

Mathematics Behind the Options Calculator

Pheneas Newman

October 7, 2025

1 European Call

$$C = Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\boxed{Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)} \quad (\text{Black-Scholes identity})$$

C Option price

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

1.1 Delta

$$\boxed{\Delta = \frac{\partial C}{\partial S} = e^{-qT}\Phi(d_1)}$$

$$\Delta = \frac{\partial C}{\partial S} = e^{-qT}\Phi(d_1) + Se^{-qT}\phi(d_1)\left(\frac{\partial d_1}{\partial S}\right) - Ke^{-rT}\phi(d_2)\left(\frac{\partial d_2}{\partial S}\right)$$

$$= e^{-qT}\Phi(d_1) + \underbrace{Se^{-qT}\phi(d_1)\frac{1}{S\sigma\sqrt{T}} - Ke^{-rT}\phi(d_2)\frac{1}{S\sigma\sqrt{T}}}_{=0}$$

$$\Delta = e^{-qT}\Phi(d_1)$$

1.2 Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = e^{-qT} (\phi(d_1) \frac{1}{S\sigma\sqrt{T}})$$

$$\begin{aligned}\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} \\ &= e^{-qT} \frac{\partial}{\partial S} (\Phi(d_1)) \\ \Gamma &= e^{-qT} \phi(d_1) \frac{1}{S\sigma\sqrt{T}}\end{aligned}$$

1.3 Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S e^{-qT} \phi(d_1) \sqrt{T}$$

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S e^{-qT} \phi(d_1) \left(\frac{\partial d_1}{\partial \sigma} \right) - K e^{-rT} \phi(d_2) \left(\frac{\partial d_2}{\partial \sigma} \right)$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}} \right) + \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{T(r-q)}{\sigma\sqrt{T}} \right) - \frac{1}{2}\sqrt{T}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T}$$

$$\mathcal{V} = S e^{-qT} \phi(d_1) \sqrt{T}$$

1.4 Theta

$$\Theta = \frac{\partial C}{\partial T} = -e^{-qT} \frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rK e^{-rT} \Phi(d_2) + qS e^{-qT} \Phi(d_1)$$

$$\begin{aligned}
\Theta &= \frac{\partial C}{\partial T} = S \left(-qe^{-qT}\Phi(d_1) + e^{-qT}\phi(d_1)\frac{\partial d_1}{\partial T} \right) - K \left(-re^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial T} \right) \\
\frac{\partial d_1}{\partial T} &= \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} := A \\
\frac{\partial d_2}{\partial T} &= \frac{(\sigma^2 + 2r - 2q)T - 2\ln(\frac{S}{K})}{4\sigma T^{\frac{3}{2}}} - \frac{\sigma}{2\sqrt{T}} := A - \frac{\sigma}{2\sqrt{T}} \\
&= Se^{-qT}(-q\Phi(d_1) + \phi(d_1)A) - Ke^{-rT}(-r\Phi(d_2) + \phi(d_2)(A - \frac{\sigma}{2\sqrt{T}})) \\
&= -qSe^{-qT}\Phi(d_1) + rKe^{-rT}\Phi(d_2) + Ke^{-rT}\phi(d_2)\frac{\sigma}{2\sqrt{T}} + \underbrace{A(Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2))}_{=0} \\
&= Ke^{-rT}\phi(d_2)\frac{\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(d_2) - qSe^{-qT}\Phi(d_1)
\end{aligned}$$

Using the Black-Scholes identity $Se^{-qT}\phi(d_1) = Ke^{-rT}\phi(d_2)$, and expressing Θ as the negative:

$$\Theta = -e^{-qT}\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + qSe^{-qT}\Phi(d_1)$$

1.5 Rho

$$\boxed{\rho = \frac{\partial C}{\partial r} = KTe^{-rT}\Phi(d_2)}$$

$$\begin{aligned}
\rho &= \frac{\partial C}{\partial r} = Se^{-qT}(\phi(d_1)\frac{\partial d_1}{\partial r}) - K(-Te^{-rT}\Phi(d_2) + e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial r}) \\
\frac{\partial d_1}{\partial r} &= \frac{T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma} = \frac{\partial d_2}{\partial r} \\
&= Se^{-qT}\phi(d_1)\frac{\sqrt{T}}{\sigma} + KTe^{-rT}\Phi(d_2) - Ke^{-rT}\phi(d_2)\frac{\sqrt{T}}{\sigma} \\
&= \frac{\sqrt{T}}{\sigma} \left(\underbrace{Se^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2)}_{=0} \right) + KTe^{-rT}\Phi(d_2) \\
\rho &= KTe^{-rT}\Phi(d_2)
\end{aligned}$$

3 Binary Call (Cash-or-Nothing)

$$C_{\text{bin}} = Q e^{-rT} \Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

C_{bin} Option price

Q Fixed cash payout if $S_T > K$

S Spot price

K Strike price

q Dividend rate

T Time to maturity

r Risk-free rate

σ Volatility

Φ Standard normal CDF

ϕ Standard normal pdf

3.1 Delta

$$\Delta = \frac{\partial C_{\text{bin}}}{\partial S} = Q e^{-rT} \frac{\phi(d_2)}{S \sigma \sqrt{T}}$$

3.2 Gamma

$$\Gamma = \frac{\partial^2 C_{\text{bin}}}{\partial S^2} = -Q e^{-rT} \frac{d_1 \phi(d_2)}{S^2 \sigma^2 T}$$

3.3 Vega

$$\mathcal{V} = \frac{\partial C_{\text{bin}}}{\partial \sigma} = -Q e^{-rT} d_1 \phi(d_2)$$

