# Mathematics Behind the Options Calculator

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## 1 European Call

$$C = Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$$
 (Black-Scholes identity)

- C Option price
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- $\sigma$  Volatility
- Φ Standard normal CDF
- $\phi$  Standard normal pdf

#### 1.1 Delta

$$\Delta = \frac{\partial C}{\partial S} = e^{-qT} \Phi(d_1)$$

$$\Delta = \frac{\partial C}{\partial S} = e^{-qT} \Phi(d_1) + Se^{-qT} \phi(d_1) \left(\frac{\partial d_1}{\partial S}\right) - Ke^{-rT} \phi(d_2) \left(\frac{\partial d_2}{\partial S}\right)$$

$$= e^{-qT} \Phi(d_1) + \underbrace{Se^{-qT} \phi(d_1) \frac{1}{S\sigma\sqrt{T}} - Ke^{-rT} \phi(d_2) \frac{1}{S\sigma\sqrt{T}}}_{=0}$$

$$\Delta = e^{-qT} \Phi(d_1)$$

#### 1.2 Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = e^{-qT} (\phi(d_1) \frac{1}{S\sigma\sqrt{T}})$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

$$= e^{-qT} \frac{\partial}{\partial S} (\Phi(d_1))$$

$$\Gamma = e^{-qT} \phi(d_1) \frac{1}{S\sigma\sqrt{T}}$$

### 1.3 Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\sqrt{T}$$

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial \sigma}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial \sigma}\right)$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) + \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) - \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T}$$

$$\mathcal{V} = Se^{-qT}\phi(d_1)\sqrt{T}$$

#### 1.4 Theta

$$\Theta = \frac{\partial C}{\partial T} = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

$$\begin{split} \Theta &= \frac{\partial C}{\partial T} = S \left( -qe^{-qT}\Phi(d_1) + e^{-qT}\phi(d_1) \frac{\partial d_1}{\partial T} \right) - K \left( -re^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2) \frac{\partial d_2}{\partial T} \right) \\ &\frac{\partial d_1}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} := A \\ &\frac{\partial d_2}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} - \frac{\sigma}{2\sqrt{T}} := A - \frac{\sigma}{2\sqrt{T}} \\ &= Se^{-qT}(-q\Phi(d_1) + \phi(d_1)A) - Ke^{-rT}(-r\Phi(d_2) + \phi(d_2)(A - \frac{\sigma}{2\sqrt{T}})) \\ &= -qSe^{-qT}\Phi(d_1) + rKe^{-rT}\Phi(d_2) + Ke^{-rT}\phi(d_2) \frac{\sigma}{2\sqrt{T}} + \underbrace{A(Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2))}_{=0} \\ &= Ke^{-rT}\phi(d_2) \frac{\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(d_2) - qSe^{-qT}\Phi(d_1) \end{split}$$

Using the Black-Scholes identity  $Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$ , and expressing  $\Theta$  as the negative:

$$\Theta = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

$$\rho = \frac{\partial C}{\partial r} = KTe^{-rT}\Phi(d_2)$$

$$\rho = \frac{\partial C}{\partial r} = Se^{-qT}(\phi(d_1)\frac{\partial d_1}{\partial r}) - K(-Te^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial r})$$

$$\frac{\partial d_1}{\partial r} = \frac{T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma} = \frac{\partial d_2}{\partial r}$$

$$= Se^{-qT}\phi(d_1)\frac{\sqrt{T}}{\sigma} + KTe^{-rT}\Phi(d_2) - Ke^{-rT}\phi(d_2)\frac{\sqrt{T}}{\sigma}$$

$$= \frac{\sqrt{T}}{\sigma}\left(\underbrace{Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2)}_{=0}\right) + KTe^{-rT}\Phi(d_2)$$

$$\rho = KTe^{-rT}\Phi(d_2)$$

## 2 European Put

$$P = Ke^{-rT} \Phi(-d_2) - Se^{-qT} \Phi(-d_1)$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

- P Option price
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- $\sigma$  Volatility
- $\Phi$  Standard normal CDF
- $\phi$ Standard normal pdf

#### 2.1 Delta

$$\Delta = \frac{\partial P}{\partial S} = -e^{-qT}\Phi(-d_1)$$

#### 2.2 Gamma

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT} \frac{\phi(d_2)}{S^2\sigma\sqrt{T}}$$

Same as European Call derivation.

## 2.3 Vega

$$\mathcal{V} = \frac{\partial^2 P}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT} \frac{\phi(d_2)}{S^2\sigma\sqrt{T}}$$

# 2.4 Theta

$$\Theta = \frac{\partial P}{\partial T} = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) - qSe^{-qT}\Phi(-d_1)$$

$$\rho = \frac{\partial P}{\partial r} = TKe^{-rT}\Phi(-d_2)$$

# 3 Binary Call (Cash-or-Nothing)

$$C_{\text{bin}} = Q e^{-rT} \Phi(d_2)$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

 $C_{\rm bin}$  Option price

- Q Fixed cash payout if  $S_T > K$
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- $\sigma$  Volatility
- $\Phi$  Standard normal CDF
- $\phi$  Standard normal pdf

### 3.1 Delta

$$\Delta = \frac{\partial C_{\text{bin}}}{\partial S} = Q e^{-rT} \frac{\phi(d_2)}{S \sigma \sqrt{T}}$$

#### 3.2 Gamma

$$\Gamma = \frac{\partial^2 C_{\text{bin}}}{\partial S^2} = -Q e^{-rT} \frac{d_1 \phi(d_2)}{S^2 \sigma^2 T}$$

## 3.3 Vega

$$\mathcal{V} = \frac{\partial C_{\text{bin}}}{\partial \sigma} = -Q e^{-rT} d_1 \phi(d_2)$$

## 3.4 Theta

$$\Theta = \frac{\partial C_{\text{bin}}}{\partial T} = Q e^{-rT} \left[ -r \Phi(d_2) + \frac{\phi(d_2)}{2T} \left( \frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right]$$

$$\rho = \frac{\partial C_{\text{bin}}}{\partial r} = Q e^{-rT} \left( -T \Phi(d_2) + \frac{\sqrt{T}}{\sigma} \phi(d_2) \right)$$

# 4 Binary Put (Cash-or-Nothing)

$$P_{\text{bin}} = Q e^{-rT} \Phi(-d_2)$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

 $P_{\rm bin}$  Option price

- Q Fixed cash payout if  $S_T < K$
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- $\sigma$  Volatility
- $\Phi$  Standard normal CDF
- $\phi$ Standard normal pdf

#### 4.1 Delta

$$\Delta = \frac{\partial P_{\text{bin}}}{\partial S} = -Q e^{-rT} \frac{\phi(d_2)}{S \sigma \sqrt{T}}$$

#### 4.2 Gamma

$$\Gamma = \frac{\partial^2 P_{\text{bin}}}{\partial S^2} = Q e^{-rT} \frac{d_1 \phi(d_2)}{S^2 \sigma^2 T}$$

# 4.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial P_{\text{bin}}}{\partial \sigma} = Q e^{-rT} d_1 \phi(d_2)}$$

## 4.4 Theta

$$\Theta = \frac{\partial P_{\text{bin}}}{\partial T} = Q e^{-rT} \left[ -r \Phi(-d_2) - \frac{\phi(d_2)}{2T} \left( \frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right]$$

$$\rho = \frac{\partial P_{\text{bin}}}{\partial r} = Q e^{-rT} \left( -T \Phi(-d_2) - \frac{\sqrt{T}}{\sigma} \phi(d_2) \right)$$

# 5 American Call

$$C^{A} = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau} \left( S_{\tau} - K \right)^{+} \right]$$

(equals the European call when q = 0, since early exercise has no value).

 $\mathbb{C}^A$  American call option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

 $\sigma$  Volatility

 $\tau$  Optimal stopping time (0  $\leq \tau \leq T)$ 

 $\mathbb{E}^{\mathbb{Q}}$  Expectation under the risk-neutral measure

### 5.1 Delta

### 5.2 Gamma

## 5.3 Vega

#### 5.4 Theta

# 6 American Put

$$P^{A} = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau} \left( K - S_{\tau} \right)^{+} \right]$$

 $P^A$  American put option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

 $\sigma$  Volatility

 $\tau$  Optimal stopping time (0  $\leq \tau \leq T)$ 

 $\mathbb{E}^{\mathbb{Q}}$  Expectation under the risk-neutral measure

#### 6.1 Delta

### 6.2 Gamma

6.3 Vega

6.4 Theta