# Opening Quote

Hi everyone,

Before we start, I’d like to leave you with this quote for just a moment.

# Intro

Hey everyone.

Today I want to talk to you about a topic I’m very passionate about: Geometric Algebra.

I’ll be giving you an introduction through the lens of quantum mechanics.

In doing so, I’d like us to explore foundational concepts of vector spaces that you typically gloss over in any quantum course, and instead offer you a geometric interpretation… using a language that does not require neither matrices nor tensors—a language that is much simpler and way more elegant.

# Bivectors

So, to start, I think everyone here is already familiar with scalars and vectors.

In geometric algebra, we can construct higher-dimensional, or, higher-grade objects, for example bivectors.

To do that, we use the outer product, often called the wedge product.

The bivector u wedge v is simply the oriented area given by the parallelogram formed by u and v.

The orientation depends on which vector you start with. So for u wedge v, you curl your fingers from u towards v. And you can easily see that this outer product operation is anti-symmetric. Meaning, u wedge v = - v wedge u.

# Tri-vectors

And we can extend this idea to even higher dimensions. Say we are in 3D space, with basis vectors e\_1, e\_2, e\_3, then we can construct the unit cubic volume by wedging all three.

I’m intentionally omitting the orientation for simplicity.

When we wedge all basis vectors together like this, we call the resulting unit volume the pseudoscalar. And it will come up time and time again in our discussions.

And you can imagine extending this notion even further, constructing multi-vectors, or, hyper volumes using the outer product.

So now, whenever you see the outer product, you clearly know what it means.

# Geometric Product

Now, at the core of geometric algebra is a multiplication operation, called the geometric product.

And it might look intimidating, adding scalars (using the inner product) to bivectors. Two different grade objects.

But this is no different from complex numbers, which we don’t often question. Say we have z\_1 and z\_2, we simply add them by adding their real and imaginary components, respectively.

Often times though, this geometric product simplifies. For example, if we are working in 2D, and we take the geometric product of the orthonormal basis vectors e\_1 and e\_2, then we know that their inner product vanishes. In this case, the geometric product is simply the outer product of the two.

If however, we bring e\_2 towards e\_1, the area between them vanishes. And the geometric product, e\_1 e\_1, reduces to the inner product, which we know is 1, because these vectors are normalized.

Now, this generalizes to any orthonormal basis vectors i and j, where i is not equal to j.

This simplification will be our bread and butter in geometric algebra, and we will employ it quite frequently in what is to come.

# The Cross Product

Now, in order to see geometric algebra in action, I’d like to anchor us to something familiar: the cross product. For vectors A and B, the product A cross B is a vector which lies in the plane perpendicular to A and B.

Perhaps, not so famous, is this definition of the cross product. Where I employ the Levi-Civita tensor, epsilon\_ijk. I’m also using summation notion, where you would expand A and B in terms of our basis set as follows.

If we replace A and B with any two basis vectors, we get this.

Clearly, this Levi-Civita tensor is quite involved. And while it will often make an appearance, my goal today is not to explain exactly but that object represents, but rather to highlight how it connects to everything else. My goal is to only spark you interest in the topic by giving you the necessary ingredients—but you have to cook!

# The Levi-Civita

Usually, Levi-Civita is defined in terms of cyclic permutations, so this might be familiar to you.

For example, epsilon\_123 would be one. But if you switch the 3 around you pick up a minus sign.

If you switch it around again, you pick up another minus sign, so you get 1 again.

And if you repeat any two indices, you get zero.

# Levi-Civita: Geometric Algebra

For now, I’d like you to see how we define this tensor in GA. And as we move long, a geometric interpretation might makes itself apparent to you.

For now though, let’s explain the terms in this formula and check if it works.

First, we are taking the outer product of e\_i with e\_j with e\_k, then we are multiplying by the reverse of the pseudoscalar, denoted by the dagger symbol. So instead of e\_1 wedge e\_2 wedge e\_3, we have e\_3, e\_2, e\_1… and the outer product here is simply the geometric product because all of these are orthogonal.

Let’s see if this works for the simplest case when we have two repeating indices, so, say epsilon\_112. As we discussed before, there’s no area between e\_1 and itself, so the wedge product vanishes, and epsilon\_112 goes to zero.

For the case of cyclic permutations, say epsilon\_123, first we write out the reverse pseudoscalar, then all these wedge products become geometric products. Now, you see the e\_3 e\_3 in the middle, well that becomes 1 since its just the inner product between the two. And you can see the pattern for the e\_2 e\_2 and then the e\_1 e\_1. So epsilon\_123 = 1.

Now, for anti-cyclic permutations, say epsilon\_132, again we expand the pseudoscalar and drop all these wedges. But, now I must reorder this e\_2 e\_3 in the middle for things to cancel out. But if I do, I pick up a minus sign, because the wedge product of two vectors is anti-symmetric. And now we can cancel out same as before. And we end up with a -1.

So, you see, we retrieve the old definition using the geometric algebra definition.

# The Outer Product

Now that we have a handle on the algebra, we are almost in a position to revisit the cross product. But first, I’d like to make a stop at the outer product to showcase the simplicity of GA.

We can first rewrite A and B in terms of the basis vectors. Then simply expand. From here, all we need to do is distribute.

Now, any basis vector wedged with itself goes to zero. That takes care of all these diagonal terms.

And I’d like to collect terms with opposite wedge products, so I will shuffle things around.

Now, let’s bring the scalars up to the front. And flip these outer products at the cost of a minus sign.

Bringing all these scalars together, we get this. These coefficients look familiar?

Focusing on this e\_1 wedge e\_2 term for a moment, that’s just the geometric product of the two. Now I can multiply by e\_3 e\_3, cause that’s just 1. And I can shuffle it twice, picking up two minus signs.

Now, I multiply by epsilon\_123, which is just 1. And remembering the formula we had for the cross product, the epsilon\_123 e\_3 becomes e\_1 cross e\_2. And this e1 e2 e3, is just our pseudoscalar, I. And Im gonna use this relation to rewrite all the wedge products up there.

Now, we know how to take the cross product between any of our basis vectors , so we can write those vectors instead.

Notice, all those coefficients, along with this minus sign over here… This is just the cross product! With a factor of I to the side.

# Cross Product Revisited

Now, going back to the cross product, we know that A cross B lies in the plane perpendicular to both A and B. But why not this vector?

Well, it can’t be, because, for A wedge B, we need to start curling our hands from A towards B, so our thumb points up.

As for the magnitude, well that’s given by the area of the parallelogram, A wedge B, which also happens to be A B sin(theta), which we learn early on.

So it is really the outer product that is fundamental, and you can see that because the cross product exists only in 3D.

One thing we haven’t talked about though is the pseudoscalar. It seems to relate the plane A wedge B to an orthogonal vector. You might have many questions. So do I. But let’s tackle a simple one: what is the action of this pseudoscalar in other dimensions, say 2D?

# Duality

So let’s consider e\_1. What is perpendicular to e\_1?

Well, we know its e\_2. Now, Im gonna multiply by e\_1 e\_1, which is 1 on the right here. Then im going to switch here, getting a minus sign. And then switch again. But this term is just the pseudoscalar in 2 dimensions. Hrmm….

What about e\_2? Well, let’s copy this. Then multiply by the pseudoscalar on both sides. Now here we have I^squared. This I squared is just e\_1 e\_2 multiplied by itself. So again, we do a switch here and pick up a minus sign. The e\_1 e\_1 go away. So do the e\_2 e\_2. And woah, we have I squared is -1.

PAUSE… looks familiar? (10 sec)

Moving on… So up there we have a minus 1 now. Let’s expand the pseudoscalar. Do a switch here on the left, picking up a minus sign. Now, this term is the reverse of the pseudoscalar. The minus sign goes away on both sides, and we get this.

Very interesting this pseudoscalar…

Now, what about in higher dimensions? To elucidate why this must generalize, let’s actually attempt to construct this orthogonal vector ourselves. So say, we want a basis vector e^j that is orthogonal to all other basis vectors in our algebra. In other words… (POINT TO EQUATION) this.

The pseudoscalar of our algebra can now have n terms for n-dimensions.

What we can do is take the outer product of all the elements in our algebra then multiply by the reverse of the pseudoscalar. But to not let this all reduce to a scalar (1), we exclude the term e\_j from our product.

This looks awfully reminiscent of the Levi-Civita definition. Except we are missing a term. To compensate for that, we need a sign factor. Here, its job is to preserve the handedness of our set. But ignoring that for now, let’s see if this works.

Let’s go to 3 dimensions as an example, and look for e^2. The sign term here becomes -1.

From our formula we must wedge e\_1 with e\_3 and then multiply with the reversed pseudoscalar—with a sign factor. We expect to get something orthogonal to both e\_1 and e\_3. Either this vector, or that vector.

Expanding the pseudoscalar and dropping the wedge, we get this. The e\_3 e\_3 go away. We do a switch here, and the e\_1 e\_1 goes away. And we get e\_2.

So, you might think this is redundant. We were looking for a vector that like e\_2 (WE SAY, THE DUAL TO e\_2) in the sense that its orthogonal to e\_1 and e\_3, but we ended up right back at e\_2.

So you might think that the dual vector e\_i = e^i. But that’s not always the case.

# Spacetime Geometric Algebra

To illustrate this, let us consider a different algebra: the 4-dimensional Spacetime Geometric Algebra, given by the familiar Minkowski metric. Here, I’ll be using the mostly minus version.

For this algebra, we have one timelike and 3 spacelike basis vectors that form an orthonormal set. The orthogonality can be written as \_\_\_.

Now here, the inner product is defined according to the metric. So for our signature, taking the spacelike gamma\_2 inner product with itself we get -1.

This is important because we previously defined duality using this relation. So the inner product of gamma\_2 with its dual has to be 1. But in order for both of these to be true, we can’t simply say that gamma\_2 is equal to its dual. The only way to satisfy both relations is if the dual of gamma\_2 is equal to negative gamma\_2.

But this is not the case for all the basis vectors. For the timelike gamma\_0, the inner product is 1. And you can use a similar line of reasoning to see that gamma\_0 is equal to its dual.

So its really the metric that defines the inner product, and in turn the relations between vectors and their duals. In the language of tensors, we say that “the metric tensor is used to raise and lower the indices”.

# The Even Subalgebra

Now, I’d like to explore another layer of complexity for the spacetime geometric algebra. Given our basis vectors, we can only form grade-2 elements, bivectors, using either of these combinations. Here, i denotes Im using a spacelike vector with index 1, 2, or 3.

And I’d like to focus on this set of bivectors, sigma\_i.

From this set we can form scalars by squaring sigma\_i. That is just gamma\_i gamma\_0 multiplied by itself. If I do a switch, I pick up a minus sign. But the spacelike gamma\_i square to -1. So we just end up with 1.

The pseudoscalar for this algebra is then sigma\_1 sigma\_2 sigma\_3. And the levi-civita can be written as sigma\_i wedge sigma\_j wedge sigma\_k multiplied by the pseudoscalar reversed. These wedges, again, go away. And I’m gonna multiply by the pseudoscalar on the right.

This, I\_dagger I, when we expand it, we can see that, using the property above, the sigma\_1 sigma\_1 go away. And so do the sigma\_2’s and the sigma\_3’s. So I\_dagger I is just 1.

Using this result, this, up here, goes away. And I’m gonna multiply by sigma\_k on the right. Sigma\_k sigma\_k goes away, and we have this.

Because of the anti-symmetry of the levi-civita under the switch of any two indices, we have that sigma\_i sigma\_j = - sigma\_j sigma\_i.

So I’m gonna add two sigma\_i sigma\_j’s together. And then flip the order of these two, picking up a minus sign. But we all know what this is. That’s the commutator. Substituting our result from the right over there, we get this!

Woah… (PAUSE 20 SEC, ASK CLASS)

I have not made any mention of physics or quantum mechanics, yet these are precisely the commutator relations defining the Pauli algebra. Does anyone have an idea how these got here?

# Rotations

Moving on, I’d like to explore what we can do with these sigma bivectors by making a correspondence with quantum mechanics.

Perhaps familiar to everyone is this equation reminiscent of Euler’s formula, but using the sigma\_z operator.

I’d like to convince you that the geometric algebra analog is also true. To prove that, we could go through the math, but, instead, I’d like to remind you that our sigmas square to 1.

Also, and this is something I swept under the rug, our pseudoscalar, I, squares to -1, acting exactly like the imaginary unit! And that all we need to say that, if this is true, then this too must be true.

Now, you’ve probably also seen this come into play in a sandwich product of this form. This simplifies to a just a cos theta and a sin theta.

And again, using the same argument, the GA equivalent also holds.

If you want to see exactly how the math simplifies, I’ll have the math for the GA version in my notes, since most all of you are probably familiar with the former.

Now, in GA, we call this a rotor, because, as you can see, it takes sigma\_1 and moves it along the unit circle in the sigma\_1 sigma\_2 plane. Infact, this I sigma\_3 term can be expanded. And the sigma\_3 sigma\_3 go away, leaving us with the sigma\_1 sigma\_2 plane—the plane of rotation!

We can see this in action by substituting for theta. For example when theta = pi/4 or pi/2. In fact, this sandwich product enables us to move anywhere along the unit sphere.

Now, this should ring a bell.

Perhaps your mind goes to the familiar Bloch sphere. Or, in the context of quantum optics, maybe the Poincare sphere describing polarization. But this sphere should really be attributed to Riemann, who introduced complex geometry to the world.

In quantum mechanics, a vector inscribed in this sphere is used to represent a two-state system, the qubit, with two complex numbers, v and w. But because these are complex numbers, shouldn’t we be using two spheres?

The key lies in global phase invariance. We say that scaling our state with a complex number lambda, leaves it unchanged. So we can scale our state by one over v to get this. Where u is w over v. But then a problem occurs when v, here, goes to zero. And we say that “its all fine and good” if we accept that when v goes to zero, u goes to infinity.

But what does that mean? And what does that say about the vector space inhabited by our quantum state?

# Projective Geometry

Before we dive into these spaces, let me illustrate that this idea is not so bizarre.

Take the function x^2 + 1 for example. And imagine standing at the origin, looking at it straight ahead. From your perspective, It would seem that very far away, the parabola is looking more like an ellipse! As y goes to this “point at infinity”, x goes to zero.

This more so resembles our world because of how we see. But what about our quantum state? What vector space does it inhabit?

# Real Projective Line

To explain, I need to introduce you to the real projective line.

Imagine an artist sitting at the origin with his easel sitting at y=1.

To project objects in space onto his painting, he simply imagines lines connecting these objects to himself, then finds where these lines intersect his painting.

Now instead of the artist imagine a special camera. In addition to projecting objects beyond the screen, this camera can also project objects between itself and the screen. It can also look behind and project objects from behind, onto the screen.

Mathematically, to project any point we simply need to scale both x and y by some factor, lambda. For the point at (4,2) this is simply one half. For the point (-4, -2), this would be minus a half.

And while this factor changes, notice that the ratio between x and y does not for any point sitting on the same line. That’s because these lines we drew pass through the origin, having zero y-intercept. So only the slope, m, is needed. And scaling both y and x by some lambda, does not change the slope.

This is analogous to our situation with the quantum state. But instead of u, we have m.

Now, I’m gonna raise the screen to illustrate my second point. And have this blue dot here to track the projection of a point on a line.

As we sweep across, the projection point moves to the left. But notice what happens when we get too far. As we approach the horizontal axis, the projection point goes very very far away, but as we exceed it, the projection point returns from “the other side”.

So even though the lines don’t intersect at y=0, the projection point wobbles back and forth from the “point of no intersection”, which corresponds with the “point at infinity”. Now we have a geometric interpretation of this point at infinity.

One last thing I’d like to show here. Take a look at this, we do a quarter rotation… followed by another quarter rotation. That’s a sweep up there. Another half rotation, and we get another sweep up here. Let’s pause for a second, anybody have an idea why this is happening? (PAUSE 30)

Before we finish, remember, we have one complex ratio, U, defining our state, so we need two real ratios.

To solve this last issue, we can extend the real projective line by, going up a dimension, to the Real Projective plane. Now instead of this circle we have many of these circles, each corresponding to a screen and a parallel line through the origin. Collectively, we have a sphere and two planes.

Now, in this space, again, only the ratio matters. So we can scale our points arbitrarily, say with 1/y, and then only the ratios x/y and z/y are needed to specify planes in our projective space.

And our complex ratio, u, would correspond to two real ratios.

So there’s a correspondence between the complex projective line and the real projective plane. And the correspondence is that of a sphere—the Riemann sphere!

Thank you…