Mathematics Behind the Options Calculator

Pheneas Newman

October 5, 2025

1 European Call

$$C = Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$$
 (Black-Scholes identity)

- C Option price
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- σ Volatility
- Φ Standard normal CDF
- ϕ Standard normal pdf

1.1 Delta

$$\Delta = \frac{\partial C}{\partial S} = e^{-qT} \Phi(d_1)$$

$$\Delta = \frac{\partial C}{\partial S} = e^{-qT} \Phi(d_1) + Se^{-qT} \phi(d_1) \left(\frac{\partial d_1}{\partial S}\right) - Ke^{-rT} \phi(d_2) \left(\frac{\partial d_2}{\partial S}\right)$$

$$= e^{-qT} \Phi(d_1) + \underbrace{Se^{-qT} \phi(d_1) \frac{1}{S\sigma\sqrt{T}} - Ke^{-rT} \phi(d_2) \frac{1}{S\sigma\sqrt{T}}}_{=0}$$

$$\Delta = e^{-qT} \Phi(d_1)$$

1.2 Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = e^{-qT} (\phi(d_1) \frac{1}{S\sigma\sqrt{T}})$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S}$$

$$= e^{-qT} \frac{\partial}{\partial S} (\Phi(d_1))$$

$$\Gamma = e^{-qT} \phi(d_1) \frac{1}{S\sigma\sqrt{T}}$$

1.3 Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\sqrt{T}$$

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial \sigma}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial \sigma}\right)$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) + \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}}\right) - \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T}$$

$$\mathcal{V} = Se^{-qT}\phi(d_1)\sqrt{T}$$

1.4 Theta

$$\Theta = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

$$\begin{split} \Theta &= \frac{\partial C}{\partial T} = S \left(-qe^{-qT}\Phi(d_1) + e^{-qT}\phi(d_1) \frac{\partial d_1}{\partial T} \right) - K \left(-re^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2) \frac{\partial d_2}{\partial T} \right) \\ &\frac{\partial d_1}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} \coloneqq A \\ &\frac{\partial d_2}{\partial T} = \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} - \frac{\sigma}{2\sqrt{T}} \coloneqq A - \frac{\sigma}{2\sqrt{T}} \\ &= Se^{-qT}(-q\Phi(d_1) + \phi(d_1)A) - Ke^{-rT}(-r\Phi(d_2) + \phi(d_2)(A - \frac{\sigma}{2\sqrt{T}})) \\ &= -qSe^{-qT}\Phi(d_1) + rKe^{-rT}\Phi(d_2) + Ke^{-rT}\phi(d_2) \frac{\sigma}{2\sqrt{T}} + \underbrace{A(Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2))}_{=0} \\ &= Ke^{-rT}\phi(d_2) \frac{\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(d_2) - qSe^{-qT}\Phi(d_1) \end{split}$$

Using the Black-Scholes identity $Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$, and expressing Θ as the negative:

$$\Theta = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

$$\rho = KTe^{-rT}\Phi(d_2)$$

2 European Put

$$P = Ke^{-rT} \Phi(-d_2) - Se^{-qT} \Phi(-d_1)$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

- P Option price
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- σ Volatility
- Φ Standard normal CDF
- ϕ Standard normal pdf

2.1 Delta

$$\Delta = \frac{\partial P}{\partial S} = -e^{-qT}\Phi(-d_1)$$

2.2 Gamma

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT} \frac{\phi(d_2)}{S^2\sigma\sqrt{T}}$$

Same as European Call derivation.

2.3 Vega

$$\mathcal{V} = \frac{\partial^2 P}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT} \frac{\phi(d_2)}{S^2\sigma\sqrt{T}}$$

2.4 Theta

$$\Theta = \frac{\partial P}{\partial T} = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) - qSe^{-qT}\Phi(-d_1)$$

$$\boxed{\rho = \frac{\partial^2 P}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{S\sigma\sqrt{T}} = Ke^{-rT} \frac{\phi(d_2)}{S^2\sigma\sqrt{T}}}$$

3 Binary Call (Cash-or-Nothing)

$$C_{\text{bin}} = Q e^{-rT} \Phi(d_2)$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

 $C_{\rm bin}$ Option price

- Q Fixed cash payout if $S_T > K$
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- σ Volatility
- Φ Standard normal CDF
- ϕ Standard normal pdf

3.1 Delta

$$\Delta = \frac{\partial C_{\text{bin}}}{\partial S} = Q e^{-rT} \frac{\phi(d_2)}{S \sigma \sqrt{T}}$$

3.2 Gamma

$$\Gamma = \frac{\partial^2 C_{\text{bin}}}{\partial S^2} = -Q e^{-rT} \frac{d_1 \phi(d_2)}{S^2 \sigma^2 T}$$

3.3 Vega

$$\mathcal{V} = \frac{\partial C_{\text{bin}}}{\partial \sigma} = -Q e^{-rT} d_1 \phi(d_2)$$

3.4 Theta

$$\Theta = \frac{\partial C_{\text{bin}}}{\partial T} = Q e^{-rT} \left[-r \Phi(d_2) + \frac{\phi(d_2)}{2T} \left(\frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right]$$

$$\rho = \frac{\partial C_{\text{bin}}}{\partial r} = Q e^{-rT} \left(-T \Phi(d_2) + \frac{\sqrt{T}}{\sigma} \phi(d_2) \right)$$

4 Binary Put (Cash-or-Nothing)

$$P_{\text{bin}} = Q e^{-rT} \Phi(-d_2)$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

 $P_{\rm bin}$ Option price

- Q Fixed cash payout if $S_T < K$
- S Spot price
- K Strike price
- q Dividend rate
- T Time to maturity
- r Risk-free rate
- σ Volatility
- Φ Standard normal CDF
- ϕ Standard normal pdf

4.1 Delta

$$\Delta = \frac{\partial P_{\text{bin}}}{\partial S} = -Q e^{-rT} \frac{\phi(d_2)}{S \sigma \sqrt{T}}$$

4.2 Gamma

$$\Gamma = \frac{\partial^2 P_{\text{bin}}}{\partial S^2} = Q e^{-rT} \frac{d_1 \phi(d_2)}{S^2 \sigma^2 T}$$

4.3 Vega

$$\boxed{\mathcal{V} = \frac{\partial P_{\text{bin}}}{\partial \sigma} = Q e^{-rT} d_1 \phi(d_2)}$$

4.4 Theta

$$\Theta = \frac{\partial P_{\text{bin}}}{\partial T} = Q e^{-rT} \left[-r \Phi(-d_2) - \frac{\phi(d_2)}{2T} \left(\frac{\ln(\frac{S}{K}) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right]$$

$$\rho = \frac{\partial P_{\text{bin}}}{\partial r} = Q e^{-rT} \left(-T \Phi(-d_2) - \frac{\sqrt{T}}{\sigma} \phi(d_2) \right)$$

5 American Call

$$C^{A} = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \left(S_{\tau} - K \right)^{+} \right]$$

(equals the European call when q = 0, since early exercise has no value).

 \mathbb{C}^A American call option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

 σ Volatility

 τ Optimal stopping time (0 $\leq \tau \leq T)$

 $\mathbb{E}^{\mathbb{Q}}$ Expectation under the risk-neutral measure

5.1 Delta

5.2 Gamma

5.3 Vega

5.4 Theta

6 American Put

$$P^{A} = \sup_{\tau \in \mathcal{T}[0,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \left(K - S_{\tau} \right)^{+} \right]$$

 P^A American put option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

 σ Volatility

 τ Optimal stopping time (0 $\leq \tau \leq T)$

 $\mathbb{E}^{\mathbb{Q}}$ Expectation under the risk-neutral measure

6.1 Delta

6.2 Gamma

6.3 Vega

6.4 Theta