\cdot)\|_{W^{1,\infty}(\Omega)}\leqslant C, \quad \text{for all} \quad t\in(0,T_{max}). \tag{3}$$

Proof We have demonstrated that L^{p+1} integrability of u for some $p+1>n/2$. In the following, the method of proving Lemma 4.3 in Ref. [1] is referred to in order to prove the Lemma. Let $p+1>n$. Multiplying the first equation of (1) by u^p and integrating over Ω, by [[2],p579,(3.15)] and [[3],p1248,(3.1)] we obtain

$$\begin{aligned} &\frac{1}{p+1}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \int_\Omega D^{\frac{\alpha}{2}}u^p D^{\frac{\alpha}{2}}\left(v^{-k}u\right)dx \\ &\leq \int_\Omega u^p\,{}_0^C\mathrm{D}_t^\beta u dx + \int_\Omega u^p\left(-\Delta\right)^{\alpha/2}\left(\rho(v)u\right)dx = 0. \end{aligned} \tag{4}$$

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Using further Eilertsen equality and "Cordoba-Cordoba" type inequalities from [2] to deflate $\int_\Omega D^{\frac{\alpha}{2}} u^p D^{\frac{\alpha}{2}}\left(v^{-k}u\right)dx$, for $\rho(v)$ and u with the same monotonicity, we get

$$\begin{aligned}
&\int_\Omega D^{\frac{\alpha}{2}} u^p D^{\frac{\alpha}{2}}\left(v^{-k}u\right)dx \le \int_\Omega p u^{p-1} D^{\alpha/2}u\left[v^{-k}D^{\alpha/2}u + uD^{\alpha/2}v^{-k}\right]dx \\
&-\int_\Omega p u^{p-1}D^{\alpha/2}uA_{\alpha/2}\int_\Omega \frac{\left(v^{-k}(x)-v^{-k}(y)\right)\left(u(x)-u(y)\right)}{|x-y|^{n+\alpha}}dx \\
&\le \int_\Omega p u^{p-1} D^{\alpha/2}u\left[v^{-k}D^{\alpha/2}u + uD^{\alpha/2}v^{-k}\right]dx \\
&\le \int_\Omega p v^{-k}u^{p-1}\left|D^{\alpha/2}u\right|^2 dx - pk\int_\Omega \frac{1}{v^{k+1}}u^p D^{\alpha/2}uD^{\alpha/2}v dx.
\end{aligned} \tag{5}$$

Combining (4) and (5), we obtain the following inequality

$$\frac{1}{p(p+1)}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \int_\Omega \frac{1}{v^k}u^{p-1}\left|D^{\alpha/2}u\right|^2 dx \le k\int_\Omega \frac{1}{v^{k+1}}u^p D^{\alpha/2}uD^{\alpha/2}v dx. \tag{6}$$

By applying Young's inequality to (6), we derive

$$\begin{aligned}
&\frac{1}{p(p+1)}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \int_\Omega \frac{1}{v^k}u^{p-1}\left|D^{\alpha/2}u\right|^2 dx \\
&\le \frac{1}{2}\int_\Omega \frac{1}{v^k}u^{p-1}\left|D^{\alpha/2}u\right|^2 dx + \frac{k^2}{2}\int_\Omega \frac{1}{v^{k+2}}u^{p+1}\left|D^{\alpha/2}v\right|^2 dx.
\end{aligned}$$

Due to $v_* \le v \le v^*$ and [[2],Theorem 3.2], there exists $C>0$ such that

$$\frac{1}{p(p+1)}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \frac{1}{C}\int_\Omega \left|D^{\alpha/2}u^{\frac{p+1}{2}}\right|^2 dx \le C\int_\Omega u^{p+1}\left|D^{\alpha/2}v\right|^2 dx. \tag{7}$$

Using Young's inequality again for the right-hand side of equation (7) yields

$$\frac{1}{p(p+1)}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \frac{1}{C}\int_\Omega \left|D^{\alpha/2}u^{\frac{p+1}{2}}\right|^2 dx \le C\int_\Omega u^{p+2}dx + C\int_\Omega \left|D^{\alpha/2}v\right|^{2(p+2)}dx.$$

Furthermore, applying Nash-Gagliardo-Nirenberg-type inequality (see [3],Lemma 5.3), standard elliptic regularity theorey and $\|v\|_{L^\infty(\Omega)} \leqslant C$, we obtain

$$\left\|D^{\alpha/2}v\right\|_{L^{2(p+2)}(\Omega)}^{2(p+2)} \le C\|v(t,\cdot)\|_{L^\infty(\Omega)}^{p+2}\|v(t,\cdot)\|_{W^{2,p+2}(\Omega)}^{p+2} \le C\|u(t,\cdot)\|_{L^{p+2}(\Omega)}^{p+2}.$$

Therefore, we have

$$\frac{1}{p(p+1)}{}_0^C\mathrm{D}_t^\beta\int_\Omega u^{p+1}dx + \frac{1}{C}\int_\Omega \left|D^{\alpha/2}u^{\frac{p+1}{2}}\right|^2 dx \le C\int_\Omega u^{p+2}dx. \tag{8}$$

Using Nash-Gagliardo-Nirenberg-type inequality again, we notice that

$$\int_\Omega u^{p+2}dx = \left\| u^{\frac{p+1}{2}} \right\|_{L^{\frac{2(p+2)}{p+1}}(\Omega)}^{\frac{2(p+2)}{p+1}} \le C \left\| u^{\frac{p+1}{2}} \right\|_{L^1(\Omega)}^{\frac{2(p+2)}{p+1}(1-\lambda^*)} \left\| D^{\alpha/2} u^{\frac{p+1}{2}} \right\|_{L^2(\Omega)}^{\frac{2(p+2)}{p+1}\lambda^*}, \tag{9}$$

with $\lambda^* = \frac{n+1}{n+3}\frac{p+3}{p+2} \in (0,1)$. By L^{p+1} integrability of u for some $p+1 > \frac{n}{2}$, we choose $p+1 > n$ satisfying $\|u(t,\cdot)\|_{L^{\frac{p+1}{2}}(\Omega)} \le C$, for all $t \in (0, T_{max})$. Thus, the inequality (9) can be expressed as

$$C\int_\Omega u^{p+2}dx \le C \left\| D^{\alpha/2} u^{\frac{p+1}{2}} \right\|_{L^2(\Omega)}^{\frac{2(p+2)}{p+1}\lambda^*}. \tag{10}$$

Due to $\frac{2(p+2)}{p+1}\lambda^* < 2$, by applying Young's inequality to (10), then from (8) we deduce that

$$\frac{1}{p(p+1)}\,{}_0^C\mathrm{D}_t^\beta \int_\Omega u^{p+1}dx + \frac{1}{2C}\int_\Omega \left| D^{\alpha/2} u^{\frac{p+1}{2}} \right|^2 dx \le C.$$

Analogous to (9), Nash-Gagliardo-Nirenberg-type inequality gives us that there exist a number $C > 0$, such that

$$\frac{1}{p(p+1)}\,{}_0^C\mathrm{D}_t^\beta \int_\Omega u^{p+1}dx + \int_\Omega u^{p+1}dx \le C.$$

Let $w(t) = \int_\Omega u^{p+1}dx$, and with reference to [[4],Corollary 2.1], we obtain for some $p+1 > n$ that

$$w(t) = \|u(t,\cdot)\|_{L^{p+1}(\Omega)} \le w(0)E_\beta(-t^\beta) - C\left(E_\beta(-t^\beta) - 1\right) \le C, \tag{11}$$

for any $t \in (0, T_{max})$. The final deflation of the above inequality makes use of the fact that there exists $t_1 > 0,\ \varepsilon > 0$, such that Mittag-Leffler function $w(0)E_\beta(-t^\beta) < \varepsilon$ for any $t > t_1$.

Finally, (3) is obtained by using standard elliptic regularity theorey (see,e.g.[5]) for (11). And (2) can be deduced by standard iteration argument (see,e.g [6]). This completes the proof.

Reference

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