Analytical solutions for Vibrating Membrane

Applied Mathematics Department

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PARTIAL DIFFERENTIAL EQUATIONS

MATH - 302

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Table of contents

Contents

1. Abstract

The analytical & numerical solutions of vibration response of a membrane exposed to finite deformations and a time-varying lateral pressure is thoroughly examined in this study.

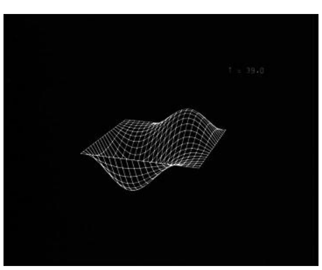
The Hamilton's Principle is used to generate the membrane's governing differential equation. The problem can then be solved by applying the technique of separation of variables.

2. Introduction

Due to its use in several engineering fields, including as space applications, actuators, sensors, robotics, bioengineering devices, and civil engineering structures, membranes have received a lot of attention recently. An overview of the literature on the theoretical, experimental, and dynamic behavior of membranes, whether or not they are hyperelastic, with a focus on practical applications. Furthermore, membranes are important in nature. In-depth study has been done in recent years on the creation of novel membrane materials, such as shape memory polymers and dielectric elastomers. These cutting-edge materials are appealing for applications in bioengineering, thin-film compliant microdevices, sensors, and vibration control.

Wave second order differential equations describe the behavior and propagation of certain wave and can be applied in a variety of real-life applications depending on the structure of the equations and its boundary conditions and exciting factors. In its two-dimensional form, it contains a time variable (t), two spatial variables (x, y), and a function of these variables, u, that describes the displacement of a medium through which waves are propagating over time.

The general solution of certain PDE depends mainly on the structure of its variables and the set of boundary and initial conditions and mainly for 2D, it can be solved by the double integral and summations as will be illustrated below. These type of PDE can be applied in a wide range of real-life applications like in drums, receiver and wave guides.



3. Problem definition

Using the wave function equation and the separation of variables technique, mathematical procedures can be arranged and established to obtain the exact real solution to a vibrating membrane by solving analytically vibrating circular and rectangular membranes.

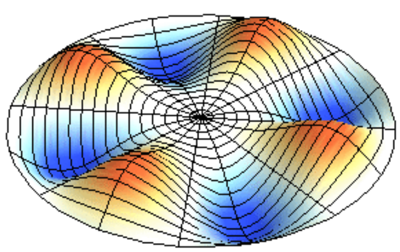
General differences

Rounded rectangular membranes: This class of membranes with rounded outer corners has a great advantage over membranes with a rectangular platform wave propagation at the boundary being greatly diffused. As a result, such membranes have a great potential for use in practical engineering applications, especially in waveguides-based structures.

Circular membranes: explains percussion instruments such as drums and timpani. However, there is also a biological application in the working of the eardrum and light/wave guides.

4. Methodology

4.1 Circular membrane



By considering the general case of -which depend on theta- vibrating circular membrane of radius c:

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}+\frac{\partial^2 u}{\partial \theta^2}\right)= \frac{\partial^2 u}{\partial t^2}, 0<r<c, t>0$$

Boundary conditions

$$u(c,\theta,t)=0, \qquad 0<\theta<2\pi, \qquad t>0$$

Initial conditions

$$u(r,\theta,0) = f(r,\theta), \qquad 0 < r < c, \qquad 0 \ < \ \theta \ < \ 2\pi$$

$$\frac{\partial u(r,\theta,0)}{\partial t} = g(r,\theta), \qquad 0 < r < c, \qquad 0 \ < \ \theta \ < \ 2\pi$$

Periodic conditions

$$u(r,\pi,t) = u(r,-\pi,t), \qquad 0 < r < c, \qquad t > 0$$

$$\frac{\partial u(r,\pi,t)}{\partial \theta} = \frac{\partial u(r,-\pi,t)}{\partial \theta}, \qquad 0 < r < c, \qquad t > 0$$

By using separation of variables method

let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Then

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{T''(t)}{a^2T(t)} = -\lambda$$

$$T''(t) + \lambda a^2 T(t) = 0 \quad \textbf{(1)}$$

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + r^2\lambda = \beta^2$$

$$r^2R''(r) + rR'(r) + (r^2\lambda - \beta^2)R(r) = 0 \quad \textbf{(2)}$$

$$\Theta''(\theta) + \Theta(\theta)\beta^2 = 0 \quad \textbf{(3)}$$

By solving equ.3, then the characteristic equation is

$$\mathrm{m}^2 = -\beta^2$$

Case I: When $\beta < 0$, it will give *trivial solution.*

Case II: When $\beta = 0$, then

$$\Theta(\theta) = C_1\theta + C_2$$

Using periodic conditions

$$\Theta(\pi) = \Theta(-\pi)$$

$$C_1\pi + C_2 = -C_1\pi + C_2$$

Thus

$$C_1 = 0$$

$$\Theta(\theta) = C_2$$

Case III: When β > 0, then

$$\Theta(\theta) = C_3\sin(\beta\theta) + C_4\cos(\beta\theta)$$

Using periodic conditions

$$\Theta(\pi) = \Theta(-\pi)$$

$$C_3\sin(\beta\pi) + C_4\cos(\beta\pi) = -C_3\sin(\beta\pi) + C_4\cos(\beta\pi)$$

Thus

$$C_3\sin(\beta\pi) = 0$$

$$\Theta'(\pi) = \Theta'(-\pi)$$

$$\beta C_4\sin(\beta\pi) + \beta C_3\cos(\beta\pi) = -\beta C_4\sin(\beta\pi) + \beta C_3\cos(\beta\pi)$$

Thus

$$C_4\sin(\beta\pi) = 0$$

For avoiding trivial solution, $C_4 \neq 0, \; C_3 \neq 0$, then

$$\sin(\beta\pi) = \sin(\mathrm{n}\pi)$$

Thus,

$$\boldsymbol{\beta = \mathrm{n}}$$

So,

$$\Theta_{\mathrm{n}}(\theta) = C_3\sin(\mathrm{n}\theta) + C_4\cos(\mathrm{n}\theta) \qquad \textbf{(4)}$$

Due vibrational nature of the problem, let $\lambda = \alpha^2$, then for equ.2:

$$r^2R''(r) + rR'(r) + (r^2\alpha^2 - n^2)R(r) = 0$$

This is Bessel's equation of order n, then

$$R(r) = C_5 J_n(\alpha r) + C_6 Y_n(\alpha r)$$

As Y_n is unbounded when r = 0, therefore C_6 must be zero to maintain the boundness of the system. Thus

$$R(r) = C_5 J_n(\alpha r)$$

From boundary conditions

$$R(c) = 0$$

Then

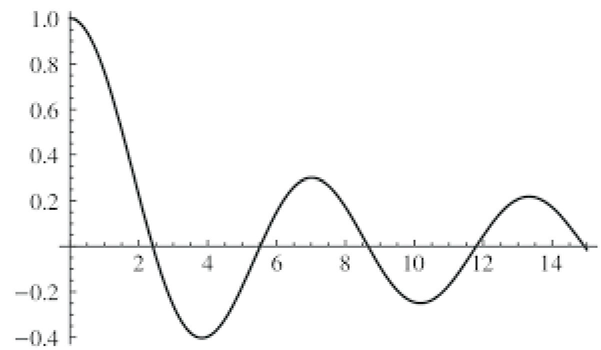
$$C_5 J_n(\alpha c) = 0$$

To avoid trivial solution, $C_5 \neq 0$. Then

$$J_n(\alpha c) = 0$$

Thus

$$\alpha c = x_m, so\ \boldsymbol{\alpha} = \frac{\mathbf{x_m}}{\mathbf{c}}$$



Therefore

$$R_n(r) = C_5 J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}} r\right),\ n = 1, 2, 3, \dots \dots \quad \mathbf{(5)}$$

For equ.1

$$T''(t) + \alpha^2 a^2 T(t) = 0$$

Thus

$$T_n(t) = C_7 \cos(\alpha_n a t) + C_8 \sin(\alpha_n a t) \quad \mathbf{(6)}$$

So, from equations **4, 5, 6**, and as $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, then

$$u(r, \theta, t) = \sum_{n=1}^{\infty} [[C_3 \sin(n\theta) + C_4 \cos(n\theta)]\ C_5 J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}} r\right) C_7 \cos(\alpha_n a t)] + \sum_{n=1}^{\infty} [[C_3 \sin(n\theta) + C_4 \cos(n\theta)]\ C_5 J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}} r\right) C_8 \sin(\alpha_n a t)]$$

Let $C_3 C_5 C_7 = \boldsymbol{A_n}, C_4 C_5 C_7 = \boldsymbol{B_n}, C_3 C_5 C_8 = \boldsymbol{C_n}, C_4 C_5 C_8 = \boldsymbol{D_n}$

Thus

$$u(r,\theta,t) = \sum_{n=1}^{\infty}[[A_n\sin(n\theta) + B_n\cos(n\theta)]\, J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathrm{r}\right)\cos(\alpha_n at)] + \sum_{n=1}^{\infty}[[C_n\sin(n\theta) + D_n\cos(n\theta)]\, J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathrm{r}\right)\sin(\alpha_n at)]$$

From Initial conditions:

$u(r,\theta,0) = f(r,\theta), \qquad 0 < r < c, \qquad 0 < \theta < 2\pi$

Then

$$f(r,\theta) = \sum_{n=1}^{\infty}[A_n\sin(n\theta) + B_n\cos(n\theta)]\, J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathrm{r}\right)$$

As

$$\left\| J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathrm{r}\right)\right\|^2 = \frac{c^2}{2}J^2{}_{n+1}(\mathbf{x_n})$$

Then

$$\boldsymbol{A_n} = [\frac{\mathbf{2}}{\boldsymbol{c^2J^2}_{n+1}(\mathbf{x_n})}\int_{\mathbf{0}}^{\boldsymbol{2\pi}}\int_{\mathbf{0}}^{\boldsymbol{c}}\boldsymbol{rf(r,\theta)J_n}\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathbf{r}\right)\boldsymbol{drd\theta}]\frac{\mathbf{1}}{\mathbf{sin(n\boldsymbol{\theta})}}$$

$$\boldsymbol{B_n} = [\frac{\mathbf{2}}{\boldsymbol{c^2J^2}_{n+1}(\mathbf{x_n})}\int_{\mathbf{0}}^{\boldsymbol{2\pi}}\int_{\mathbf{0}}^{\boldsymbol{c}}\boldsymbol{rf(r,\theta)J_n}\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathbf{r}\right)\boldsymbol{drd\theta}]\frac{\mathbf{1}}{\mathbf{cos(n\boldsymbol{\theta})}}$$

$\dfrac{\partial u(r,\theta,0)}{\partial t} = g(r,\theta), \qquad 0 < r < c, \qquad 0 < \theta < 2\pi$

Then

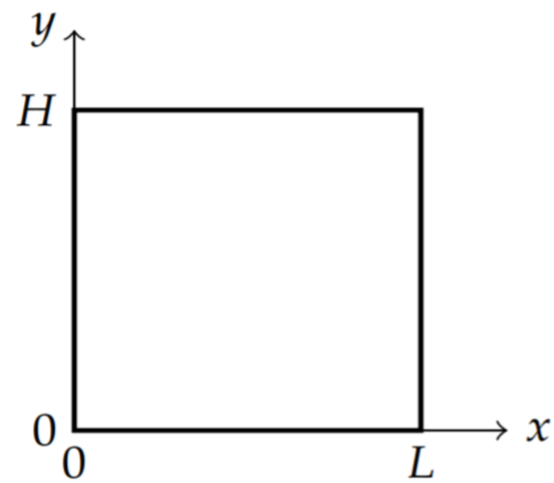
$$g(r,\theta) = \sum_{n=1}^{\infty}\alpha_n a[C_n\sin(n\theta) + D_n\cos(n\theta)]\, J_n\left(\frac{\mathbf{x_n}}{\mathbf{c}}\mathrm{r}\right)$$

Thus

$$C_n = \left[\frac{2}{c^2 J^2{}_{n+1}(x_n)} \int_0^{2\pi}\int_0^{c} r g(r,\theta) J_n\left(\frac{x_n}{c} r\right) dr d\theta\right] \frac{1}{\alpha_n a \sin(n\theta)}$$

$$D_n = \left[\frac{2}{c^2 J^2{}_{n+1}(x_n)} \int_0^{2\pi}\int_0^{c} r g(r,\theta) J_n\left(\frac{x_n}{c} r\right) dr d\theta\right] \frac{1}{\alpha_n a \cos(n\theta)}$$

4.2 Rectangular Membrane



The rectangular membrane of length LL and width HH. There are fixed boundary conditions along the edges.

The PDE is given in the following form:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, \quad 0 < x < L, \quad 0 < y < H$$

a set of boundary conditions

$$u(0, y, t) = 0, u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H$$

$$u(x, 0, t) = 0, u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L$$

and a pair of initial conditions (since the equation is second order in time)

$$u(x, y, 0) = f(x, y), u_t(x, y, 0) = g(x, y), \quad 0 < x < L, \quad 0 < y < H$$

The first step is to separate the variables:

$$u(x, y, t) = X(x)Y(y)T(t)$$

Then resub in the DE to get:

$$X(x)Y(y)T''(t) = c^2[X''(x)Y(y)T(t) + X(x)Y''(y)T(t)]$$

Dividing both by $X(x)Y(y)T(t)$ *and* c^2

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}+\frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda$$

Thus, the function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we choose the constant λ to be positive, λ>0 After that we generate these two equations:

$$T'' + c^2\lambda T = 0$$
$$\frac{X''}{X}+\frac{Y''}{Y} = -\lambda$$

The first equation is easily solved. We have

$$T(t) = a\cos(\omega t) + b\sin(\omega t) \text{ where } \omega = c\sqrt{\lambda}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\mathrm{v} = \frac{\omega}{2\pi} = \frac{\mathrm{c}}{2\pi} * \sqrt{\lambda}\ eq.1$$

Next, we solve the spatial equation. We need carry out another separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y}-\lambda}_{\text{Function of } y} = -\mu.$$

By the same manner as we evaluated T(t) we can induce the solution of X and Y and apply our boundary conditions to evaluate our constants.

$$X'' + \mu \mathrm{X} = 0$$
$$\mathrm{Y}'' + (\lambda-\mu)Y = 0$$

Then, by having Dirichlet boundary condition

$$X_n(x) = \sin\frac{n\pi x}{L}, \mu_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3, \ldots$$
$$Y_m(y) = \sin\frac{m\pi x}{H}, \lambda - \mu_m = \left(\frac{m\pi}{H}\right)^2, m = 1,2,3, \ldots$$

At this point we need to be careful about the indexing of the separation constants. So far, we have seen that μ depends on n and that the quantity κ=λ−μ depends on m . Solving for λ, we should write $\lambda_{nm} = \mu_n + \kappa_m$

Getting the value of λ which equals k+ μ

$$\lambda_{nm} = \left(\frac{n\pi}{T_1}\right)^2 + \left(\frac{m\pi}{H}\right)^2, n, m = 1,2, \ldots$$

Previously at equation (1) so by substitute for λ

$$v_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2}, n, m = 1,2, \ldots$$

The solution simply can be summarized in the following equation:

$$u_{nm} = (a\cos\ \omega_{nm}t + b\sin\ \omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$$

and the most general solution is written as a linear combination of the product solutions,

$$u(x, y, t) = \sum_{n,m}(a_{nm}\cos\ \omega_{nm}t + b_{nm}\sin\ \omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

For completeness, we now return to the general solution and apply the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}$$

The first initial condition is $u(x,y,0) = f(x,y)$ by Setting t=0 in the general solution, we obtain

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}.

The coefficients a_{nm} can be found knowing what we already know about Fourier sine series. We can write the initial condition as the single sum

$$f(x,y) = \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{L},$$

$$where\ , A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin \frac{m\pi y}{H}.$$

These are two Fourier sine series. the coefficients of Fourier sine series can be computed as integrals, we have

$$A_n(y) = \frac{2}{L} \int_0^L f(x,y) \sin \frac{n\pi x}{L} dx,$$

$$a_{nm} = \frac{2}{H} \int_0^H A_n(y) \sin \frac{m\pi y}{H} dy.$$

Inserting the integral for $A_n(y)$ into that for a_{nm}. we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} dxdy.$$

Again, we have a double Fourier sine series. But now we can quickly determine the Fourier coefficients using the above expression for a_{nm} to find that

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} dxdy$$

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

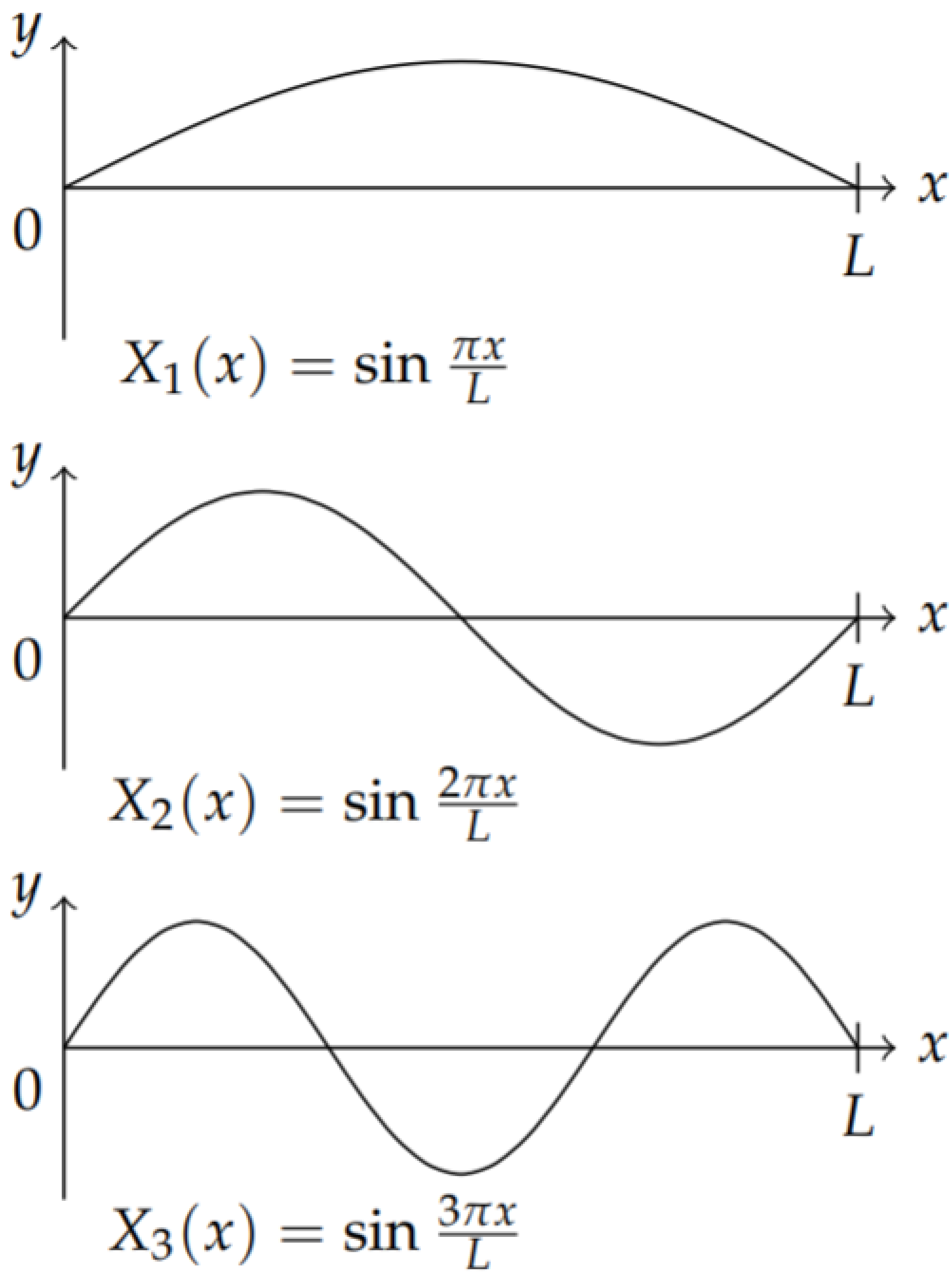
$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos\, \omega_{nm} t + b_{nm} \sin\, \omega_{nm} t) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

$$where, \qquad a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} dxdy$$

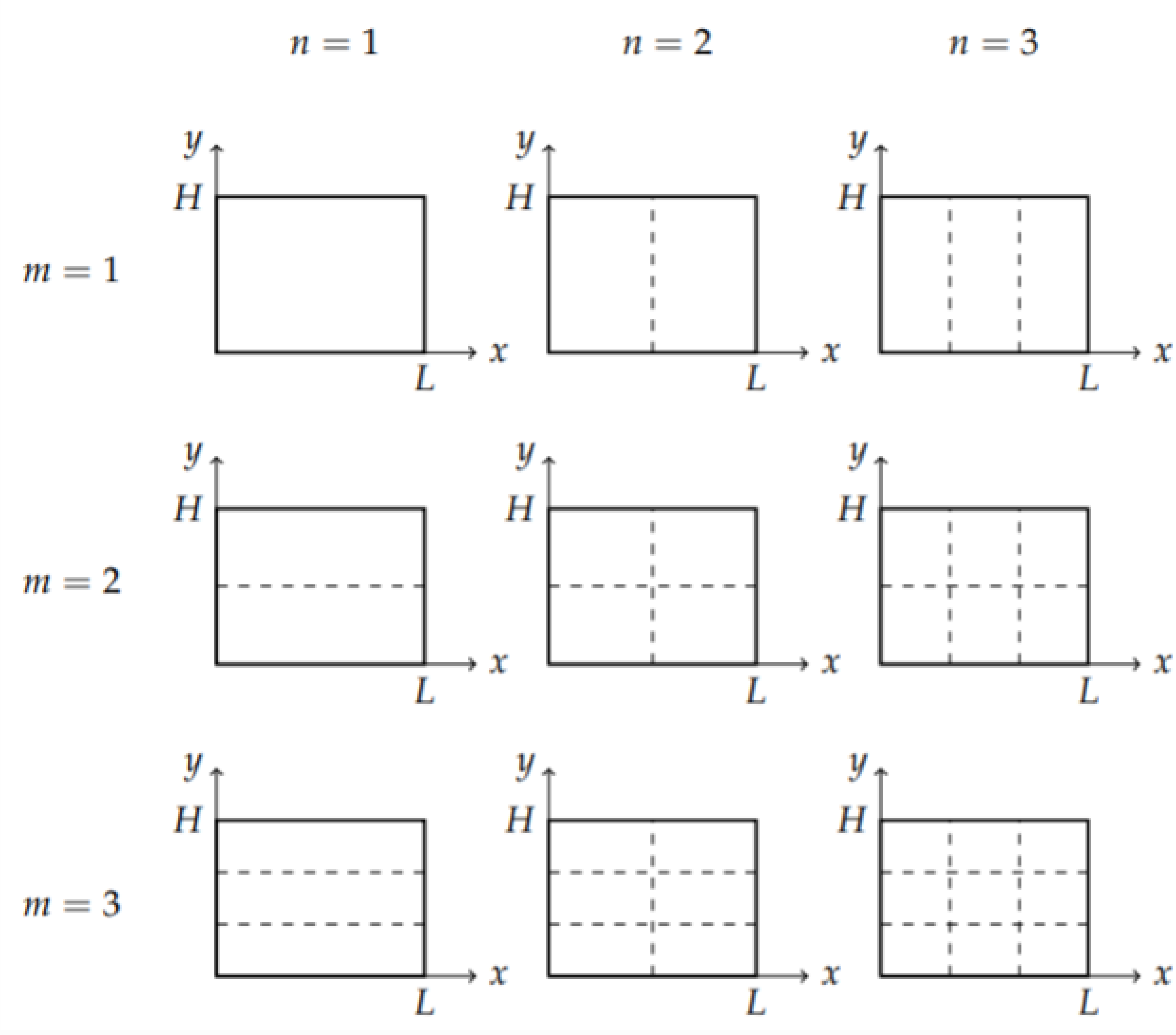
$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} dxdy$$

and the angular frequencies are given by

$$\omega_{nm} = c_n \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



The actual shapes of the harmonics are sketched by locating the nodes, or places on the string that do not move.



Instead of nodes, we will look for the nodal curves, or nodal lines. These are the points (x,y) at which $\phi_{nm}(x,y)=0$. Of course, these depend on the indices, n and m.

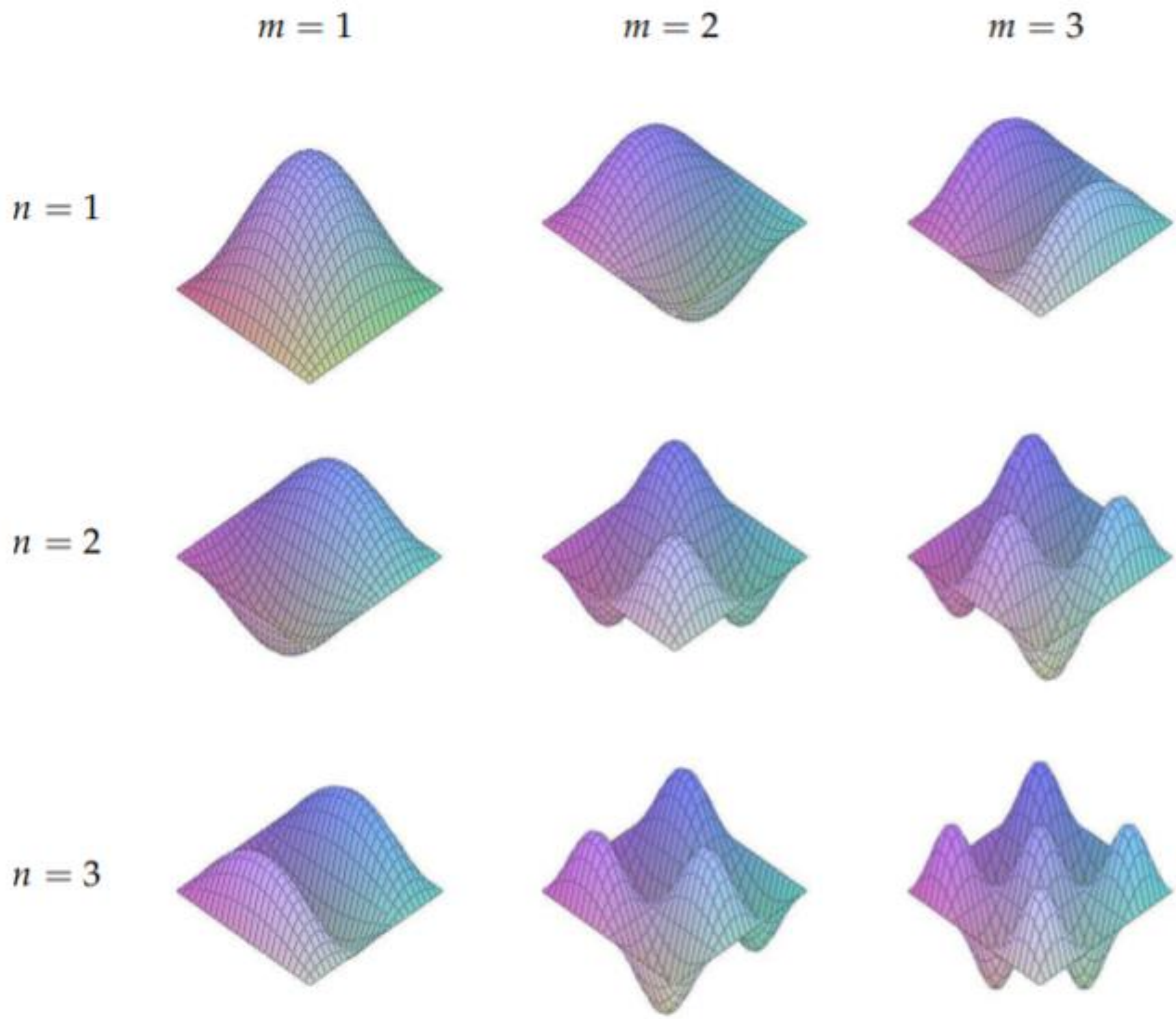
For example, when n=1 and m=1, we have

$$\sin\frac{\pi x}{L}\sin\frac{\pi y}{H} = 0$$

These are zero when either

$$\sin\frac{\pi x}{L} = 0, \text{ or } \sin\frac{\pi y}{H} = 0$$

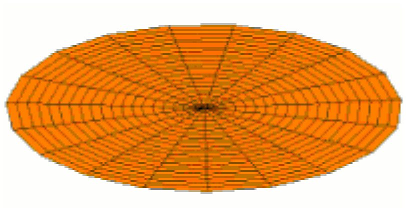
Of course, this can only happen for x=0, L, and y=0, H. Thus, there are no interior nodal lines.

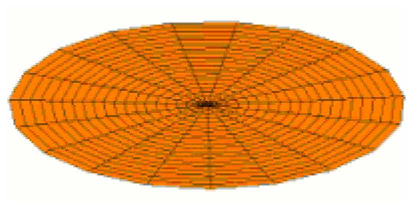


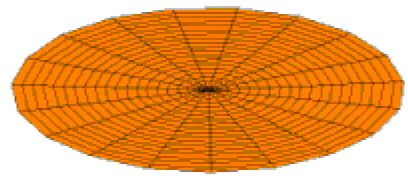
A three-dimensional view of the vibrating rectangular membrane for the lowest modes

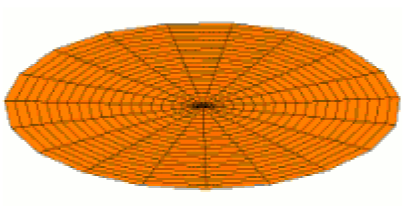
5. Results from the Simulator (Featool_Multiphysics)

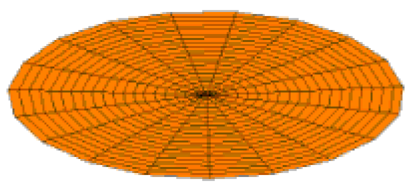
5.1 For Circular Membrane

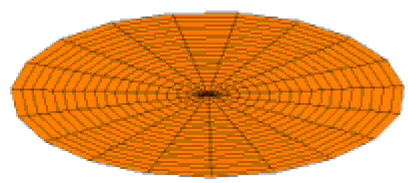

Mode u_{01}

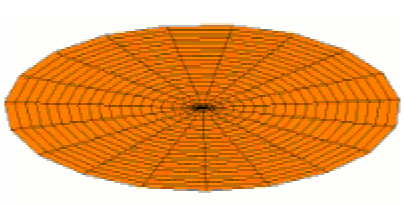

Mode u_{02}

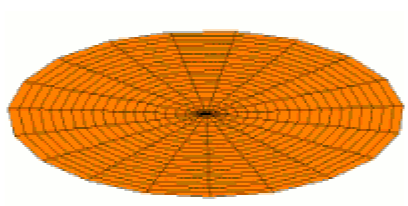

Mode u_{03}

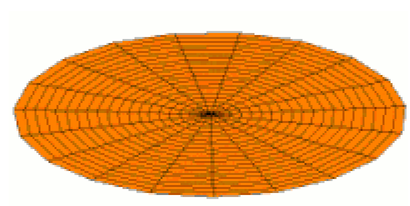

Mode u_{11}

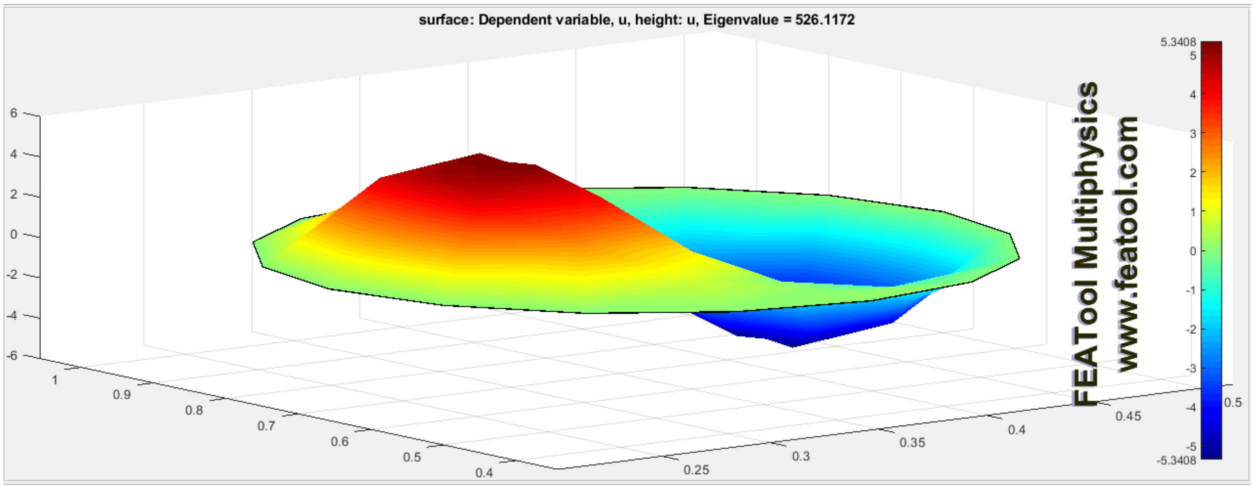

Mode u_{12}


Mode u_{13}


Mode u_{21}

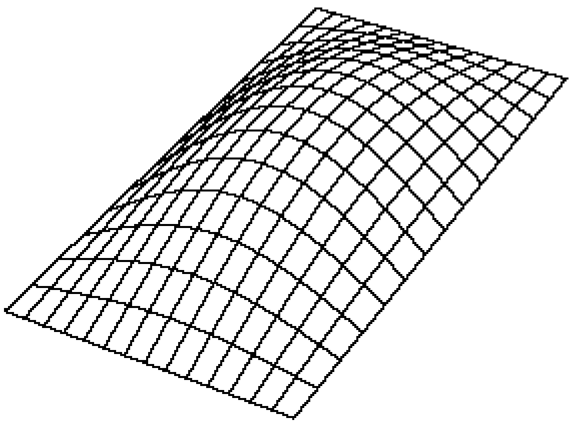

Mode u_{22}


Mode u_{23}

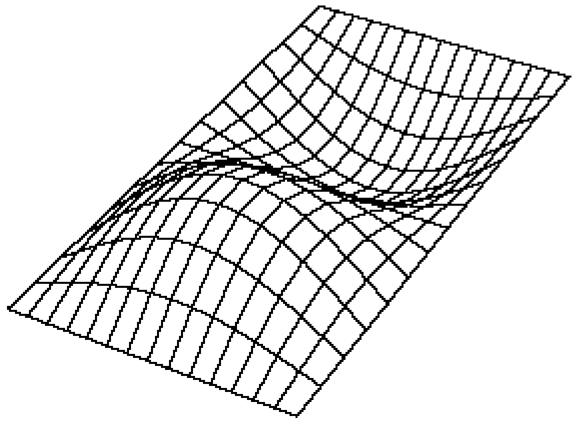


5.2 For Rectangular Membrane

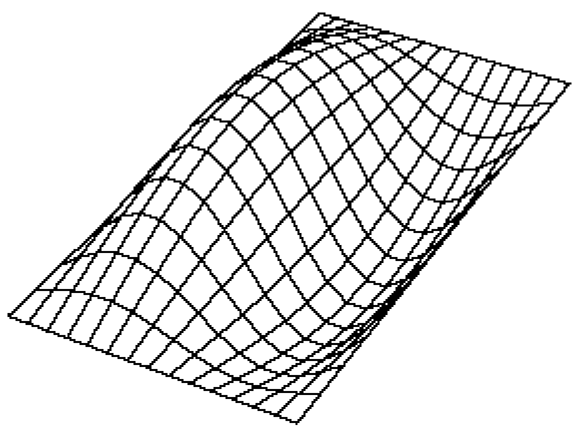
The (n_x=1, n_y=1) Solution



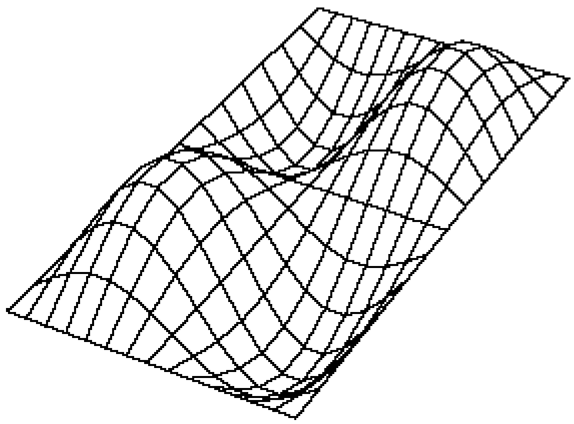
The (n_x=1, n_y=2) Solution

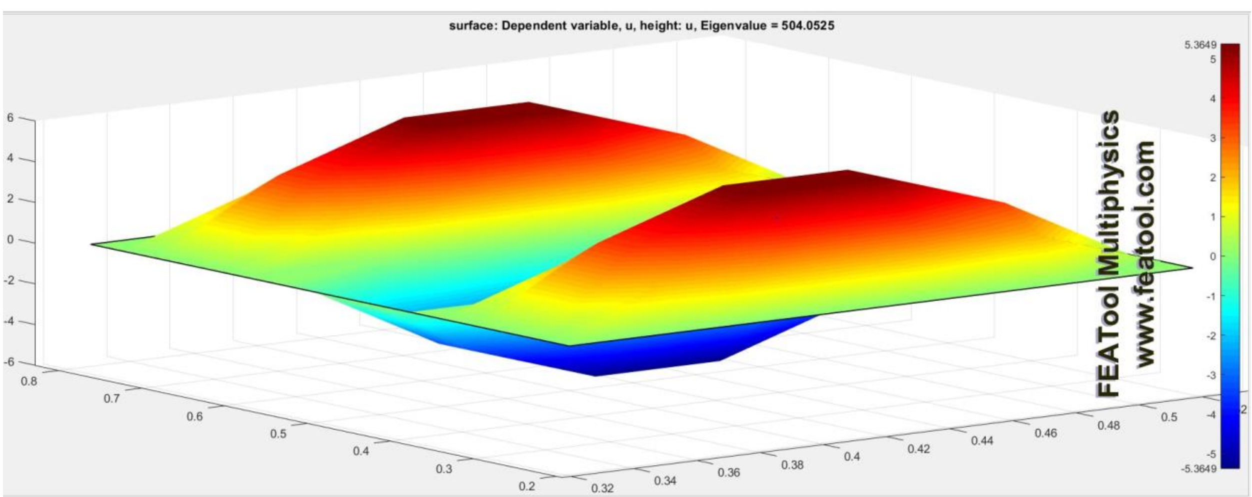


The (n_x=2, n_y=1) Solution



The (n_x=2, n_y=2) Solution





6. Conclusion

Vibrations of membranes play critical role in engineering real-life application especially in wave guides for communications field and biological application in the working of the eardrum or wave guides used in surgeries.

The solution of these membranes depends mainly on the shape of these membranes which will control which dimensions the solution will walk through, and boundary conditions which will change the eigenfunctions and values.

As illustrated from the results how the eigen values control the motion of the wave and how it behave, so by constructing the mathematical model of certain application and apply our initial conditions, we can predict the solutions and outcomes of the project and maximize the revenue as possible as we can.

7. References

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