1 Introduction

ADD 5 REFERENCES!!!

This lecture is intended to introduce Geometric Algebra (GA). My goal is to show that the complementarity of the spin $\frac{1}{2}$ system arises not from something physical, but rather the even subalgebra, hidden within the GA of spacetime. This should be sufficient to entice the reader to ask the question:

”What is the connection between spin-1/2 systems and geometry?”

In approaching this goal, I shall dive deeper into concepts commonly overlooked in any quantum mechanics course. These include:

1. the cross product, outer product, and their relationship to duality,
2. the Levi-Cevita tensor (only to hint at more fundamental relationships),
3. the inner product and its relationship to the metric tensor,
4. the even subalgebra of spacetime and its relationship to Pauli matrices–there out called Pauli vectors,
5. and finally, deriving the commutator relationships of the even subalgebra, giving a brief outlook on its relationship to spin-1/2 systems.

2 The Core of Geometric Algebra

2.1 Vectors, Bivectors, and Multivectors

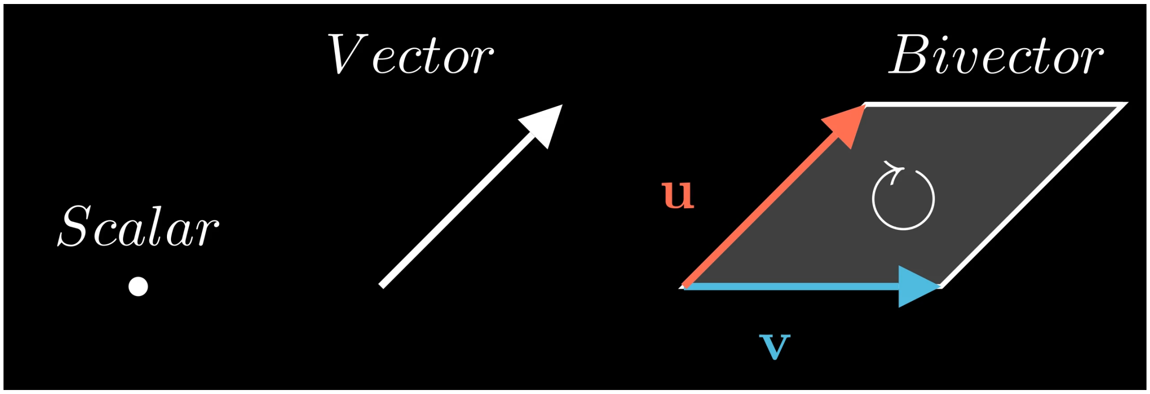

Figure 1: The scalar, vector, and bivector formed by the parallelogram of two vectors. The orientation of the area is given by curling your hand from the first vector to the second.

I assume here that the reader is familiar with scalars and vectors. In GA, we can build higher-dimensional (called higher-grade) objects, such as bivectors. A bivector is the directed area, given by the parallelogram formed by two vectors. The grade 0 scalar, grade 1 vector, and grade 2 bivector are shown in Figure 1. The mathematical operation that forms bivectors from vectors is called the outer product–often referred to as the wedge product. This operation is anti-symmetric so that

$$\boxed{\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}}, \tag{1}$$

for any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. This anti-symmetry is important and I shall be employing it frequently in the future.

As an aside, one could construct even higher grade objects, oriented volumes or hypervolumes, called trivectors and multivectors, respectively. However, I shall not need more than to show, without rigour, how these objects come about, namely via successive applications of the outer product. For instance, taking the outer product of the orthonormal basis $\{\boldsymbol{e}_i\} \vee i \in \{1, 2, 3\}$, corresponding to $\{\hat{i}, \hat{j}, \hat{k}\}$:

$$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = \mathbb{I} \tag{2}$$

yields the cube of unit volume in 3D space shown in Figure 2, where $\mathbb{I}$ is called the **pseudoscalar**–a very important quantity we shall encounter time and again in our discussions of GA.

Now you have a geometric idea of the outer product–no more a mysterious or purely algebraic operation. The outer product is a grade-raising operation, as we saw in the examples for bivectors and trivectors. For multivectors, you can imagine more of the same.

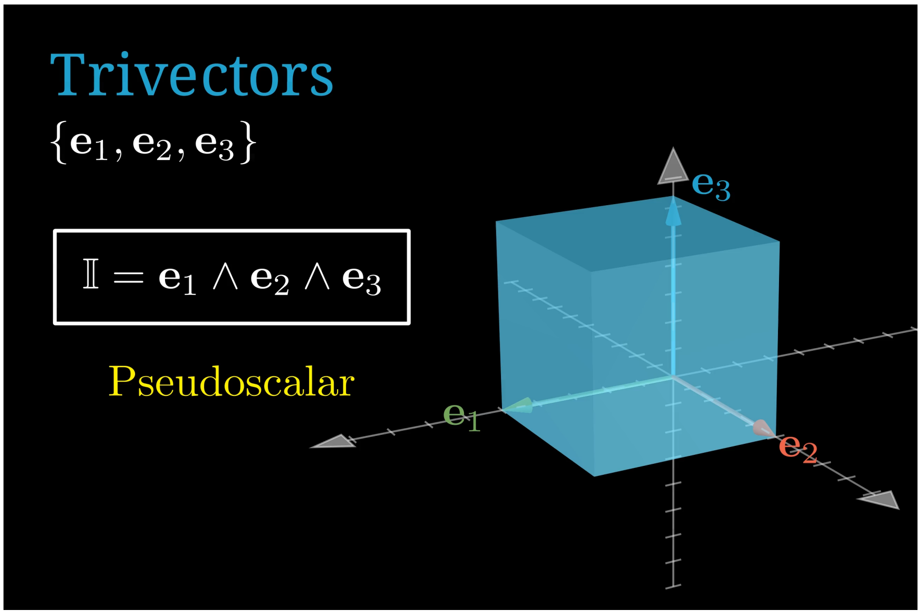

Figure 2: The trivector $\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3}$ is the unit volume of 3D space, referred to as the pseudoscalar. The volume orientation is omitted in the figure for simplicity.

2.2 The Geometric Product

At the core of any algebra is an operation akin to multiplication. In linear algebra, we multiply matrices via specific rules. For elements in a GA, the rule is simple. For any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their geometric product is defined as

$$\boxed{\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}}, \tag{3}$$

where $\boldsymbol{u} \cdot \boldsymbol{v}$ resembles the dot product, but I shall call it the inner product from now on. This terminology does not alter the mathematics in a strict sense at least for now, and the reader should interpret this term in the usual sense. Namely, the inner product of two vectors is a scalar–a measure of how two vectors are aligned in space, scaled by their lengths. Thus, the inner product term produces a grade-0 object in this case.

The geometric product could be slightly unnerving at first glance, given the indoctrination of linear algebra typical of any physicist. However, we can interpret it using an analogy with complex numbers. Much like we add the complex numbers:

$$(a + ib) + (c - id) = (a + c) + (b - d)i,$$

one can think of same-grade components in a multi-vector adding together. Apples add with apples and oranges with oranges.

We need not contend too much with this idea for the moment, since, most of the time the geometric product reduces to either the inner or outer product term. As an example, consider the orthonormal basis for $\mathbb{R}^2$, $\{\boldsymbol{e_1}, \boldsymbol{e_2}\}$. The product

$$\boldsymbol{e_1}\boldsymbol{e_2} = \boldsymbol{e_1} \wedge \boldsymbol{e_2}, \tag{4}$$

because $\boldsymbol{e_1} \cdot \boldsymbol{e_2} = 0$, since the basis vectors are orthogonal. This applies to any of the basis vectors. On the other hand,

$$\boldsymbol{e_1}\boldsymbol{e_1} = \boldsymbol{e_1} \cdot \boldsymbol{e_1} = 1, \tag{5}$$

since the area $\boldsymbol{e_1} \wedge \boldsymbol{e_1}$ vanishes and the basis vectors are normalized. This simplification scheme will be our bread and butter in what is to come.

3 The Cross Product, Levi-Cevita, and Duality

3.1 The Cross Product

Now that we have the basics, I would like to anchor our discussion in something familiar: the cross product. I would like us to examine a not-so-famous expression for the cross product between two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{A} \times \boldsymbol{B} = \epsilon_{ijk} a_i b_j \boldsymbol{e_k}, \tag{6}$$

where I have used Einstein summation notation, since

$$\boldsymbol{A} = \sum_i a_i \boldsymbol{e_i}, \qquad \boldsymbol{B} = \sum_j b_j \boldsymbol{e_j}.$$

You might have previously seen equation 6. If not, I intentionally leave it to the reader to see that it computationally matches the linear algebra definition. The nominal definition of the Levi-Cevita is described in terms of cyclic and anti-cyclic permutations of 1, 2, 3:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{cyclic permutations of 1,2,3} \\ -1 & \text{anti-cyclic permutations of 1,2,3} \\ 0 & \text{repeating indices} \end{cases} \tag{7}$$

The appearance of the Levi-Cevita in equation 6 implicitly implies that for a right-handed orthonormal basis set

$$\boldsymbol{e_i} \times \boldsymbol{e_j} = \epsilon_{ijk}\boldsymbol{e_k}. \tag{8}$$

I shall intentionally refrain from commenting about the geometrical interpretation of the Levi-Cevita in equation 6 (and by extension 8). Indeed, this appearance is no coincidence, with deep geometrical implications. However, I shall not give away the entire answer here. Rather, I plan to entice the reader(s) to explore the answer for themselves. In the following, I shall give some of the necessary ingredients. A full geometric treatment of Levi-Cevita, however, requires a lecture of its own. In that spirit, I shall only quote and verify the GA definition of Levi-Cevita ([1]):

$$\epsilon_{ijk} = \boldsymbol{e_i} \wedge \boldsymbol{e_j} \wedge \boldsymbol{e_k}\mathbb{I}^\dagger, \tag{9}$$

where $\mathbb{I}^\dagger = \boldsymbol{e_3} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_1} = \boldsymbol{e_3}\ \boldsymbol{e_2}\ \boldsymbol{e_1}$, the reverse of equation 2. To demonstrate the simplicity of the algebra, we can verify that this definition is valid by comparing with the more familiar definition.

1. For the case where any two indices repeat, for example:

$$\epsilon_{112} = \boldsymbol{e_1} \wedge \boldsymbol{e_1} \wedge \boldsymbol{e_2}\mathbb{I}^\dagger$$

   the term $\boldsymbol{e_1} \wedge \boldsymbol{e_1} = 0$, yields $\epsilon_{112} = 0$. This result generalizes, since $\boldsymbol{e_i} \wedge \boldsymbol{e_i} = 0$:

$$\epsilon_{iij} = \boldsymbol{e_i} \wedge \boldsymbol{e_i} \wedge \boldsymbol{e_j}\mathbb{I}^\dagger = 0.$$

2. For cyclic permutations, e.g.,

$$\epsilon_{123} = \boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3}\mathbb{I}^\dagger = \boldsymbol{e_1 e_2 e_3 e_3 e_2 e_1} = 1,$$

   since $\boldsymbol{e_i e_i} = \boldsymbol{e_i} \cdot \boldsymbol{e_i} = 1$, and we see that the reversion of the pseudoscalar, $\mathbb{I}$, leads to the cyclic property of the Levi-Civita tensor.

3. And for anti-cyclic permutations, our orthogonal basis set ensures anti-commutation:

$$\begin{aligned}
\epsilon_{132} &= \boldsymbol{e_1} \wedge \boldsymbol{e_3} \wedge \boldsymbol{e_2 e_3 e_2 e_1} \\
&= \boldsymbol{e_1 e_3 e_2 e_3 e_2 e_1} \\
&= -\boldsymbol{e_1 e_2 e_3 e_3 e_2 e_1} \\
&= -\epsilon_{123} = -1,
\end{aligned}$$

   ensuring that the Levi-Civita is completely anti-symmetric under the swapping of any two indices.

Therefore, we recover the definition of equation (7) using that of equation (9).

3.2 The Outer Product

I shall return to the cross product shortly. For now, let us get more familiar with GA computations. I would like to highlight that all one needs to know here is associativity and distributivity–no matrix manipulations. To demonstrate, let us compute the outer product of $\boldsymbol{A}$ and $\boldsymbol{B}$, which can be simply expanded via

$$\begin{aligned}\boldsymbol{A} \wedge \boldsymbol{B} &= (a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + a_3\boldsymbol{e_3}) \wedge (b_1\boldsymbol{e_1} + b_2\boldsymbol{e_2} + b_3\boldsymbol{e_3}) \\ &= a_1\boldsymbol{e_1} \wedge b_1\boldsymbol{e_1} + a_1\boldsymbol{e_1} \wedge b_2\boldsymbol{e_2} + a_1\boldsymbol{e_1} \wedge b_3\boldsymbol{e_3} \\ &+ a_2\boldsymbol{e_2} \wedge b_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} \wedge b_2\boldsymbol{e_2} + a_2\boldsymbol{e_2} \wedge b_3\boldsymbol{e_3} \\ &+ a_3\boldsymbol{e_3} \wedge b_1\boldsymbol{e_1} + a_3\boldsymbol{e_3} \wedge b_2\boldsymbol{e_2} + a_3\boldsymbol{e_3} \wedge b_3\boldsymbol{e_3}.\end{aligned}$$

Notice that terms where two parallel bases vectors are wedged vanish ($\boldsymbol{e_i} \wedge \boldsymbol{e_i} = 0$), and we are left with parts that are totally anti-symmetric. Thus the outer product reads

$$\begin{aligned}\boldsymbol{A} \wedge \boldsymbol{B} &= (a_1b_2 - a_2b_1)\, \boldsymbol{e_1} \wedge \boldsymbol{e_2} \\ &+ (a_2b_3 - a_3b_2)\, \boldsymbol{e_2} \wedge \boldsymbol{e_3} \\ &+ (a_1b_3 - a_3b_1)\, \boldsymbol{e_1} \wedge \boldsymbol{e_3}\end{aligned}$$

Focusing for now on the first term $(a_1b_2 - a_2b_1)\, \boldsymbol{e_1} \wedge \boldsymbol{e_2}$, the wedge can be manipulated as such:

$$\boldsymbol{e_1} \wedge \boldsymbol{e_2} = \boldsymbol{e_1e_2e_3e_3} = \boldsymbol{e_3e_1e_2e_3},$$

where I have implanted $\boldsymbol{e_3e_3} = 1$ to the right. I then swapped $\boldsymbol{e_3}$ to the left twice, picking up a minus sign each time since $\boldsymbol{e_ie_j} = \boldsymbol{e_i} \wedge \boldsymbol{e_j} = -\boldsymbol{e_j} \wedge \boldsymbol{e_i} = -\boldsymbol{e_je_i}$. Now I shall utilize the fact that $\epsilon_{123} = 1$ along with equation (8) to write:

$$\begin{aligned}\boldsymbol{e_1} \wedge \boldsymbol{e_2} &= \epsilon_{123}\boldsymbol{e_3e_1e_2e_3} \\ &= \boldsymbol{e_1} \times \boldsymbol{e_2}\mathbb{I}.\end{aligned}$$

We can use this result but to rewrite all the outer products as

$$\begin{aligned}\boldsymbol{A} \wedge \boldsymbol{B} &= (a_1b_2 - a_2b_1)\, (\boldsymbol{e_1} \times \boldsymbol{e_2})\mathbb{I} \\ &+ (a_2b_3 - a_3b_2)\, (\boldsymbol{e_2} \times \boldsymbol{e_3})\mathbb{I} \\ &+ (a_1b_3 - a_3b_1)\, (\boldsymbol{e_1} \times \boldsymbol{e_3})\mathbb{I},\end{aligned}$$

or,

$$\begin{aligned}\boldsymbol{A} \wedge \boldsymbol{B} &= (a_1b_2 - a_2b_1)\, (\boldsymbol{e_3})\mathbb{I} \\ &+ (a_2b_3 - a_3b_2)\, (-\boldsymbol{e_2})\mathbb{I} \\ &+ (a_1b_3 - a_3b_1)\, (\boldsymbol{e_1})\mathbb{I}.\end{aligned}$$

Here, we see the cross product makes an appearance! These scalar numbers are exactly the components of the cross product. Also, notice the minus sign next to $\boldsymbol{e_2}$, which would have been given by the determinant of the matrix defining the cross product. It might be of value to trace exactly where that minus sign comes from in our mathematics so far. Suffices to say, the Levi-Cevita is the culprit.

Rewriting our result, the outer product can now be expressed in a compact form

$$\boxed{\boldsymbol{A} \wedge \boldsymbol{B} = (\boldsymbol{A} \times \boldsymbol{B})\mathbb{I}}. \tag{10}$$

Equation (10) makes the definition of the cross product abundantly clear. Both the orientation and the magnitude of the cross product are given by the outer product, as shown in Figure 3. Unlike the cross product, however, the outer product generalizes to higher dimensions, while the cross product only exists in 3D space.

Before concluding this section, I would like to point out that in equation (10), the pseudoscalar, $\mathbb{I}$, seems to relate the oriented parallelogram, $\boldsymbol{A} \wedge \boldsymbol{B}$, to an orthogonal vector, $\boldsymbol{A} \times \boldsymbol{B}$. Let us invesitgate whether this action of the pseudoscalar generalizes.

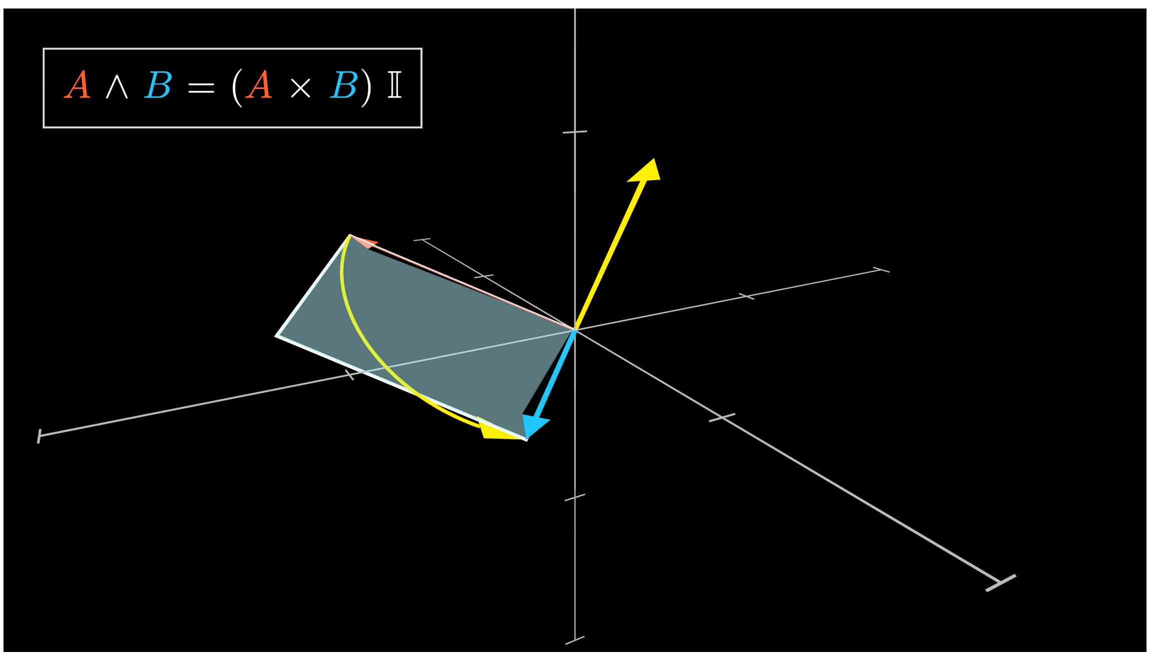

Figure 3: The cross product of two vectors $\boldsymbol{A}$, shown in red, and $\boldsymbol{B}$, shown in blue, is a vector perpendicular to the parallelogram spanned by the two vectors, namely, $\boldsymbol{A} \wedge \boldsymbol{B}$. The orientation is simply given by the outer product, as shown with the curved yellow arrow. Curling our hands in this orientation results in the vector $\boldsymbol{A} \times \boldsymbol{B}$, shown in yellow.

3.3 Duality

The result of equation (10) is profound and has a clear geometric interpretation. As we already know, the vector $\boldsymbol{A} \times \boldsymbol{B}$ is perpendicular to the plane spanned by $\boldsymbol{A} \wedge \boldsymbol{B}$. What about in 2D? What is perpendicular to $\boldsymbol{e_1}$? Well, there is only $\boldsymbol{e_2}$. Coincidently,

$$\boldsymbol{e_2} = \boldsymbol{e_2}\ 1 = \boldsymbol{e_2}\ \boldsymbol{e_1}\boldsymbol{e_1} = \boldsymbol{e_1}\ \boldsymbol{e_1}\boldsymbol{e_2} = \boldsymbol{e_1}\ \mathbb{I}.$$

But what is perpendicular to $\boldsymbol{e_2}$?

$$\begin{aligned}
\boldsymbol{e_2} &= \boldsymbol{e_1}\ \mathbb{I} \\
\boldsymbol{e_2}\mathbb{I} &= \boldsymbol{e_1}\ \mathbb{I}\mathbb{I} \\
\boldsymbol{e_2}\mathbb{I} &= \boldsymbol{e_1}\ (\mathbb{I})^2 = \boldsymbol{e_1}\ (\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3})^2 \\
\boldsymbol{e_2}\mathbb{I} &= \boldsymbol{e_1}\ (\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}) \\
\boldsymbol{e_2}\mathbb{I} &= \boldsymbol{e_1}\ (\boldsymbol{e_2}\boldsymbol{e_3}\boldsymbol{e_1}\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}) \\
\boldsymbol{e_2}\mathbb{I} &= \boldsymbol{e_1}\ (-\boldsymbol{e_3}\boldsymbol{e_2}\boldsymbol{e_2}\boldsymbol{e_3}) \\
&\boxed{\boldsymbol{e_2}\mathbb{I} = -\boldsymbol{e_1}} \qquad (11)
\end{aligned}$$

or,

$$\begin{aligned}
-\boldsymbol{e_1} &= \boldsymbol{e_2}\mathbb{I} \\
-\boldsymbol{e_1} &= \boldsymbol{e_2}\boldsymbol{e_1}\boldsymbol{e_2} = -\boldsymbol{e_2}\boldsymbol{e_2}\boldsymbol{e_1} \\
&\boxed{\boldsymbol{e_1} = \boldsymbol{e_2}\mathbb{I}^\dagger}. \qquad (12)
\end{aligned}$$

This result is interesting to say the least. Ignoring the miraculous, very suspicious, $\mathbb{I}^2 = -1$, I tackle the more immediate task at hand: does this orthogonality by mere multiplication with the pseudoscalar, $\mathbb{I}$, generalize to higher dimensions? Well, it has to. Let us say we want to pick out an element of the space that is perpendicular to all other elements, then we certainly can. Here is how we do it. We can construct from our space a vector $\boldsymbol{e^j}$ perpendicular to all vectors $\boldsymbol{e_i}$ where $i \neq j$,

$$\boxed{\boldsymbol{e_i} \cdot \boldsymbol{e^j} = \delta_i^j}, \qquad (13)$$

where δ_i^j is the usual Kronecker delta. This can be achieved if we consider the unit volume spanned by our basis vectors (the pseudoscalar of our algebra)

$$\mathbb{I} \equiv \boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} ... \wedge \boldsymbol{e_n}, \qquad (14)$$

for some n-dimensional space, spanned by $\boldsymbol{e_n}$. Then we can find $\boldsymbol{e^j}$ via

$$\boldsymbol{e^j} = (-1)^{j-1}\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge ...\hat{e}_j \wedge ...e_n\mathbb{I}^\dagger, \qquad (15)$$

where $\hat{e}_j$ in equation (15) just implies that term is missing from the product. The appearance of the $(-1)^{j-1}$ term should not come as a surprise, given our definition of the Levi-Civita in equation (9), and because we are now missing a term. Ignoring it for now, let us focus on the remainder of equation (15).

Let us consider what equation (15) tells us. It says, to find a vector $\boldsymbol{e}^j$ perpendicular to all other basis elements, you first wedge all those elements to construct a hypervolume, but one which does NOT include the original vector $\boldsymbol{e}_j$.[1] After wedging all the elements, you multiply with the reversed pseudoscalar, $\mathbb{I}^\dagger$, like we did in equation (12) to pick out an element perpendicular to all those wedged elements.

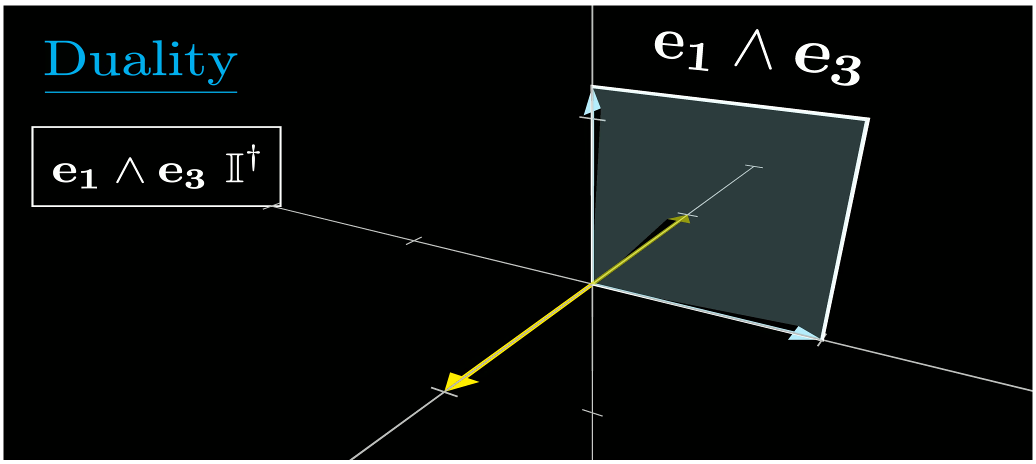

Figure 4: We expect the action of the pseudoscalar to take $\boldsymbol{e}_2$ and spit out a vector perpendicular to the other basis elements, namely, $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$, for $\mathbb{R}^3$.

As an example, consider $\boldsymbol{e}_2$, but now in 3D space. For our procedure to work, we must consider the elements of the algebra orthogonal to $\boldsymbol{e}_2$, namely $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$, and multiply them with the pseudoscalar. The result should be an element orthogonal to both $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$, as shown in Figure 4. Let us test this algebraicly:

$$\begin{aligned}
\boldsymbol{e}_2 \cdot \boldsymbol{e}^2 &= \boldsymbol{e}_2 \cdot ((-1)^1 \boldsymbol{e}_1 \wedge \boldsymbol{e}_3 \mathbb{I}^\dagger) \\
&= \boldsymbol{e}_2 \cdot (-\boldsymbol{e}_1\boldsymbol{e}_3 \mathbb{I}^\dagger) \\
&= \boldsymbol{e}_2 \cdot (-\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3 \mathbb{I}^\dagger) \\
&= \boldsymbol{e}_2 \cdot (\boldsymbol{e}_2 \mathbb{I}\mathbb{I}^\dagger) \\
&= \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1.
\end{aligned}$$

where $\mathbb{I}\mathbb{I}^\dagger = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1 = 1$. Now, one might argue that this result is redundant. We tried to look for an element much like $\boldsymbol{e}_2$ in the space, in the sense that it is perpendicular to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. However, we ended up

[1] Notice the subscript vs. superscript notation used to denote the difference between a vector vs. covector, respectively. Fancy names do not mean much, but we need some nomenclature to differentiate between the two.

right back at $\boldsymbol{e_2}$. Although this result implies that

$$\boldsymbol{e_i} = \boldsymbol{e^i},$$

one cannot take that to be the case for all vector spaces. As we shall see in the next section, the metric of the space dictates the rules that relate a vector to its covector, or, a vector to its dual. Now the term "dual" is no longer mysterious and has a grounded, geometrical meaning.

4 An Algebra within an Algebra

4.1 Spacetime Geometric Algebra

In the last section, I introduced the idea of duality, and said that it is not always the case that the basis vector and their duals are identical–a somewhat redundant result. To illustrate that this is not always the case, let us consider a slightly more sophisticated algebra, the Spacetime Geometric Algebra (SGA), which inherits the 4D Minkowski metric from special relativity. I assume the reader is already familiar with that tensor. I will be using the mostly negative version, the one with the signature: $(+, -, -, -)$.

The orthonormal basis of SGA consists of one timelike vector γ_0 and three spacelike vectors γ_1, γ_2, and γ_3. Like with normal Euclidean space, the basis vectors follow two important rules. The first identifies orthogonality:

$$\boxed{\boldsymbol{\gamma_\alpha \gamma_\beta = -\gamma_\beta \gamma_\alpha}}, \tag{16}$$

while the second, and most important, dictates the operation of the inner product via:

$$\boxed{\boldsymbol{\gamma_\alpha \gamma_\alpha = \gamma_\alpha \cdot \gamma_\alpha} = \eta_{\alpha\alpha}}, \tag{17}$$

since the outer product $\boldsymbol{\gamma_\alpha} \wedge \boldsymbol{\gamma_\alpha}$ vanishes. I cannot overstress how fundamental equation (17) is. The metric defines the structure of the geometry. By defining the norm, it also implicitly defines the rules of dual transformations–the reason as to why, I shall leave to the reader. As for how, consider the spacelike basis vector resembling $\hat{y}$, namely $\boldsymbol{\gamma_2}$, and its norm: [2]

$$\boldsymbol{\gamma_2} \cdot \boldsymbol{\gamma_2} = \eta_{22} = -1. \tag{18}$$

Because of equation (13), which defines duality, and equation (18) above, we must have that:

$$\boldsymbol{\gamma_2} \cdot \boldsymbol{\gamma^2} = 1 \Longrightarrow \boldsymbol{\gamma_2} \cdot (-\boldsymbol{\gamma_2}) = -\boldsymbol{\gamma_2} \cdot \boldsymbol{\gamma_2} = -(-1) = 1,$$

meaning,

$$\boxed{\boldsymbol{\gamma_i = -\gamma^i}}. \tag{19}$$

[2] By norm, I refer to the inner product; length would be the square root of that.

But that only holds for $i \in \{1, 2, 3\}$. For the timelike vector,

$$\boxed{\boldsymbol{\gamma_0} = \boldsymbol{\gamma^0}}. \tag{20}$$

It is a good exercise to see why the latter must be true using a similar argument to the one described above. Conversely, if our metric was of the mostly plus signature, $(-, +, +, +)$, we would have:

$$\boldsymbol{\gamma^0} = -\boldsymbol{\gamma_0} \qquad\qquad \boldsymbol{\gamma^i} = \boldsymbol{\gamma_i}.$$

Therefore, the inner product, defined by the metric tensor, sets the rules of the algebra. As we say in the language of tensors: "the metric tensor is used to raise or lower indices.":

$$\boldsymbol{\gamma_\alpha} = \boldsymbol{\gamma^\beta}\,\eta_{\alpha\beta} \qquad\qquad \boldsymbol{\gamma^\alpha} = \boldsymbol{\gamma_\beta}\,\eta^{\alpha\beta}$$

4.2 The Even Subalgebra

The last section should have been indicative of the complexity of SGA, given its metric. Another aspect of this intricacy arises when we consider the amount of grade-2 elements we can generate using our basis set $\{\boldsymbol{\gamma_0}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}\}$. Considering that scalars form via the products $\boldsymbol{\gamma_0}\boldsymbol{\gamma_0} = 1$ and $\boldsymbol{\gamma_i}\boldsymbol{\gamma_i} = -1$, the only generators for bivectors are

$$\{\boldsymbol{\gamma_i}\boldsymbol{\gamma_0}\} \quad \text{and} \quad \{\boldsymbol{\gamma_i}\boldsymbol{\gamma_j}\},$$

each of which generates 3 independent bivectors (given their anti-commutativity), for a total of 6 bivectors in our spacetime geometric algebra. I would like to focus on the bivector algebra formed by the set $\{\boldsymbol{\sigma_i}\} \equiv \{\boldsymbol{\gamma_i}\boldsymbol{\gamma_0}\}$, from which you can generate scalars via

$$(\boldsymbol{\sigma_i})^2 = \boldsymbol{\sigma_i}\boldsymbol{\sigma_i} = \boldsymbol{\gamma_i}\boldsymbol{\gamma_0}\boldsymbol{\gamma_i}\boldsymbol{\gamma_0} = -\boldsymbol{\gamma_i}\boldsymbol{\gamma_i}\boldsymbol{\gamma_0}\boldsymbol{\gamma_0} = 1, \tag{21}$$

and the pseudoscalar via,

$$\begin{aligned} \mathbb{I} &= \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\boldsymbol{\sigma_3} \\ &= \boldsymbol{\gamma_1}\boldsymbol{\gamma_0}\boldsymbol{\gamma_2}\boldsymbol{\gamma_0}\boldsymbol{\gamma_3}\boldsymbol{\gamma_0} \\ &= -\boldsymbol{\gamma_0}\boldsymbol{\gamma_1}\boldsymbol{\gamma_2}\boldsymbol{\gamma_0}\boldsymbol{\gamma_3}\boldsymbol{\gamma_0} \\ &= \boldsymbol{\gamma_0}\boldsymbol{\gamma_1}\boldsymbol{\gamma_2}\boldsymbol{\gamma_3}\boldsymbol{\gamma_0}\boldsymbol{\gamma_0}, \end{aligned}$$

$$\boxed{\mathbb{I} = \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\boldsymbol{\sigma_3} = \boldsymbol{\gamma_0}\boldsymbol{\gamma_1}\boldsymbol{\gamma_2}\boldsymbol{\gamma_3}}. \tag{22}$$

This means that the subalgebra and SGA share the same pseudoscalar.

Borrowing on from equation (9), within this subalgebra, the Levi-Civita symbol is defined in terms of subalgebra's pseudoscalar as

$$\begin{aligned}
\epsilon_{ijk} &= \boldsymbol{\sigma_i} \wedge \boldsymbol{\sigma_j} \wedge \boldsymbol{\sigma_k} \mathbb{I}^\dagger \\
&= \boldsymbol{\sigma_i \sigma_j \sigma_k \sigma_3 \sigma_2 \sigma_1} \\
\epsilon_{ijk}\mathbb{I} &= \boldsymbol{\sigma_i \sigma_j \sigma_k \sigma_3 \sigma_2 \sigma_1 \sigma_1 \sigma_2 \sigma_3} \\
\epsilon_{ijk}\mathbb{I} &= \boldsymbol{\sigma_i \sigma_j \sigma_k} \\
\epsilon_{ijk}\mathbb{I}\boldsymbol{\sigma_k} &= \boldsymbol{\sigma_i \sigma_j \sigma_k \sigma_k}
\end{aligned}$$

$$\boxed{\boldsymbol{\sigma_i \sigma_j} = \epsilon_{ijk}\mathbb{I}\boldsymbol{\sigma_k}} \tag{23}$$

This is no trivial result. Because the Levi-Civita is totally anti-symmetric, equation (23) implies

$$\boxed{\boldsymbol{\sigma_i \sigma_j} = -\boldsymbol{\sigma_j \sigma_i}}, \tag{24}$$

which implies

$$\boldsymbol{\sigma_i \sigma_j} = \frac{1}{2}\left(\boldsymbol{\sigma_i \sigma_j} - \boldsymbol{\sigma_j \sigma_i}\right) = \frac{1}{2}\left[\boldsymbol{\sigma_j}, \boldsymbol{\sigma_j}\right]. \tag{25}$$

Finally, combining equations (23) with (25), we arrive at the commutator relationship governing the famous Pauli matrices:

$$\boxed{\left[\boldsymbol{\sigma_j}, \boldsymbol{\sigma_j}\right] = 2\epsilon_{ijk}\mathbb{I}\boldsymbol{\sigma_k}}. \tag{26}$$

Since the product of any of the $\boldsymbol{\sigma_i \sigma_j}$ reduces to 1 if $i = j$ (see equation (21)) and results in the third orthogonal bivector otherwise (see equation (1)), the algebra is summarized by

$$\boxed{\boldsymbol{\sigma_i \sigma_j} = \delta_{ij} + \epsilon_{ijk}\mathbb{I}\boldsymbol{\sigma_k}}. \tag{27}$$

Looks familiar?... Indeed, this is the algebra Pauli introduced to describe quantum spin! However, I would like us to pause here and ponder a little, since, up to this point, I had made no mention of quantum mechanics, nor did I introduce any physics, observables, measurements, etc. We arrived here purely via the geometric algebra of spacetime. Does this mean quantum spin is inextricably linked to geometry? While, I cannot fully answer this question, nor did I find anyone to claim they can in the literature, I can certainly start to pierce the veil–show you a glimpse of the geometry of quantum spin.

5 Geometry of Spin

5.1 Rotations

In the last section, I discussed the Pauli algebra, and how it arises from the bivector algebra within SGA. I would like to expand on this here, by first making a correspondence with a familiar operator. Consider

$$e^{i\hat{\sigma}_z\theta/2} = \cos(\theta/2) + i\hat{\sigma}_z\sin(\theta/2), \tag{28}$$

where $\hat{\sigma_z}$ is the quantum operator (sigma matrix) analog to our $\boldsymbol{\sigma_3}$. Equation (28) is relatively known in quantum mechanics, and can be verified by expanding the exponential function. The proof is simple and can be found in many textbooks. I would like to make the argument that the GA analog also holds, namely

$$e^{\mathbb{I}\boldsymbol{\sigma_3}\theta/2} = \cos(\theta/2) + \mathbb{I}\boldsymbol{\sigma_3}\sin(\theta/2). \tag{29}$$

Instead of going through the math, I would like to remind the reader of the fundamental result we stumbled upon while deriving the dual in equation (11):

$$\mathbb{I}\mathbb{I} = \mathbb{I}^2 = -1, \tag{30}$$

which I swept under rug right momentarily. This result alludes to a hidden complex structure embedded within GA. Equation (30) holds not only for the pseudoscalar of 3D space which was used in deriving equation (11), but also for the 4D Minkowski space (as one quickly check), and, by extension, for the even subalgebra. This result is precisely why GA

Additionally, I have shown in the previous section that the Pauli vectors are unitary (see equation (21)). Hence, we have all the ingredients necessary to assert that if equation (28) holds equation (29) must also hold.

Finally, using equation (23), I would like to rewrite $\mathbb{I}\boldsymbol{\sigma_3}$.

$$\boldsymbol{\sigma_i}\boldsymbol{\sigma_j} = \epsilon_{ijk}\mathbb{I}\boldsymbol{\sigma_k} = \epsilon_{ij3}\mathbb{I}\boldsymbol{\sigma_3},$$

which implies,

$$\mathbb{I}\boldsymbol{\sigma_3} = \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}.$$

Rewriting equation (29), we have

$$e^{\mathbb{I}\boldsymbol{\sigma_3}\theta/2} = \cos(\theta/2) + \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\sin(\theta/2). \tag{31}$$

Here, I would like to examine the sandwich product

$$e^{-\mathbb{I}\boldsymbol{\sigma_3}\theta/2}\ \boldsymbol{v}\ e^{\mathbb{I}\boldsymbol{\sigma_3}\theta/2}, \tag{32}$$

for some vector $\boldsymbol{v}$ in our subalgebra. To illustrate, let us use $\boldsymbol{v} = \boldsymbol{\sigma_1}$:

$$e^{-\mathbb{I}\boldsymbol{\sigma_3}\theta/2}\ \boldsymbol{\sigma_1}\ e^{\mathbb{I}\boldsymbol{\sigma_3}\theta/2} = \left(\cos\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\sin\left(\frac{\theta}{2}\right)\right)\boldsymbol{\sigma_1}\left(\cos\left(\frac{\theta}{2}\right) + \boldsymbol{\sigma_1}\boldsymbol{\sigma_2}\sin\left(\frac{\theta}{2}\right)\right)$$

$$
\begin{aligned}
&= \cos\left(\frac{\theta}{2}\right)\boldsymbol{\sigma_1}\cos\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1\sigma_2}\sin\left(\frac{\theta}{2}\right)\boldsymbol{\sigma_1}\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\boldsymbol{\sigma_1\sigma_1\sigma_2}\sin\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1\sigma_2}\sin\left(\frac{\theta}{2}\right)\boldsymbol{\sigma_1\sigma_1\sigma_2}\sin\left(\frac{\theta}{2}\right)\\
&= \boldsymbol{\sigma_1}\cos^2\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1\sigma_2}\,\boldsymbol{\sigma_1}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \boldsymbol{\sigma_1\sigma_1\sigma_2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2}\sin^2\left(\frac{\theta}{2}\right)\\
&= \boldsymbol{\sigma_1}\cos^2\left(\frac{\theta}{2}\right) + \boldsymbol{\sigma_2\sigma_1}\,\boldsymbol{\sigma_1}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \boldsymbol{\sigma_2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2}\sin^2\left(\frac{\theta}{2}\right)\\
&= \boldsymbol{\sigma_1}\cos^2\left(\frac{\theta}{2}\right) + 2\boldsymbol{\sigma_2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \boldsymbol{\sigma_1}(\boldsymbol{\sigma_1\sigma_2})^2\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}
$$

or,

$$
e^{-\mathbb{I}\boldsymbol{\sigma_3}\theta/2}\;\boldsymbol{\sigma_1}\;e^{\mathbb{I}\sigma_z\theta/2} = \boldsymbol{\sigma_1}\cos(\theta) + \boldsymbol{\sigma_2}\sin(\theta), \tag{33}
$$

where, in the last step, I made use of the half-angle trigonometric identities and that $(\boldsymbol{\sigma_1\sigma_2})^2 = \boldsymbol{\sigma_1\sigma_2\sigma_1\sigma_2} = -\boldsymbol{\sigma_1\sigma_2\sigma_2\sigma_1} = -1$, again, behaving like the imaginary unit. The latter is no coincidence. In fact, the Pauli algebra is isomorphic to the algebra of quaternions, where

$$
\boldsymbol{\sigma_1}\mathbb{I} \sim i \qquad\qquad -\boldsymbol{\sigma_2}\mathbb{I} \sim j \qquad\qquad \boldsymbol{\sigma_3}\mathbb{I} \sim k.
$$

Hence, the quaternion algebra, fundamentally related to spatial rotations, is also a subalgebra of the SGA, and the imaginary unit i is represented by the bivector $\boldsymbol{\sigma_1\sigma_2}$. Equation (33) describes a rotation of the vector $\boldsymbol{\sigma_1}$ about the $\boldsymbol{\sigma_3}$ axis by an angle θ. Now we can visualize the rotation as being brought about by the action of the exponential of the bivector $\boldsymbol{\sigma_1\sigma_2}$, which is a rotation in the plane defined by the two vectors $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$. The sense of rotation would then be opposite if we use the bivector $\boldsymbol{\sigma_2\sigma_1} = -\boldsymbol{\sigma_1\sigma_2}$, a purely geometric intuition!

It is because of this anti-commutativity that we can write

$$
R_{\boldsymbol{\sigma_3}}(\theta) = e^{-\mathbb{I}\boldsymbol{\sigma_3}\theta/2}, \tag{34}
$$

and,

$$
R_{\boldsymbol{\sigma_3}}(\theta)^\dagger = e^{\mathbb{I}\boldsymbol{\sigma_3}\theta/2}, \tag{35}
$$

since a swap (or reversion, denoted by the dagger) of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ in the exponent of equation (34) gives a minus sign. Finally, one can check that this operation is indeed a rotation. For instance, substituting $\theta = \pi/2$ in equation (33) gives

$$
R_{\sigma_3}\left(\frac{\pi}{4}\right)\,\boldsymbol{\sigma_1}\,R_{\sigma_3}\left(\frac{\pi}{4}\right)^\dagger = \boldsymbol{\sigma_2}.
$$

Substituting $\theta = \pi/4$ gives

$$
R_{\sigma_3}\left(\frac{\pi}{2}\right)\,\boldsymbol{\sigma_1}\,R_{\sigma_3}\left(\frac{\pi}{2}\right)^\dagger = \frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma_1} + \boldsymbol{\sigma_2}\right).
$$

Indeed, we are moving around the unit circle in the plane defined by $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$! Thus, using these rotations, we can move around the unit sphere, commonly referred to as see Figure 5. No surprise, this unit sphere corresponds with the famous Bloch sphere, popularized by Bloch in Nuclear Magnetic Resonance, or the Poincaré sphere, more appropriate in the context of quantum optics, used to describe the polarization of light. However, this mathematical entity should be rightly attributed to Riemann, who first introduced it in the context of complex geometry. In the next section, I shall explore how this sphere relates to the mathematical description of the qubit.

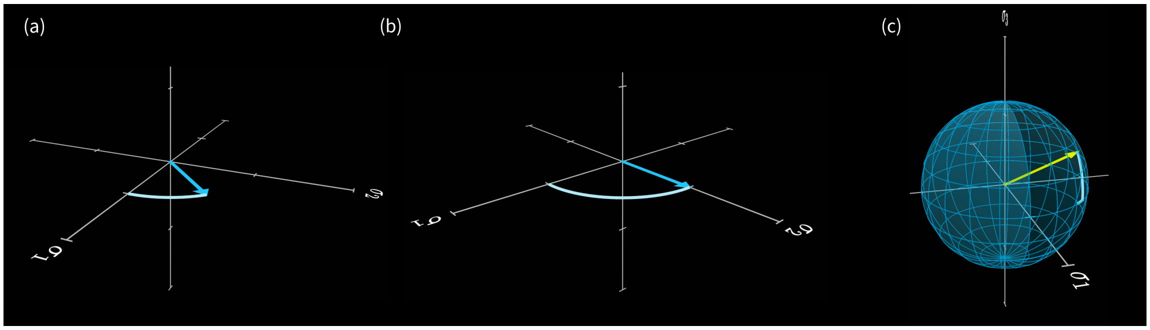

Figure 5: (a) a rotations of $\theta = \pi/4$ about $\boldsymbol{\sigma_3}$, (b) a rotation of $\theta = \pi/2$ about $\boldsymbol{\sigma_3}$, and (c) how arbitrary rotations can allow us to move around the unit sphere.

5.2 The Riemann Sphere

5.3 The Qubit

In quantum mechanics, we would associate a state to the vector shown in Figure 5(c). One could write this arbitrary state as

$$|\nearrow\rangle = v\,|\uparrow\rangle + w\,|\downarrow\rangle\,, \tag{36}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the basis states, and v and w are complex numbers. The latter implies we need 4 real numbers to describe the state, which would more appropriatly correspond with two spheres! The correspondence, therefore, is subtle, and relies, most importantly on the idea of global phase invariance. Namely, if we scale the state by a complex number $\lambda = e^{i\phi}$, the state is unchanged:

$$\lambda\,|\nearrow\rangle = |\nearrow\rangle\,. \tag{37}$$

This allows us to scale the state by $\frac{1}{v}$ such that

$$\frac{1}{v}\,|\nearrow\rangle = |\uparrow\rangle + u\,|\downarrow\rangle\,, \tag{38}$$

where $u = \frac{w}{v}$.

References

[1] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. en. 1st ed. Cambridge University Press, May 2003. ISBN: 978-0-521-48022-2 978-0-521-71595-9 978-0-511-80749-7. DOI: 10.1017/CBO9780511807497. URL: https://www.cambridge.org/core/product/identifier/9780511807497/type/book.