

Minimal bending energies of bilayer polyhedra

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Motivated by recent experiments on bilayer polyhedra composed of amphiphilic molecules, we study the elastic bending energies of bilayer vesicles forming polyhedral shapes. Allowing for segregation of excess amphiphiles along the ridges of polyhedra, we find that bilayer polyhedra can indeed have lower bending energies than spherical bilayer vesicles. However, our analysis also implies that, contrary to what has been suggested on the basis of experiments, the snub dodecahedron, rather than the icosahedron, generally represents the energetically favorable shape of bilayer polyhedra.

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In an aqueous environment, amphiphilic molecules such as lipids are observed to self-organize into bilayer vesicles [1, 2], thus forming the physical basis for cell membranes. Bilayer vesicles generally exhibit shapes with constant or smoothly varying curvature [3, 4]. But in recent experiments [5, 6], bilayer vesicles with polyhedral shape, consisting of flat faces connected by ridges and vertices with high local curvature, have been observed. In these experiments, two types of oppositely charged, single-tailed amphiphiles were used, with a slight excess of one amphiphile species over the other. At high temperatures, the amphiphiles were found to form spherical bilayer vesicles. However, provided that the number of excess, unpaired amphiphiles was tuned to some optimal range, cooling the system below the chain melting temperature yielded polyhedral bilayer vesicles. It was reported that the bilayer polyhedra were stable over weeks, and that their shape was consistently reproduced upon thermal cycling. Furthermore, it was suggested [5, 6] that the observed polyhedra had icosahedral symmetry, although some uncertainty regarding the polyhedral symmetry remained.

What is the mechanism governing the formation and symmetry of bilayer polyhedra? It was argued [5, 6] on the basis of the experimental phenomenology that elastic contributions to the polyhedron free energy dominate over entropic or electrostatic contributions, and that minimization of elastic bending energy alone determines the shape of bilayer polyhedra. In this Letter, we take these intriguing observations as our starting point and address the general problem of finding polyhedral shapes with minimal bending energy. Questions regarding the minimal energy shape of bilayer vesicles are commonly answered using a variational approach [1–3]. However, a given polyhedral shape is defined by the geometric parameters characterizing its vertices and ridges. Thus, polyhedra are inherently of singular nature, which severely restricts the applicability of variational calculus. We therefore employ a complementary method, in which we allow for ridges and vertices with arbitrary geometric properties and, on this basis, calculate polyhedron bending energies as a function of polyhedron symmetry. In the remainder of this Letter, we first consider the most straightforward case of bilayers with *uniform* composition, and then turn to the richer case in which there is segregation of excess amphiphiles.

The solution of the two-dimensional equations of elasticity is a formidable challenge, and has only been achieved for the vertices and ridges of polyhedra in certain limiting cases [7–11]. Thus, in order to determine the elastic energies of arbitrary polyhedral shapes, we mainly employ simple expressions based on the Helfrich-Canham-Evans free energy of bending [1–3],

$$G = \frac{K_b}{2} \int dS \left(\frac{1}{R_1} + \frac{1}{R_2} - H_0 \right)^2, \quad (1)$$

where K_b is the bilayer bending rigidity, R_1 and R_2 are the two principal radii of curvature, and H_0 is the spontaneous curvature. The resulting expressions for ridge and vertex energies are intuitive and only involve a few parameters but, ultimately, are purely phenomenological. We assess their validity by making comparisons to the aforementioned limiting cases, which allow for stretching as well as bending deformations.

Figure 1(a) shows schematic illustrations of a bilayer bending gradually (left panel) and sharply (right panel) along a ridge with dihedral angle α_i . The first model is inspired by the electron micrographs of bilayer polyhedra in Refs. [5, 6, 12], while the second model provides a more faithful representation of the polyhedral geometry. Based on the picture presented in the left panel of Fig. 1(a), we approximate ridges by a bilayer bending partially around a cylinder of radius $R_1 = d/(\pi - \alpha_i)$, where d is the arc length. Following the right panel, we discretize the bilayer with a lattice spacing b , and assume a harmonic potential for the angle between adjacent bond vectors. Upon setting $d = b$ one finds [13] from either approach a ridge energy similar [14] to the expression used in Ref. [6],

$$G_r = \frac{\bar{K}_b}{2} (\pi - \alpha_i)^2 l, \quad (2)$$

where $\bar{K}_b = K_b/b$, l is the ridge length, and we have assumed that $H_0 = 0$. A simple expression for the bending energy associated with closed bilayer vertices [see Fig. 1(b)] can be obtained [13] from Eq. (1) following analogous steps, leading to the vertex energy

$$G_v = \frac{K_b}{2} \sum_j (\pi - \beta_j)^2, \quad (3)$$

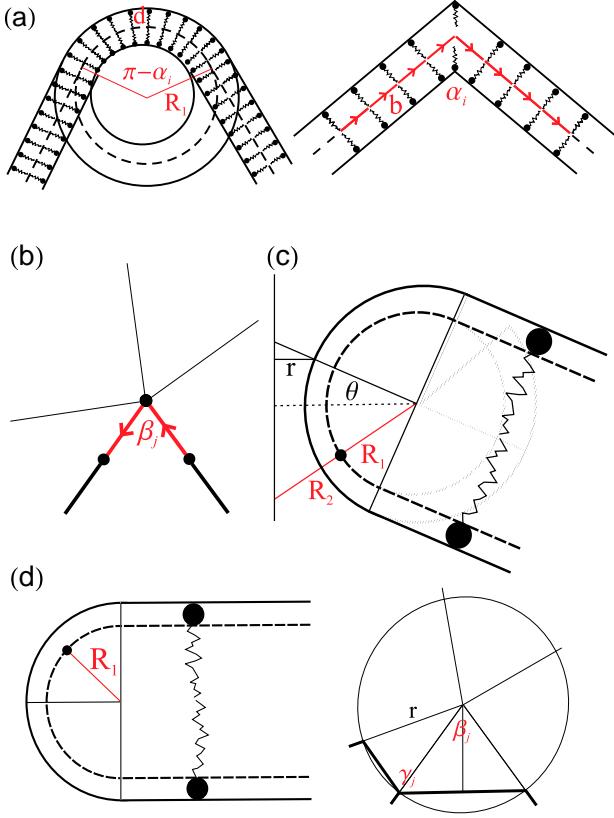


FIG. 1: (color online). Illustration of the contributions to the elastic bending energies of polyhedra [13]: (a) Side view of a ridge with dihedral angle α_i , (b) top-down view on a vertex with face angle β_j , (c) side view of a pore around the tip of a cone with apex angle $\pi - 2\theta$ and radius r , (d) side view (left panel) and top-down view (right panel) of a pore composed of straight edges along each face.

where β_j denotes the face angle subtended by two ridges meeting at a given polyhedron vertex.

Closed bilayer vertices may break up to form pores [5, 6, 12], which was suggested [5] as a mechanism for avoiding the curvature singularity associated with closed vertices. Figure 1(c,d) show two models for pores at the vertices of polyhedra which, similarly as before, are inspired by the experimental images in Refs. [5, 6, 12] [Fig. 1(c)] and a stricter interpretation of the polyhedral geometry of bilayer vesicles [Fig. 1(d)]. In our first model [see Fig. 1(c)] we approximate the vertex of a given polyhedron by a cone with apex angle $\pi - 2\theta$, where $\theta = \pi/2 - \arccos(1 - \Omega/2\pi)$ for a solid angle Ω subtended by the polyhedron vertex. In our second model we assume that, along each face, the pore consists of a straight cylindrical edge [see Fig. 1(d), left panel], which bends through an angle γ_j across a ridge from one face to a neighboring face [see Fig. 1(d), right panel]. In both cases, the elastic pore energy G_p can be evaluated on the basis of Eq. (1) and is found [13] to depend on the pore radius, the monolayer bending rigidity, K_b^* , and the monolayer spontaneous curvature, H_0^* , as well as the face angle and solid angle characterizing the geometry of a given polyhedron. For physi-

cally relevant parameter ranges, G_p increases with decreasing H_0^* and Ω .

We now turn to the presence of excess amphiphiles in bilayer polyhedra. The expulsion of excess amphiphiles from flat bilayers and the resulting molecular segregation, together with the high spontaneous curvature of single-tailed excess amphiphiles, are thought [5, 6] to have two principal effects on the bending energies of polyhedra. On the one hand, excess amphiphiles can seed pores into bilayers [5, 6] and, thus, pores may have a role beyond reducing the elastic energy of polyhedron vertices. In particular, it has been suggested [6] that excess amphiphiles produce pores in the spherical bilayer vesicles from which bilayer polyhedra originate upon cooling. On the other hand, it has been found [6] that excess amphiphiles preferentially accumulate along the ridges of polyhedra. As a result, molecular segregation can decrease the bending energy of the outer monolayer at ridges.

The above observations suggest a simple description of how vertex and ridge energies are modified by the presence of excess amphiphiles. Ideally, excess amphiphiles are arranged along ridges such that they induce an anisotropic spontaneous curvature commensurate with the dihedral angle. Assuming such “perfect segregation”, only the bending energy of the inner layer must be considered when computing ridge energies. We therefore obtain [13] a lower bound on the modified ridge energy which takes a similar form as Eq. (2), but with the rescaled *bilayer* bending modulus \bar{K}_b replaced by the rescaled *monolayer* bending modulus $\bar{K}_b^* = K_b^*/b$. Thus, provided that the optimal amount of excess amphiphiles is present [5, 6], the ridge energy is lowered by a factor K_b^*/K_b , with the number of pores seeded into spherical vesicles equal to or greater than the number of polyhedron vertices. Experiments [5, 6] and simulations [15] suggest that $K_b^*/K_b \lesssim 10^{-2}$.

From the simple model for perfectly segregated bilayer polyhedra described above one finds [13] that the optimal ratio of the amphiphile species in excess to the total amphiphile content is given by $r_I \approx 0.52$ for the polyhedron sizes observed in experiments [5, 6]. This optimal value for r_I is a direct result of amphiphile and polyhedron geometry and, hence, does not depend on any elastic parameters. The corresponding experimental estimate is $r_I \approx 0.57$ [5, 6]. We expect that in experiments not all excess amphiphiles are segregated along the ridges and vertices of polyhedra as a result of, for instance, entropic mixing within bilayer polyhedra or the formation of micelles [6]. Thus, our theoretical estimate for r_I is in broad agreement with experimental observations.

For certain limits of the equations of elasticity, approximate solutions corresponding to polyhedron vertex [7–9] and ridge [10, 11] energies have been obtained. In particular, it has been found [7] that 5-fold disclinations in hexagonal lattices are accommodated for small lattice sizes through a stretching of lattice vectors. However, for large enough lattice sizes it becomes energetically favorable to buckle out of the plane [7], in which case the energetics of the system are dominated by bending. This behavior is characterized by a dimensionless quantity known as the Föppl-von Kármán number

TABLE I: Vertex energies G_v and ranges of pore energies G_p [13] at zero pore radius in units of bilayer bending modulus K_b for the five Platonic solids with parameter values corresponding to the experiments in Refs. [5, 6].

Platonic solid	G_v	G_p [Fig. 1(c)]	G_p [Fig. 1(d)]
Tetrahedron	6.6	$8.7 \frac{K_b^*}{K_b} - 16 \frac{K_b^*}{K_b}$	$3.3 \frac{K_b^*}{K_b} - 8.1 \frac{K_b^*}{K_b}$
Cube	3.7	$6.0 \frac{K_b^*}{K_b} - 11 \frac{K_b^*}{K_b}$	$7.4 \frac{K_b^*}{K_b} - 16 \frac{K_b^*}{K_b}$
Octahedron	8.8	$6.4 \frac{K_b^*}{K_b} - 12 \frac{K_b^*}{K_b}$	$4.4 \frac{K_b^*}{K_b} - 11 \frac{K_b^*}{K_b}$
Dodecahedron	2.4	$3.9 \frac{K_b^*}{K_b} - 8.1 \frac{K_b^*}{K_b}$	$11 \frac{K_b^*}{K_b} - 21 \frac{K_b^*}{K_b}$
Icosahedron	11	$4.3 \frac{K_b^*}{K_b} - 8.7 \frac{K_b^*}{K_b}$	$5.5 \frac{K_b^*}{K_b} - 14 \frac{K_b^*}{K_b}$

$\Gamma = YR^2/K_b$, where Y is the two-dimensional Young's modulus and R is the lattice size. As $\Gamma \rightarrow \infty$, pronounced ridges develop between the 5-fold disclination sites of icosadeltaedral triangulations of the sphere [8, 9], and the overall shape resembles that of the icosahedron. In this limit, the total elastic energy is dominated by ridges and was determined in Refs. [10, 11] to be of the form $K_b(\pi - \alpha)^{7/3}l^{1/3}f(Y/K_b)$, with $0.1 \lesssim f(Y/K_b) \lesssim 0.4 \text{ nm}^{-1/3}$ for bilayer polyhedra [13]. Thus, the asymptotic expression for the ridge energy found in Refs. [10, 11] leads to a similar dependence on the dihedral angle and proportionality factor as in Eq. (2), while increasing sub-linearly with the ridge length l .

In Table I we compile the values of the bilayer vertex energy G_v in Eq. (3) for the five Platonic solids, and compare these values to the corresponding elastic energies of pores with zero radius. The lowest energy states of icosadeltaedral triangulations of the sphere are found to resemble icosahedra for $\Gamma \gtrsim 10^7$ [8, 9], which corresponds to a vertex energy greater than $8K_b$ with, for instance, a value $12K_b$ for $\Gamma = 10^{10}$. This compares quite favorably with our estimate $G_v \approx 11K_b$ for the icosahedron. Moreover, note from Table I that our two models for pores predict similar ranges for the pore energy G_p , with the competition between pores and closed bilayer vertices governed by the ratio K_b^*/K_b . Using the aforementioned estimate $K_b^*/K_b \lesssim 10^{-2}$ [5, 6, 15], we find that closed bilayer vertices will be unstable to the formation of (closed) pores, thus removing the singularity associated with polyhedron vertices. This is consistent with experimental observations [5, 6, 12] and allows adjustment of the volume of bilayer polyhedra for a fixed total area or number of amphiphiles.

Ridges impose an energetic cost and, hence, one expects that for a fixed area and dihedral angle the faces of polyhedra relax to form regular polygons. We therefore focus here on the convex polyhedra with regular polygons as faces, but we have also considered polyhedra with irregular faces [13]. The class of convex polyhedra with regular faces encompasses the five Platonic solids, the thirteen Archimedean solids, the two (infinitely large) families of prisms and antiprisms, and the 92 Johnson solids [16, 17]. It has been shown [18] that this list exhausts all convex polyhedra with regular faces. Thus, counting prisms and antiprisms as one solid each, there are exactly

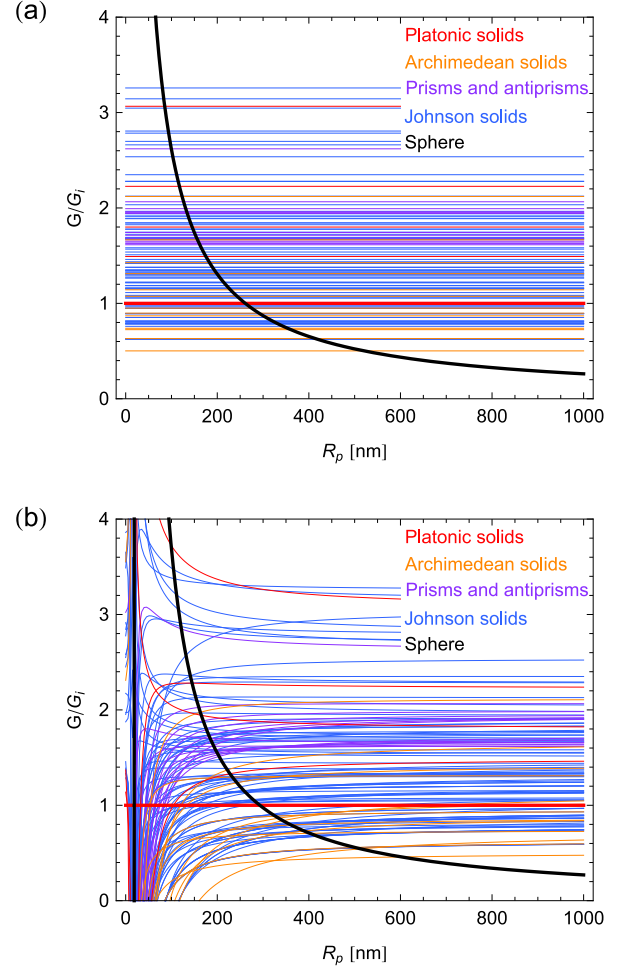


FIG. 2: (color online). Total elastic bending energies of the convex polyhedra with regular faces, obtained [13] for the case of perfect segregation of excess amphiphiles, and bending energy of the sphere, normalized by the total bending energy of the icosahedron, G_i , with (a) pores with $r = 0 \text{ nm}$ and (b) pores with $r = 20 \text{ nm}$ at each polyhedron vertex.

112 convex polyhedra with regular faces. With each of these polyhedra a specific set of parameters characterizing ridges, vertices, and faces is associated, leading to distinct contributions to the total elastic bending energy.

We have evaluated the total elastic bending energies of all convex polyhedra with regular faces. As illustrated in Fig. 2, polyhedron energies are compared by plotting elastic energy as a function of the polyhedron radius R_p [8], which is related to the polyhedron area A via $A = 4\pi R_p^2$, such that, for each value of R_p , all shapes have the same total area. While the quantitative details of the resulting energy curves depend on the particular combination of the aforementioned expressions for ridge, vertex, and pore energy used, we find [13] that all curves share the same basic qualitative features. Consistent with a previous study [6], the icosahedron [see Fig. 3(a)] minimizes bending energy among the Platonic solids. However, we also find that, in general, the icosahedron does *not* mini-

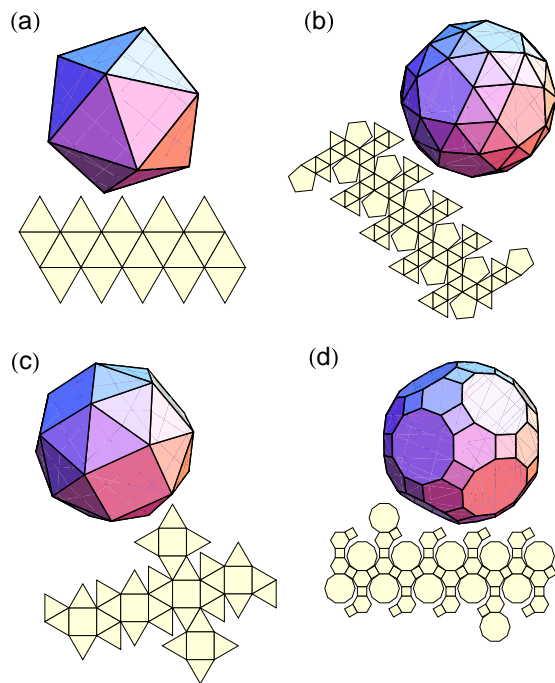


FIG. 3: (color online). Image and net representations of (a) the icosahedron, (b) the snub dodecahedron, (c) the snub cube, and (d) the great rhombicosidodecahedron. The polyhedra in (b) and (c) are chiral.

mize bending energy among arbitrary polyhedral shapes. In fact, for large enough polyhedron sizes, the snub dodecahedron [see Fig. 3(b)] is the polyhedral shape minimizing bending energy among the convex polyhedra with regular faces, and the snub cube [see Fig. 3(c)] also has a lower energy than the icosahedron in this limit.

Allowing for an optimal number of excess amphiphiles (see Fig. 2), we find that polyhedra can have lower bending energies than the sphere, but only [13] if we permit molecular segregation along *ridges* as observed in Ref. [6]. Segregation at pores, which was originally suggested in Ref. [5] as a potential mechanism stabilizing polyhedral shapes, is *not* sufficient to produce polyhedra with bending energies which are favorable compared to the sphere. Indeed, if we assume that pores are closed, bilayer polyhedra are energetically favorable for the experimentally observed polyhedron radius $R_p \approx 500$ nm [see Fig. 2(a)], with the snub dodecahedron as the minimum energy shape among the convex polyhedra with regular faces. If we allow pores of a finite size, a sequence of polyhedral shapes is obtained which minimize bending energy for smaller polyhedron radii [see Fig. 2(b)]. The most notable of these polyhedral shapes is the great rhombicosidodecahedron [see Fig. 3(d)], which surpasses the snub dodecahedron in bending energy at $R_p \approx 300$ nm. However, according to our analysis, the snub dodecahedron represents the minimum in bending energy among the convex polyhedra with regular faces for the polyhedron sizes and pore sizes ($r \approx 20$ nm) found experimentally [5, 6, 12].

In summary, we have used Eq. (1) to systematically evaluate the bending energies of bilayer polyhedra. Allowing for the segregation of excess amphiphiles along ridges, we indeed find that bilayer polyhedra can have lower bending energies than spherical vesicles. But we also find that, contrary to what has been suggested on the basis of experiments [5, 6], the snub dodecahedron and the snub cube generally have lower total elastic bending energies than the icosahedron. This result is consistent with several independent theoretical studies which suggest that the elastic energies of chiral shapes such as the snub dodecahedron and the snub cube can be favorable compared to the icosahedron [19–21] and that, even if the icosahedral shape is imposed, the minimum energy structure may still be chiral [22]. In light of our theoretical results, we suggest revisiting the symmetry and energetics of bilayer polyhedra in greater experimental detail.

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