Contents

1. The Standard Model and Physics Beyond it.

To comprehend several of the calculations carried out in this dissertation, it is important to lay out a strong foundation of several important concepts in particle physics. In the sections that follow, the main concepts associated with the SM are explained.[1]

1.1 Lagrangian and Hamiltonian Mechanics

1.1.1 Lagrangian Mechanics

Before 1788 physicists and mathematicians used Newton's formalism to describe all physical systems. However, its usage becomes complicated if one wants to describe multiparticle, or complex, systems. One of the most efficients solutions was found by Joseph Lagrange in 1788.

Lagrange found that it was possible to obtain second order Equations of Motion (EOM), by the means of a functional referred to as the Lagrangian. In this section we will make a brief derivation of the Lagrange equations of motions, using calculus of variations. If the reader wants to extend his knowledge on this formalism we suggest reading [1].

Let us begin with simple example of complex systems, the double pendulum of Figure 1.1.

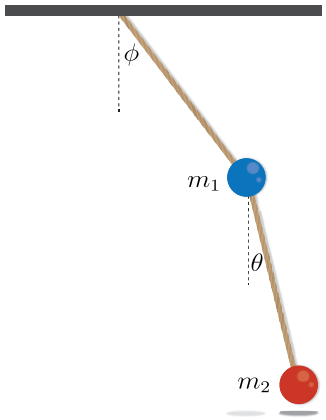


Figure 1.1: Double pendulum consisting of two masses, m_1 and m_2, at constant lengths

Note that if we used Newtonian formalism, we would need all four cartesian coordinates to describe the dynamics of the system. However, one can reduce the number of coordinates, or degrees of freedom, by noting that it is possible to describe the system, if we only use the angles θ and ϕ, which are more general and reduce the complexity of the system. In other words we say

[1] From now on, we use natural units. This means that we set $\hbar = c = 1$.

that our cartesian coordinates can be written as functions of more general parameters, referred to as generalized coordinates:

$$\vec{r}_i(x_i, y_i, t) = \vec{r}_i(\theta, \phi, t); \quad i = 1, 2 \tag{1.1}$$

Thus, we define the generalized coordinates as the set of the most general and simple degrees of freedom that describe a physical system.

This result can be extrapolated to more variables, which are useful to describe more complex systems such as the spherical pendulum. If the system can be described by a set of n generalized coordinates, the cartesian coordinates can be written as

$$\vec{r} = \vec{r}(q_1, q_2, \cdots, q_n, t) \tag{1.2}$$

Given these coordinates, it becomes natural to wonder how to derive the equations of motion in terms of them. To do so, let us consider a particle moving in 2D, from a point a to a point b, as seen in Figure 1.2. This particle can take any possible path, so it is natural to wonder which of these paths is the one that takes the least distance.

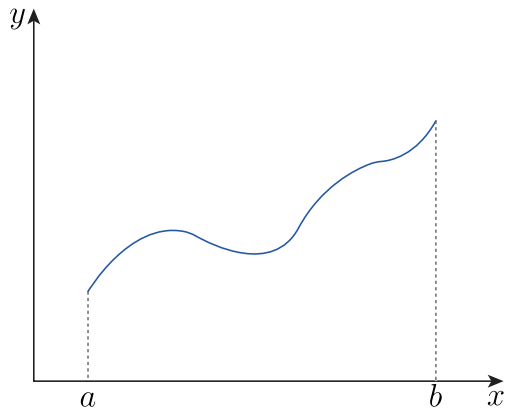


Figure 1.2: Particle moving in 2D from point a to point b

To do so, let us recall from basic calculus that the traveled distance is given by the arc length

$$L = \int_a^b ds = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{1.3}$$

Note that the squared root is a function of another function y(x), i.e it is a functional f(y,y',x). In terms of functionals, the arc length takes the form

$$L = \int_a^b f(y, y', x)\mathrm{d}x \tag{1.4}$$

To solve this problem, let us consider another path $\gamma(x)$, which deviates from the correct one in

terms of a function η [1]:

$$\gamma(x) = y(x) + \alpha\eta(x) \tag{1.5}$$

From differential calculus, it is known that the correct path will be a minimum of L with respect to the deviation parameter α:

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$$

This implies

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b f(\gamma, \gamma', x)\mathrm{d}x = \int_a^b \frac{\mathrm{d}f}{\mathrm{d}\alpha}\mathrm{d}x \tag{1.6}$$

Before performing the integration, the derivative of the f functional can be calculated using chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} = \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial \alpha} + \frac{\partial f}{\partial \gamma'}\frac{\partial \gamma'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} \tag{1.7}$$

$$= \frac{\partial f}{\partial \gamma}\eta(x) + \frac{\partial f}{\partial \gamma'}\eta'(x) \tag{1.8}$$

Hence, the integral becomes

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \int_a^b \left[\frac{\partial f}{\partial \gamma}\eta(x) + \frac{\partial f}{\partial \gamma'}\eta'(x)\right] = 0 \tag{1.9}$$

However, this relation is written in terms of derivatives of the deviation function, which is still arbitrary. The best way to get rid of these derivatives is to use partial integration on the second term

$$\int_a^b \frac{\partial f}{\partial \gamma'}\eta'(x)\mathrm{d}x = \int_a^v \frac{\partial f}{\partial y'}\eta'(x)\mathrm{d}x = \frac{\partial f}{\partial y'}\eta(x)\bigg|_a^b - \int_a^b \eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x \tag{1.10}$$

$$= -\int_a^b \eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\mathrm{d}x \tag{1.11}$$

This means that Equation 1.9 becomes

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\mathrm{d}x = 0 \tag{1.12}$$

As the deviation function is arbitrary, it follows that the remaining part of the integrand must vanish

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{1.13}$$

This equation is known as the Euler-Lagrange equation. Now, if we return to physics, we can change the problem to wonder which path takes the least energy.

According to William Hamilton, this problem can be solved if the arc length is replaced by a physical quantity, the action S, and the functional f is replaced by the Lagrangian L [1]. This means that the least energy principle is equivalent to the least action. This principle can be written as

$$S = \int_{t_a}^{t_b} L(q, q', t)\mathrm{d}t \quad \rightarrow \quad \delta S = 0 \tag{1.14}$$

This implies that the Lagrangian must follow the Euler-Lagrange equation [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.15}$$

However, the Lagrangian can be any functional. This problem can be easily solved by the means of D'Alembert's principle, whose derivation we will not make, as a more sophisticated treatmet can be found in [2]. Due to this principle, it is possible to write the Lagrangian as

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, \dot{q}, t) \tag{1.16}$$

where T and V are the kinetic and potential energies as function of the generalized coordinates, and time.

In this formalism, both angular and linear momentum arise explicitly from the Euler-Lagrange equations as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.17}$$

Note that the time derivative of these momenta vanishes, if their associated coordinate q_i does not explicitly appear on the Lagrangian. In other words, momentum is conserved if it's generalized coordinate does not appear in the Lagrangian, i.e it is cyclic. The fact that conserved quantities arise from cyclic coordinates is the first gimplse of a more formal prescription of them, known as the Noether theorem.

1.1.2 Hamiltonian Mechanics

Even though the Lagrangian formalism of classical mechanics simplifies how one finds the EOM, they are second order differential equations, which usually are not easy to solve analytically. This problem can be solved if we apply a Legendre transformation to the Lagrangian [1]. In other

[2] Also referred to as the Equation Of Motion (EOM)

words, we define a new function, referred to as the Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - L \tag{1.18}$$

Note that if we are describing a conservative system, i.e with no frictional forces, then H will be the total energy of it [2]:

$$H = \sum_i p_i \dot{q}_i - L = T + V \tag{1.19}$$

Now that we have defined a new functional, we shall now try to obtain the EOM. To do this, let us analyze -in 1D- how the Hamiltonian changes with respect to both q and p:

$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} \tag{1.20}$$

Note that the first derivative on the third term is the definition of the momentum. This cancels the first and third term, leaving us with

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \tag{1.21}$$

This equation seems simpler than those obtained from the Euler-Lagrange equations. However, we want the equations of motion to depend only on the Hamiltonian, not on the Lagrangian. This can be done by using the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \rightarrow \frac{\partial L}{\partial q} = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{1.22}$$

Hence, our first equation of motion becomes

$$\frac{\partial H}{\partial q} = -\dot{p} \tag{1.23}$$

Finally, if we differentiate with respect to p, we would obtain a second EOM

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1.24}$$

These two EOM form a set of first order differential equations, known as the Hamilton equations.

Hamiltonian and Lagrangian formalisms give us new ways of finding the EOM of a given system, which also agree with Newton's second law. However, we have only studied both formalisms when the systems are formed by a countable set of particles. This means that we need to extend both our formalisms to an infinite set of particles or to more complex systems. The description of complex systems is known as Classical Field Theory (CFT), and we will study it from now on.

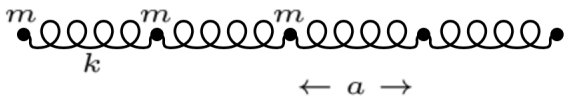


Figure 1.3: String modeled as a collection of particles of mass m, separated by springs with elasticity constant k.

1.1.3 Classical Field Theory

As our goal is to study complex systems, let us first consider the simplest of them, namely a 1D string. If we want to properly study it, let us consider it as an infinite collection of points with mass m, connected by springs with elasticity constant k, each of them separated by a distance a, as seen in Figure 1.3. Note that if we consider a displacement ϕ, the Lagrangian of the system will be given by

$$L = \frac{1}{2}\sum_{i=1} m\dot{\phi}_i^2 - k(\phi_{i+1} - \phi_i)^2 \tag{1.25}$$

This Lagrangian can be rewritten in terms of the separation, a, as

$$L = \frac{1}{2}\sum_{i} a\left[\frac{m}{a}\phi_i^2 - (ka)\left(\frac{\phi_{i+1} - \phi_i}{a}\right)\right] \tag{1.26}$$

If we let the separation to go to zero, it will be possible to us to redefine the Lagrangian in terms of the so-called Young modulus $Y = ka$, and the linear mass density $\mu = \frac{m}{a}$. It is also possible to note that the sum will be replaced by an 1D integral

$$L = \frac{1}{2}\int \mathrm{d}x\left[\mu\dot{\phi}^2 - Y\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right] \tag{1.27}$$

Note that the terms in the integral resemble a linear density for the Lagrangian. This allows us to define the 1D Lagrangian density as

$$L = \int \mathrm{d}x\mathcal{L} \tag{1.28}$$

This changes the action to a double integral:

$$S = \int \mathrm{d}t\mathrm{d}x\mathcal{L} \tag{1.29}$$

From now on we will refer to the Lagrangian density to just the Lagrangian.

Note that the ϕ value will change with the position of a specific point, as well as with time,

making it a field. We can think of a 3D field as a grid that exists through a certain region of space, as seen in Figure 1.4.

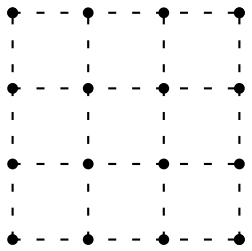


Figure 1.4: Representation of a 2D field as a grid.

In other words, there is a direct map between the generalized coordinates and fields in complex systems. This means that we can map,

$$q_i \to \phi(x) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \to \partial_\mu \qquad \dot{q}_i \to \partial_\mu \phi,$$

such that the Euler-Lagrange equations, using Einstein's notation [ref:specialrelativity], become:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial L}{\partial \phi} \tag{1.30}$$

For the moment we have seen that it is possible to have a Lagrangian description for the mechanics of classical fields. However, we also need a Hamiltonian description associated to the dynamics of a classical field. For the string that we were using before, the Legendre transformation yields:

$$H = \sum_i p_i \varphi_i - L \tag{1.31}$$

In the continuum limit $a \to 0$, we can define the canonical field as

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \tag{1.32}$$

such that we can define the Hamiltonian density, or Hamiltonian as

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L} \tag{1.33}$$

1.1.4 Noether's Theorem

When we were introducing the Lagrangian formalism, we mentioned that if a variable does not appear explicitly in the Lagrangian, then its momentum will be conserved. Now, we shall see what happens if we now consider continuous transformations of the fields. Let us consider a Lagrangian,

that depends on a field ϕ as well as from its first order derivatives

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$$

If we want to analyze continuous transformations, we need to focus on an infinitesimal transformation. However, there are two possible transformations that can be done, namely pure field transformations as well as space-time transformations. An infinitesimal transformation for the fields can is defined by [3]

$$\phi - \rightarrow \phi' = \phi + \theta \Delta\phi, \tag{1.34}$$

and will also change the Lagrangian to a new value $\mathcal{L}'$ As we are transforming the fields, we want these transformations to leave the action invariant, and thus the Euler-Lagrange equations. In other words, we need to ensure

$$\Delta S = \int \mathrm{d}^4 x (\mathcal{L}' - \mathcal{L}) = \int \mathrm{d}^4 x \Delta L = 0 \tag{1.35}$$

If we now use calculus of variations we obtain that the Lagrangian variation is given by

$$\Delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\partial(\Delta\phi) \tag{1.36}$$

This implies

$$\Delta S = \int \mathrm{d}^4 x \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\partial(\Delta\phi) = 0 \tag{1.37}$$

If we now use partial integration on the second term we obtain

$$\Delta\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi\right) + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\Delta\phi \tag{1.38}$$

As we require ϕ to be a solution of the Euler-Lagrange equations, the second term in Equation 1.38 vanishes. This gives us

$$\Delta\mathcal{S} = \int \mathrm{d}^4 x \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi\right) \tag{1.39}$$

As we require ΔS to vanish, the integrand must be zero:

$$\partial_\mu j^\mu = 0 \rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi \tag{1.40}$$

As this equation takes the form of a conservation law, we say that j^μ is then a conserved current

associated to a given symmetry. Now, let us consider space-time transformations

$$x_\mu \to x'_\mu = x_\mu - a_\mu; \quad a_\mu \in \mathbb{R}^{1,3} \tag{1.41}$$

As we are dealing with infinitesimal transformations, the Lagrangian and the field can be expanded in a Taylor series up to first order:

$$\mathcal{L}(x') = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}(x)) \qquad \phi(x') = \phi(x) + a^\mu \partial_\mu \phi(x) \tag{1.42}$$

Hence, Hamilton's principle will now read

$$\Delta S = \int \mathrm{d}^4 x (\mathcal{L}(x') - \mathcal{L}(x)) + \Delta L \tag{1.43}$$

By replacing our previous relations we obtain that the conserved current will take the form

$$J^\mu = T^\mu_\nu \Delta x^\nu \to T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{1.44}$$

Note that this new conserved current depends explicitly on a tensor, referred to as the energy-momentum tensor. Finally, if we consider that Equation 1.40 only arises from pure field transformations, and that Equation 1.44 from pure space-time transformations, we obtain a general expression for the conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + T^\mu_\nu \Delta x^\nu \tag{1.45}$$

Note that we have found a very important result, namely that each symmetry associated to a continuous transformation will give us a conserved current. This result was first found by Emmy Noether in the 19^{th} century, and is known as Noether's Theorem. Noether's theorem changed the way physicists work, giving more importance to the concept of symmetry (and therefore of group). Through this dissertation we will analyze how symmetries arise in particle physics, helping us to construct the SM.

1.2 Relativistic Quantum Mechanics

1.2.1 The Klein-Gordon equation

In the non relativistic limit, the dynamics of a particle, with given energy E and located on a certain region of space, is described by the Schrödinger equation[3]

$$\frac{-1}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = i\frac{\partial}{\partial t}\psi(\vec{r}) \tag{1.46}$$

This equation can be easily obtained from the classical Hamiltonian, and promoting the momentum, energy and position vectors to operators, as described by

$$H = \frac{p^2}{2m} + V; \quad p = -i\nabla \quad E = i\partial_t$$

Even though Schrödinger's equation describes with great precision phenomena such as the Hydrogen Atom, and its perturbative corrections, it does not describe any system in the relativistic regime, neither a system with multiple particles, nor processes of creation or annihilation of particles in the vacuum.

To describe any possible relativistic system, we require to use Einstein's energy-momentum relation

$$E^2 = p^2 + m^2 \tag{1.47}$$

If we now promote our momentum and energy variables to operators, we obtain

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2} = -\nabla^2 + m^2 \rightarrow \frac{\mathrm{d}^2}{\mathrm{d}t^2} - \nabla^2 + m^2 = 0 \tag{1.48}$$

Note that the difference of the derivatives can be reduced by using Minkowski notation for 4-vectors,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \nabla^2 = \mathrm{d}_t^2 - \sum_{i=1}^{3}\mathrm{d}_{x_i}^2 = \sum_{\mu=0}^{3}\eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu = \Box, \tag{1.49}$$

and the Minkowski metric with (+,-,-,-) signature:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \tag{1.50}$$

Thus, we can reduce Equation 1.48 into a more compact, Lorentz invariant, expression that we can evaluate on a single particle function ϕ. This gives us the so called Klein-Gordon equation of

[3]Here we use natural units where c=$\hbar$=1

motion, or KG equation, for a particle moving freely through space-time:

$$\Box\phi + m^2\phi = 0 \tag{1.51}$$

It may seem that KG equation gives us a perfect description for studying single particle quantum mechanics in the relativistic realm. However, we shall see that this prescription fails to describe single particle dynamics. To do so, let us try to build a probability conservation law, given that the KG equation has solutions of the form

$$\phi \propto e^{-ipx} = e^{ip_\mu x^\mu} = e^{i(\vec{p}\cdot\vec{x}-Et)}; \quad p^\mu = (E, \vec{p})$$

Now, let us define probability density and current as [4]

$$\rho = i(\phi^*\partial_t\phi - \phi\partial_t\phi^*); \vec{J} = -i(\phi^*\nabla\phi - \phi\nabla\phi^*) \tag{1.52}$$

such that we can algebraically reduce the KG equation to a conservation law

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\nabla\cdot\vec{J} \to \partial_\mu J^\mu = 0 \tag{1.53}$$

Nevertheless, this conservation of probability is ill defined. This can be seen if we now use a plane wave expansion of the KG wave function, and evaluate the probability density. Note that it will give us $\rho = \pm|E|$, which for negative values is not well defined.

As we found a problem with the definition of probability densities and currents, it may seem that we have to neglect the KG equation as a relativistic description of quantum systems. However, Einstein's energy-momentum equation will give us a clue on what to do. Note that Equation 1.47 implies that particles can be created, and consequently annihilated, in the same space-time point. This obligates ϕ to not be a single particle wave function, but a field.

More specifically, we say that ϕ is a scalar field, and therefore representation, due to its transformation law under the Poincaré group. But first, we shall remember that the Poincaré group is defined as the semidirect product of the Lorentz group, O(1,3), with the set of space-time translations:

$$IO(1,3) = \mathbb{R}^{1,3} \rtimes O(1,3) = \{(a,\Lambda) : a \in \mathbb{R}^{1,3}, \Lambda \in O(1,3)\} \tag{1.54}$$

Then, a Poincaré transformation of the position vector is given by

$$x_\mu \to x'_\mu = \Lambda^\mu_\nu x^\nu + a_\mu \tag{1.55}$$

Now, we say that a field ϕ is a scalar representation under the Poincaré group if it transforms as

$$\phi'(x') = \phi'(x'_\mu) = \phi(x) \tag{1.56}$$

Along this whole chapter we will study other two possible representations of the Poincaré group,

namely the vector and spinor representations.

Finally, it is our duty to build a Lorentz-invariant Lagrangian, that gives us the KG equation after applying the Euler-Lagrange equations. First, we can construct a mass term, which is purely scalar and thus Lorentz-invariant, by multiplying ϕ with itself:

$$\mathcal{L}_{mass} = \frac{1}{2}m^2\phi^2 \tag{1.57}$$

Additionally, we can construct a kinetic term, which is quadratic in the derivatives, by contracting the space-time derivative of the field with itself. As it is a contraction over space-time indices, it is also manifestly Lorentz invariant:

$$\mathcal{L}_{kin} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{1.58}$$

Both terms give us a Lorentz invariant Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 \tag{1.59}$$

To verify that this Lagrangian gives the correct EOM, we apply the Euler-Lagrange equation:

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\delta^\mu_\nu\partial^\nu\phi = \partial^\mu\phi \\ &\frac{\partial\mathcal{L}}{\partial\phi} - = m^2\phi \end{aligned} \tag{1.60}$$

Hence,

$$(\Box + m^2)\phi = 0 \tag{1.61}$$

1.2.2 The Dirac equation

The negative probability density problem that originates from the KG equation made many physicists to look for another relativistic equation that describes the quantum realm. Among these physicists was Paul Dirac, whose solution was the most successful of all. Dirac's solution describes perfectly not only the dynamics of a relativistic particle, more specifically a spin-1/2 one, but also coupled to Maxwell's electrodynamics via symmetry transformations. In this section, we will derive the Dirac equation and study how it's solutions transform under the Poincaré group.

First, let us consider Equation 1.47 in terms of the momentum vector:

$$p^\mu p_\mu - m^2 = 0 \tag{1.62}$$

As the solution of this equation failed with scalar fields, Dirac consider that there existed matrices

γ, β, such that

$$(\beta^\mu p_\mu + m)(\gamma^\nu p_\nu - m) = 0 \tag{1.63}$$

If we expand this equation we obtain

$$\beta^\mu p_\mu \gamma^\nu p_\nu + m(\gamma^\nu p_\nu - \beta^\mu p_\mu) - m^2 = 0 \tag{1.64}$$

Now, note that there are no linear momentum terms in Equation 1.62. This implies that the matrices γ and β are the same one. Hence, Einstein's relation becomes

$$\gamma^\mu \gamma^\nu p_\mu p_\nu - m^2 = 0 \tag{1.65}$$

This equation is manifestly Lorentz, and Poincaré, invariant. However, we have to expand the sums to find all the possible conditions that these matrices must follow:

$$\begin{aligned}&(\gamma^0)^2 p_0^2 + (\gamma^1)^2 p_1^2 + (\gamma^2)^2 p_2^2 + (\gamma^3)^2 p_3^2 + (\gamma^0\gamma^1 + \gamma^1\gamma^0) p_0 p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0) p_0 p_2 + \cdots \\ &\cdots + (\gamma^0\gamma^3 + \gamma^3\gamma^0) p_0 p_3 + (\gamma^1\gamma^2 + \gamma^2\gamma^1) p_1 p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1) p_1 p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2) p_2 p_3 - m^2 = 0\end{aligned}$$

From the first four terms we obtain $(\gamma^0)^2 = \mathbb{1}$, and $(\gamma^i)^2 = -\mathbb{1}$. Additionally, if we note that Einstein's relation does not have any crossed momentum terms, we are able to define an anticommutation relation, referred to as the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{1.66}$$

Note that Equation 1.66 can be taken as a defining relation for the gamma matrices, which can be taken to be 4×4 complex matrices. However, there is an infinite number of possible representations of the Clifford algebra, the one used by Dirac is known as the Pauli-Dirac representation and is given, in terms of Pauli matrices, by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{1.67}$$

In addition to these four gamma matrices, it is possible to construct an additional matrix, γ^5, whose eigenfunctions will be useful further on

$$\gamma^5 = i\prod_{\chi=0}^{3} \gamma^\chi \tag{1.68}$$

After defining the Clifford algebra, we obtain that Equation 1.65 can be reduced to.

$$\gamma^\mu p_\mu - m = 0 \tag{1.69}$$

Then, we apply first quantization to obtain a differential operator, which acts on a 4 component

function ψ, giving us the so called Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{1.70}$$

The solutions of Dirac equation are known as spinors, and carry indices $\alpha = 1, \cdots, 4$:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \psi_\alpha \tag{1.71}$$

Now, it is our duty to study how these spinors transform under the Poincaré group. To do so, let us first recall that we need to build Lorentz invariant equations of motion. Hence, the field shall transform linearly with respect to any Lorentz transformations. To do so, let us introduce the Lorentz transformation matrix S(Λ) such that a spinor transforms as

$$\psi'(x') = S(\Lambda)\psi(x) \tag{1.72}$$

where the matrix S, carries spinor indices $S = (S_{\alpha\beta})$. This relation is obviously invertible such that we can write $\psi(x) = S^{-1}(\Lambda)\psi'(x')$.

As $\psi'(x')$ has to be a solution of the Dirac equation, we consider Equation 1.70 after a Lorentz boost. In other words, we consider

$$i\gamma^\mu \partial'_\mu \psi'(x') - m\psi'(x') = 0; \partial'_\mu = \Lambda^\nu_\mu x_\nu \tag{1.73}$$

If we now replace $\psi'(x')$ by the means of Equation 1.72, we obtain

$$i\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu_\mu \partial'_\nu \psi'(x') - S^{-1}(\Lambda) m\psi'(x') = 0 \tag{1.74}$$

Multiplying by S(Λ) on both sides

$$iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu_\mu \partial_\nu \psi'(x') - m\psi'(x') = 0 \tag{1.75}$$

As the gamma matrices do not depend on the position, we can infer a transformation law for them in terms of S and Λ:

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu_\mu = \gamma^\nu \tag{1.76}$$

Given this transformation law, we get that $\psi'(x')$ is also a solution of Dirac equation. However, there could be an infinite number of 4×4 matrices that follow Equation 1.76. This can be solved by using infinitesimal Lorentz transformations []

$$\Lambda^\nu_\mu = \delta^\nu_\mu + \epsilon^\nu_\mu; S(\Lambda) = \mathbb{1} - \frac{i}{2}\epsilon^{\mu\nu} M_{\mu\nu} \tag{1.77}$$

where $\epsilon^{\mu\nu}$ is the 2-dimensional antisymmetric density. Note that S will be uniquely determined by M, if we replace this definitions on Equation 1.76 we obtain that M takes the form

$$M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}\sigma_{\mu\nu} \tag{1.78}$$

As S gives us the transformation law for ψ, and also form a representation of the Poincaré group, we refer to it as the spinor representation. This implies that the set of all solutions of the Dirac equation, transform under the spinor representation, and we call it spinors. Further on this dissertation, we will realize that Dirac spinors do not form an irreducible representation of the Poincareé group, which forces us to find them. This irreducible representations are called Weyl, or chiral, spinors.

Finally, the Lagrangian of the Dirac field is given by

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{1.79}$$

1.2.3 Maxwell and Proca equations

1.2.3.1 Maxwell equations

In previous sections we have studied how we can build relativistic field equations that couple both special relativity, and quantum mechanics, without quantizing their solutions. Additionally, we have also studied how these fields form representations of the Poincaré group. However, we have not yet included in our description any vector representation of the Poincaré group. As we will see, any vector field forms one of these representations, but we first study its dynamics, beginning with Maxwell's equations.

When one first wants to study electromagnetic phenomena, one finds that the dynamics of the electric and magnetic fields, in vacuum, are given by

$$\begin{aligned} \boldsymbol{\nabla}\cdot\vec{E} &= 0 & \boldsymbol{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \boldsymbol{\nabla}\cdot\vec{B} &= 0 & \boldsymbol{\nabla}\times\vec{B} &= \frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1.80}$$

The equations on the first column of 1.80 are known as the Gauss laws for the electric and magnetic fields respectively; whereas the other two equations are known as the Faraday and Ampere laws respectively, and together all four equations are referred to as Maxwell equations. It is possible to write the electric and magnetic fields in terms of a scalar and vector potential: [ref:griffiths].

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{1.81}$$

$$\vec{B} = \boldsymbol{\nabla} \times \vec{A} \tag{1.82}$$

If we replace these expressions into the Gauss law for electric field, and the Ampere law we obtain:

$$\nabla^2\phi = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \vec{A} \tag{1.83}$$

$$\nabla^2\vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} = \nabla\left(\boldsymbol{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t}\right) \tag{1.84}$$

Note that Equation 1.83 resembles the Poisson equation, while 1.84 does the same with the wave equation. It is possible to reduce both to simpler equations by the means of the so called gauge conditions. In basic electrodynamics there are two possible gauge conditions, known as the Coulomb and Lorenz gauges respectively, these are given by:

$$\boldsymbol{\nabla} \cdot \vec{A} = 0 \quad \text{(Coulomb gauge)} \tag{1.85}$$

$$\boldsymbol{\nabla} \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0 \quad \text{(Lorenz gauge)} \tag{1.86}$$

These equations imply that we can transform the potentials by the means of a scalar function λ. The main requirement for the addition of this λ, is that it should leave the electric and magnetic fields invariant. This can be done if the transformation laws are given by

$$\vec{A}' = \vec{A} + \boldsymbol{\nabla}\lambda \tag{1.87}$$

$$\phi' = \phi - \frac{\partial \lambda}{\partial t} \tag{1.88}$$

Note that each of the gauge conditions set different differential equations that set an specific λ.

If we want to introduce an explicit Lorentz invariant formulation of electrodynamics [ref:nachoelectro, griffiths, jackson], define the field strength tensor as,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{1.89}$$

and define the 4-potential field $A^\mu = (\phi, \vec{A})$. Note that it is possible to rewrite the field strength tensor in terms of the potential as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \tag{1.90}$$

It is also trivial to see that Equations 1.87 and 1.88, can be reduced into a single expression:

$$A'_\mu = A_\mu + \partial_\mu \lambda \tag{1.91}$$

One can also see, that the Lorenz gauge reduces to $\partial_\mu A^\mu = 0$, while the Coulomb gauge preserves its form.

Now, we shall note that the field strength tensor follows the Bianchi identity, as it is a rank 2 tensor [ref:Carroll]. This identity can be written as

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \tag{1.92}$$

However, this identity has a physical meaning. Namely, it reduces to the Gauss law for the magnetic field, and to Faraday's law:

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \qquad \longrightarrow \qquad \begin{aligned} \boldsymbol{\nabla} \cdot \vec{B} &= 0 \\ \boldsymbol{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned} \tag{1.93}$$

The other two Maxwell equations can be obtained by the contraction of F with a derivative operator:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \longrightarrow \qquad \begin{aligned} \boldsymbol{\nabla} \cdot \vec{E} &= 0 \\ \boldsymbol{\nabla} \times \vec{B} &= \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{1.94}$$

Finally, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.95}$$

1.2.3.2 Proca equations

Maxwell formulation of electrodynamics is one of the most successful theories in history. It describes that all electromagnetic phenomena originate from a vector potential, A_μ, and gives us one marvelous consequence: Light is an electromagnetic phenomenon. However, these vector potentials do not contain mass terms for the fields.

One possibility to describe massive electromagnetism is to add mass terms to 1.83 and 1.84, such that they take the form [ref:martinshaw]:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} + \boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \vec{A}\right) + m^2 \vec{A} = 0 \tag{1.96}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi - \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \vec{A}\right) + m^2 \phi = 0 \tag{1.97}$$

Note that the addition of the mass term in these equations, which are known as Proca equations, tell us that Equations 1.87 and 1.88 will not be valid transformations for the potentials. As the

equations are non longer valid under gauge transformations, we say that mass terms break gauge invariance in massive electromagnetism. If we use the Lorenz gauge both EOM reduce to the KG equation

$$\Box\vec{A} + m^2\vec{A} = 0 \tag{1.98}$$

$$\Box\phi + m^2\phi = 0 \tag{1.99}$$

Or using the relativistic field potential

$$\Box A_\mu + m^2 A_\mu = 0 \tag{1.100}$$

With Lagrangian given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu \tag{1.101}$$

Now, it is our duty to study how these vector fields, A_μ, transform under Poincaré transformations. To do so, we will use the fact that any vector field transforms via

$$A'^\mu(x') = \Lambda^\mu_\nu A^\nu(\Lambda x) \tag{1.102}$$

As it is an invertible relation, we can rewrite it as

$$A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1} x) \tag{1.103}$$

This family of transformations define an specific representation of the Poincaré group, referred to as the vector representation. Thus, all of the vector fields of interest for us will form one of these representations. If the reader wants to extend his knowledge on these representations, we suggest reading [ref:weinberg].

1.3 Relativistic Quantum Field Theory

1.3.1 Solutions of the KG and Dirac equations.

In the previous section, we found that it is possible to describe relativistic particles with spin-0, or $\frac{1}{2}$, by the means of the KG and Dirac equations. However, we have not found an specific solution to them. In this section we will derive the solutions for the KG equation and to the Dirac equation in the Pauli-Dirac representation.

1.3.1.1 Solutions of the KG equation.

Let us first recall the KG equation,

$$\Box\phi + m^2\phi = 0, \tag{1.104}$$

and note that it has the form of a inhomogeneous wave equation. Therefore, we obtain solutions given by a plane wave superposition:

$$\phi(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}[A(p)e^{-ipx} + B^\dagger(p)e^{ipx}] \tag{1.105}$$

with a and b acting as Fourier coefficients for a given momentum p. Even though this seems to be the correct solution, it does not include the mass shell condition, as well as the requirement of positive energies.

This problem can be solved by introducing the Lorentz Invariant measure,

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} \to \int \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\delta(p_0^2 - m^2)\theta(p_0), \tag{1.106}$$

where the delta function ensures that we are taking on-shell mass values, and the Heaviside function guarantees positive energy values. Using properties of the delta function, the measure reduces to

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p} \quad ; E_p = p_0 = \sqrt{p^2 + m^2} \tag{1.107}$$

With this new integration measure, the correct Fourier expansion takes the form

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}[A(\vec{p})e^{ipx}] + B^\dagger(\vec{p})e^{-ipx}] \tag{1.108}$$

Now, let us redefine our expansion coefficients as

$$a(\vec{p}) = \frac{1}{\sqrt{2E_P}}A(\vec{p}) \qquad\qquad b(\vec{p}) = \frac{1}{\sqrt{2E_P}}B(\vec{p}) \tag{1.109}$$

Such that the most general solution of the KG equation is given by

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}[a(\vec{p})e^{ipx} + b^\dagger(\vec{p})e^{-ipx}], \tag{1.110}$$

as expected from a wave equation.

1.3.1.2 Solutions of the Dirac equation

Let us recall the Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0 \tag{1.111}$$

Where the gamma matrices are in the Pauli-Dirac representation, as seen in Equation1.67. The simplest solution of this equation can be found in terms of a plane wave

$$\psi(x) = u(p)e^{-ipx} \tag{1.112}$$

Hence, the 4-component spinor, u(p), obeys the Dirac equation:

$$(\gamma^\mu p_\mu - m)u(p) = 0 \tag{1.113}$$

If we expand the contraction between γ^μ and p_μ we obtain the following differential equation:

$$\begin{pmatrix} (E-m)\mathbb{1}_2 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)\mathbb{1}_2 \end{pmatrix} u(p) = 0 \tag{1.114}$$

This allows us to write $u(p)$ as a bispinor:

$$u(p) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

The bispinor decomposition of the Dirac equation sets two different EOM, that the spinors have to obbey:

$$(\vec{\sigma}\cdot\vec{p})u_A = (E+m)u_B \tag{1.115}$$

$$(\vec{\sigma}\cdot\vec{p})u_B = (E-m)u_A \tag{1.116}$$

Note that only the first equation will give us non singular solutions for $E > 0$, so we will work with it. Additionally, there is an infinite set of possible solutions, for u_A and u_B for the positive energy solution. The simplest one can be made in terms of the 2D canonical basis

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.117}$$

These two spinors give us two different, linearly independent, solutions, which are given by

$$u_1(p) = N(p)\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+M} \end{pmatrix}; \quad u_2(p) = N(p)\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{1.118}$$

As we need four solutions to form a basis of the bispinor space, we now need two more solutions. To find them, let us realize that the second possible plane wave solution is given by

$$\psi(x) = v(p)e^{ipx} \tag{1.119}$$

Note that replacing this solution into the position space representation of the Dirac equation, we obtain that the bispinor v must obey:

$$(\gamma^\mu p_\mu + m)v(p) = 0 \tag{1.120}$$

This equation has two different, linearly independent, solutions:

$$v_1(p) = N(p)\begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2(p) = N(p)\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{1.121}$$

These v_1, v_2 are considered to be the antiparticle solutions of the Dirac equation. According to the Feynman-Stückelberg interpretation, these solutions, which originaly arise from apparent negative energies, correspond to negative energy particles moving backwards in time. This implies that they can be taken to be antiparticles, that annihilate with their corresponding particle, with positive energy and travel forwards in time.

Hence, we can build a basis of positive energy solutions of the Dirac equation with these four bispinors, namely $B = \{u_1, u_2, v_1, v_2\}$, by choosing a proper normalization factor:

$$N(p) = \sqrt{E+m} \tag{1.122}$$

In fact, the basis formed by this bispinors is complete in the sense that the following relations hold

$$\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = \not{p} + m \tag{1.123}$$

$$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = \not{p} - m \tag{1.124}$$

Therefore, any possible bispinor can be written as a linear combination of these four solutions. In other words, any bispinor can be written as

$$\psi(x) = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [c_r(p)u_r(p)e^{-ipx} + d_r^* v_r(p)e^{ipx}] \tag{1.125}$$

Note that we have found solutions to the KG and Dirac equations. However, they are still being treated as classical fields, since no quantization procedure has been carried, this will be solved in the next section.

1.3.2 Quantization of scalar and spinor fields.

In the last section we found solutions to the KG and Dirac equations. However, these solutions were purely classical, as no quantization procedure has been carried. In this section we will quantize both fields by the means of the so called second quantization procedure [ref:second quantization].

1.3.2.1 Quantization of a complex scalar field

In order to quantize a given classical field, we need to reduce its Hamiltonian into the form of a Quantum Harmonic Oscillator (QHO). Therefore, we need to calculate:

$$\mathcal{H} = \sum_i \pi_i \dot{\phi} - \mathcal{L} \tag{1.126}$$

So far we have analyzed the case of a real scalar field whose dynamics is given by Equation 1.59. However, this field has no special As we are considering a complex scalar field, there are two degrees of freedom, namely ϕ and ϕ^*. This means that we first need to calculate both conjugate momenta from the Lagrangian

$$\begin{aligned}\mathcal{L} =&\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \\ =&\partial_t \phi^* \partial_t \phi - (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) - m^2\phi^2\phi\end{aligned} \tag{1.127}$$

Hence, both momenta are given by:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*; \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \tag{1.128}$$

This implies that the Hamiltonian is

$$\begin{aligned}\mathcal{H} =&\pi\pi^* + \pi^*\pi - \mathcal{L} \\ =&\pi\pi^* + \pi^*\pi - \pi\pi^* + (\boldsymbol{\nabla}\phi^*)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^*\phi \\ =&\pi^*\pi + (\boldsymbol{\nabla}\phi^*)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^*\phi\end{aligned} \tag{1.129}$$

If now use the Fourier expansion of the KG field, we get the following terms

$$\boldsymbol{\nabla}\phi = i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\vec{p}}{\sqrt{2E_p}}[a(p)e^{-ipx} - b^*(p)e^{ipx}] \tag{1.130}$$

$$\boldsymbol{\nabla}\phi^* = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\vec{p}}{(\sqrt{2E_p}}[a^*(p)e^{ipx} - b(p)e^{-ipx}] \tag{1.131}$$

$$\pi^* = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}[a(p)e^{-ipx} - b^*(p)e^{ipx}] \tag{1.132}$$

$$\pi = i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}[a^*(p)e^{ipx} - b(p)e^{-ipx}] \tag{1.133}$$

However this does not reduce the expresion for the Hamiltonian. To do so, we impose commutation relations between the field and it's conjugated field:

$$[\phi(x), \pi(y)] = i\delta(x-y) \tag{1.134}$$

This relation induces commutation relations for the Fourier coefficients, converting them into operators, referred to as ladder operators, acting into a certain Hilbert space. The commutation relations are given by

$$[a(p), a^\dagger(p')] = (2\pi)^3\delta(p-p') \tag{1.135}$$
$$[b(p), b^\dagger(p')] = (2\pi)^3\delta(p-p') \tag{1.136}$$

Both commutation relations allow us to reduce the Hamiltonian to

$$\mathcal{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left[a^*(p)a(p) + b^*(p)b^(p) + \frac{1}{2}[a(p), a^*(p)] + \frac{1}{2}[b(p), b^*(p)] \right] \tag{1.137}$$

If we apply both commutation relations, we obtain

$$\mathcal{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left[a^\dagger(p)a(p) + b^\dagger(p)b(p) + \frac{1}{2}\delta(0) + \frac{1}{2}\delta(0) \right] \tag{1.138}$$

Note that the last two terms diverge. However, this divergence has a physical meaning as we have not set an energy reference. This means that these divergencies contain all possible ground state energies for each mode. To solve this problem we set an energy reference state, namely $|\Omega\rangle$, such that

$$H|\Omega\rangle = E_p|\Omega\rangle, \tag{1.139}$$

and it is known as the ground state of the field. Hence, the ladder operators action on the ground state is given by

$$|p\rangle = \sqrt{2E_p}a^\dagger(p)|\Omega\rangle \tag{1.140}$$

Nevertheless, the definition of a ground state does not take the divergence out. This can be done by introducing the *normal ordered* Hamiltonian:

$$:\mathcal{H}: = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p(a^\dagger(p)a(p) + b^\dagger(p)b(p)) \tag{1.141}$$

In general, a normal ordered operator is such that all the creation operators are sent to the left, and the annihilation ones to the right.

To finish with the quantization procedure, we can wonder about the propagation of a particle from point x to point y. The information of this propagation is then contained in the commutator, as measurements between two different points must not affect each other for spacelike intervals.

The commutator is then given by:

$$\langle\Omega|\left[\phi(x),\phi(y)\right]|\Omega\rangle \tag{1.142}$$

Note that the commutator is nothing but a number, hence its expectation value is analogous to calculating without considering the ground state:

$$\langle\Omega|\left[\phi(x),\phi(y)\right]|\Omega\rangle = \left[\phi(x),\phi(y)\right]\langle\Omega|\Omega\rangle = \left[\phi(x),\phi(y)\right] \tag{1.143}$$

Now, we expand the fields in terms of their corresponding Fourier transform:

$$\begin{aligned}\left[\phi(x),\phi(y)\right] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{(2E_p)}}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\left[a(p)e^{-ipx}+a^\dagger(p)e^{-ipx},a(q)e^{-iqy}+a^\dagger(q)e^{iqy}\right]\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{(2E_p)}}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\{\left[a(p)e^{-ipx},a(q)e^{-iqy}\right]+\left[a(p)e^{-ipx},a^\dagger(q)e^{iqy}\right]+\cdots\\ &\quad\cdots+\left[a^\dagger(p)e^{ipx},a(q)e^{-iqy}\right]+\left[a^\dagger(p)e^{ipx},a^\dagger(q)e^{iqy}\right]\}\end{aligned} \tag{1.144}$$

Using Equations 1.135 and 1.136, we reduce the commutator to

$$\left[\phi(x),\phi(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{(2E_p)}}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\{e^{-ipx+iqy}\left[a(p),a^\dagger(q)\right]-e^{ipx-iqy}\left[a(q),a^\dagger(p)\right]\} \tag{1.145}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{(2E_p)}}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\{e^{-ipx+iqy}(2\pi)^3\delta(p-q)-e^{ipx-iqy}(2\pi)^3\delta(q-p)\} \tag{1.146}$$

Integrating over q we obtain a simple equation for the commutator

$$\left[\phi(x),\phi(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\{e^{-ip(x-y)}-e^{ip(x-y)}\} \tag{1.147}$$

This is analogue to integrating,

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}, \tag{1.148}$$

along the contour in Figure 1.5 if $x^0 > y^0$.

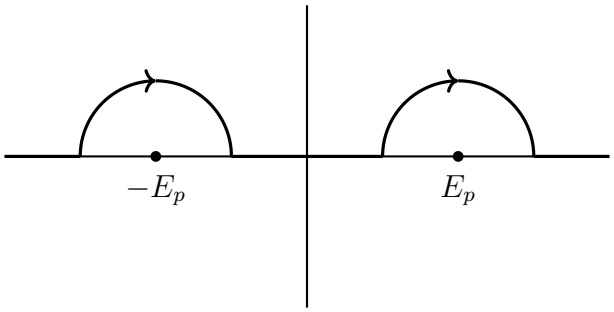


Figure 1.5: Contour associated with a retarded propagation of information.

Note that closing the contour from above, will give zero contribution. In other words, the information is propagating in a *retarded* way, as it first reaches y, rather than x. Hence, we say that this commutator is giving us a retarded propagation, and we can construct a retarded propagator -or Green's Function- by [ref:peskin]:

$$D_R(x-y) = \theta(x^0 - y^0) \langle \Omega | \left[\phi(x), \phi(y) \right] | \Omega \rangle \tag{1.149}$$

However, note that we can also consider the propagation of information for $x^0 > y^0$, i.e If the information first reaches x than y. This means that we need to choose the contour in Figure 1.6.

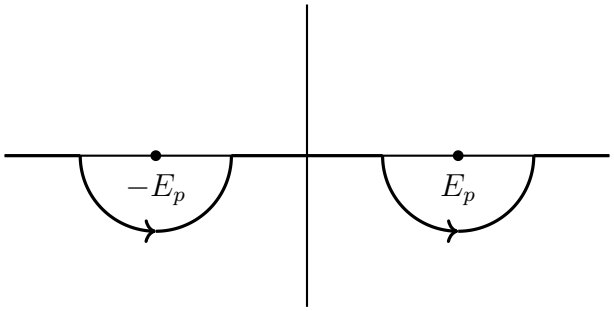


Figure 1.6: Contour associated to advanced propagation of information

By in integrating along this contour we obtain then an *advanced* propagation of information, which is formally given by

$$D_A(x-y) = \theta(y^0 - x^0) \langle \Omega | \left[\phi(x), \phi(y) \right] | \Omega \rangle \tag{1.150}$$

However, it is impossible to measure in an experiment whether the propagation was retarded or advanced. This implies that we need to find a propagator that includes a time ordering for the measurements. This can be done if we perform the integration along the contour given in Figure 1.7

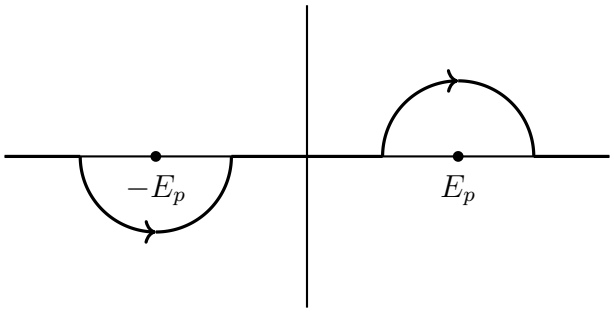


Figure 1.7: Integration contour for time ordered propagation of information.

This propagator was introduced by Feynman and is given by a shifting the poles in Equation 1.148 by a small complex factor

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{1.151}$$

In position space it is given by

$$D_F(x-y) = \langle\Omega| T\phi(x)\phi(y) |\Omega\rangle = \theta(x^0 - y^0) \langle\Omega| \phi(x)\phi(y) |\Omega\rangle + \theta(y^0 - x^0) |\Omega\rangle \tag{1.152}$$

It is straightforward to show that each of these propagators are Green functions of the Klein-Gordon differential operator:

$$(\Box + m^2)G(x-y) = -i\delta^{(4)}(x-y), \tag{1.153}$$

Applying a Fourier transform of the Green function we obtain

$$G(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} G(p) e^{-ip(x-y)} \rightarrow G(p) = \frac{i}{p^2 - m^2} \tag{1.154}$$

Note that it is possible to obtain both advanced and retarded propagators, by integrating over their corresponding contours. However the time ordered propagator is obtained by adding a shift in the poles, as we did before.

By finding the propagator, we have finished the quantization procedure for a spin-0 field. However, we still want to quantize both fermions and vector bosons. In the next section we focus on fermionic quantization.

1.3.2.2 Quantization of spinor fields

For a half-integer spinor, we found that its dynamics are contained in the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{1.155}$$

To quantize the field, we need to calculate the Hamiltonian. Hence, we expand the contractions in the Lagrangian

$$\begin{aligned}\mathcal{L} =& i\bar{\psi}\gamma^0\partial_t - i\bar{\psi}(\vec{\gamma}\cdot\boldsymbol{\nabla})\psi - m\bar{\psi}\psi \\ =& i\psi^\dagger\dot{\psi} - i\bar{\psi}(\vec{\gamma}\cdot\boldsymbol{\nabla})\psi - m\bar{\psi}\psi\end{aligned} \tag{1.156}$$

This form of the Lagrangian allows us to calculate the conjugated fields, namely $\pi(x)$, and $\bar{\pi}(x)$:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \quad ; \quad \bar{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0 \tag{1.157}$$

Hence, the Hamiltonian is

$$\begin{aligned}\mathcal{H} =& \bar{p}i\dot{\bar{\psi}} + \pi\dot{\psi} - \pi\dot{\psi} + i\bar{\psi}(\vec{\gamma}\cdot\boldsymbol{\nabla})\psi + m\bar{\psi}\psi \\ =& \bar{\psi}(i\vec{\gamma}\cdot\boldsymbol{\nabla} + m)\psi\end{aligned} \tag{1.158}$$

Now, we use Equation 1.125 to introduce both spinors in momentum space

$$\psi(x) = \sum_{r=1}^{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}[a_r(p)u_r(p)e^{-ipx} + b_r^\dagger(p)v_r(p)e^{ipx}] \tag{1.159}$$

$$\bar{\psi}(x) = \sum_{r=1}^{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}[a_r^\dagger(p)\bar{u}_r(p)e^{ipx} + b_r(p)\bar{v}_r(p)e^{-ipx}] \tag{1.160}$$

As we are dealing with half-integer spin particles, introducing commutation relations between the fields will generate a problem with causality, as the expectation value $\langle\Omega|\left[\psi(x),\bar{\psi}(x')\right]|\Omega\rangle$ will not vanish for spacelike intervals. This problem was solved by Pauli and Jordan in the 1940's, when they interchanged the commutation relations of the creation and annihilation operators for anticommutators given by

$$\{a_r(p), a_s^\dagger(p')\} = \delta_{rs}(2\pi)^3\delta(p-p') \tag{1.161}$$

Which directly induce an anticommutation relation for the fields

$$\{\psi_r(x), \bar{\psi}_s(x')\} = i\delta(x-y)\delta_{rs} \tag{1.162}$$

When analized for positive energy solutions only, the expectation value of the anticommutator will vanish for spacelike intervals, recovering causality of the theory. Note that preserving causality directly implies that spinor fields follow the Fermi-Dirac statistics, and therefore Pauli's exclusion principle.[4].

Now, if we replace Equations 1.159& 1.160 into the Hamiltonian, and use anticommutation

[4]This means that causality dictates the statistics of fields, this can be seen when proving the spin-statistics theorem. See Weinberg *The Quantum Theory of Fields Vol. 1*

relations, we find

$$\mathcal{H} = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p [a_r^\dagger(p) a_r(p) - b_r^\dagger(p) b_r(p)] \tag{1.163}$$

This Hamiltonian seems not to be bounded from below due to the minus sign associated to them. This can be solved by noting that the anticommutation relations are symmetric between $b^\dagger$ and b, such that we can relabel them by[ref:peskin]

$$\hat{b}_r(p) = b_r^\dagger(p); \quad \hat{b}_r^\dagger(p) = b_r(p) \tag{1.164}$$

By forcing them to obey the same anticommutation relations of the b's, then we get a Hamiltonian, which is bounded from below, so we can drop the hats and obtain

$$\mathcal{H} = \sum_{r=1}^{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p [a_r^\dagger(p) a_r(p) + b_r^\dagger(p) b_r(p) + (2\pi)^3 \delta(0)] \tag{1.165}$$

Once again we have a divergent term arising from the sum of all ground state energies. As we did before, we solve this problem by introducing an energy reference $|\Omega\rangle$ such that a state with momentum $\vec{p}$ and spin s can be created by

$$|\vec{p}; s\rangle = \sqrt{2E_p} a_r^\dagger(p) |\Omega\rangle \tag{1.166}$$

We will refer to particles created by $a^\dagger(p)$ as fermions, and those created by $b^\dagger(p)$ as antifermions, such that the bispinors u and v can be directly associated to them.

To finish with the quantization procedure, we calculate the propagator, this time by finding the Green function of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) G(x - y) = i\delta(x - y) \tag{1.167}$$

By the means of a Fourier transform we obtain

$$G(x - y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} G(p) e^{-ip(x-y)} \rightarrow G(p) = i(\gamma^\mu p_\mu - m)^{-1} \tag{1.168}$$

Using the Dirac equation we then find

$$G(p) = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2} \tag{1.169}$$

When trying to recover the propagator in position space, we will need to integrate along the contours in Figures 1.5, 1.6, and 1.7. Each of these contours will give us retarded, advanced and time ordered propagators respectively. As we need to focus on time ordered processes, we find that

this propagator is given by

$$G(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{1.170}$$

This finishes our quantization procedure for scalar and spinor fields. However, if we want to properly quantize all of our fields in the SM we need to quantize vector bosons. As we will see, quantization of vector bosons brings many problems to the theory in both Abelian and Non-Abelian fields. In the next section we will explore how to quantize the photon field, and we will see that our problems arise from the construction of the Hamiltonian.

1.3.3 Quantization of the electromagnetic field.

For the last section we have focused on finding Lorentz invariant EOM compatible with quantum mechanics. This introduced the concepts of scalar and spinor fields to solve the KG and Dirac equations, allowing us to use canonical quantization to quantize the field at each point in spacetime. However, we have not yet dealt with the quantization procedure for spin-1 particles, which are described by Maxwell and Proca equations for massless, and massive particles respectively. First, we will study how to quantize the electromagnetic field, leaving massive vector fields to later sections[5].

From Section 1.2.3.2 we know that homogeneous Maxwell equations, in Lorenz gauge, are analogous to solve both wave equations

$$\Box\phi = 0 \tag{1.171}$$

$$\Box\vec{A} = 0 \tag{1.172}$$

However, these solutions do not transform trivially under Lorentz transformations.This obligates us to go to Minkowski space, where we work with the field A_μ. As the field transforms as a vector under the Lorentz group, we can make a Fourier expansion in terms of polarization vectors:

$$A^\mu(x) = \sum_{r=0}^{3} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [\epsilon_r^\mu(p) a_r(p) e^{-ipx} + \epsilon_r^{\mu *} a_r^\dagger(p) e^{ipx}] \tag{1.173}$$

Note that this expansion is rather problematic as it has four polarization vectors instead of two. In other words, there are two unphysical polarizations included in the Fourier expansion, causing it to be redundant. Additionally, this inclusion of two extra polarization vectors there are two more problems, namely that the propagator is does not exist, and that one of the four conjugated fields also vanishes.

Let us try to see this problems one by one. First, let us calculate the propagator of the photon

[5]Different approaches to this problem can be found in *Florian Scheck-Quantum Mechanics* and *F. Mandl & G. Shaw-Quantum Field Theory*

field. Let us remember that homogeneous Maxwell equations are analogous to

$$\partial_\mu F^{\mu\nu} = 0 \tag{1.174}$$

In terms of the photon field, this can be rewritten as

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \tag{1.175}$$

Factorizing a field and using the definition of the D'Alembert operator, one can find that the EOM can be written as

$$(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)A_\mu = 0 \tag{1.176}$$

As we want to find the propagator, we can use that it is indeed a Green's function of the differential operator defined above

$$(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)G(x-y) = \delta^\mu_\nu\delta(x-y) \tag{1.177}$$

Now, we use that it is possible to make a Fourier expansion of the Green's function in momentum space, such that

$$G(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ipx}\tilde{G}(p) \tag{1.178}$$

Inserting this expression into Equation 1.177 sets that our Green's function has the form

$$\begin{aligned} -(\eta^{\mu\nu}p^2 - p^\nu p^\mu)\tilde{G}(p) &= \delta^\mu_\nu \\ \tilde{G}(p) &= -\delta^\mu_\nu(\eta^{\mu\nu}p^2 - p^\nu p^\mu)^{-1} \end{aligned} \tag{1.179}$$

In other words, the Green's function needs to be a linear combination of the metric and projections into the momenta[ref:Lahiri],

$$\tilde{G}(p) = a\eta_{\mu\nu} + bp_\mu p_\nu, \tag{1.180}$$

where a and p can be either constants or momentum dependent coefficients. Replacing in Equation 1.179 we obtain a linear relation that helps us to determine de coefficients:

$$-[ap^2\delta^\mu_\nu + bp^2p^\mu p_\nu - ap^\mu p_\nu - bp^2p^\mu p_\nu] = \delta^\mu_\nu \tag{1.181}$$

Note that all terms containing b will vanish, this leaves us with

$$[ap^2\delta^\mu_\nu - ap^\mu_\nu] = -\delta^\mu_\nu \tag{1.182}$$

For b we have one problem as all terms vanish. This implies that b can be any parameter. If we leave b behind, we also find that all a terms give us a secondary problem. This can be seen by

comparing terms in Equation 1.182, here we obtain that a can have two possible values

$$a = -\frac{1}{p^2} \quad \text{or} \quad a = 0 \tag{1.183}$$

As it is not possible to properly determine both a and b, the propagator does not exist.

For the second problem, let us expand the Lagrangian associated to the photon field

$$\begin{aligned} \mathcal{L} =& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ =& -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ =& -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \end{aligned} \tag{1.184}$$

This being said, we can calculate the conjugated field via

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \tag{1.185}$$

Substituting the Lagrangian we obtain

$$\pi^\mu = F^{\mu 0} \tag{1.186}$$

This gives us the following expressions for the conjugated fields

$$\pi^i = F^{i0} = E_i \tag{1.187}$$

$$\pi^0 = 0 \tag{1.188}$$

As $\pi^0 = 0$ it is not possible to define the Hamiltonian. Fortunately for us, this problem goes hand to hand with the existence of redundant degrees of freedom. This can be seen by noting that Equations 1.80 imply that only two out of 4 components of the field are physically relevant and independent, and this is reflected on the vanishing value of π^0. In addition to this, we can note that all possible gauge conditions can be used to eliminate redundant degrees of freedom, and we have not yet introduced it in our Lagrangian formalism.

As we are dealing with a manifestly Lorentz invariant theory, we can only introduce the Lorenz gauge. This means that to introduce the Lorenz gauge, we need to explicitly add a term to the Lagrangian. This can be done with the term

$$\mathcal{L}_g = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2, \tag{1.189}$$

where the term ξ is known as the gauge fixing parameter. Note that all possible gauge fixing values are equivalent with Lorenz gauge, which is now fixed.Additionally, one can verify that this new term induces restrictions on the photon field degrees of freedom without changing Maxwell equations or the electric and magnetic fields.

After gauge fixing, the total lagrangian for the photon field takes form

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu A_\nu F^{\mu\nu} + \frac{1}{\xi}(\partial_\mu A^\mu)^2)\right] \tag{1.190}$$

Hence, the EOM becomes

$$\begin{aligned}\left[\Box A^\nu - \left(1 - \frac{1}{\xi}\right)\partial^\nu \partial^\mu A_\mu)\right] = 0 \\ \left[\eta^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\nu \partial^\mu\right] A_\mu = 0\end{aligned} \tag{1.191}$$

This implies that the propagator, in momentum space, has form

$$\tilde{G}(p) = -\delta^\mu_\nu \times \left[\eta^{\mu\nu}p^2 - \left(1 - \frac{1}{\xi}\right)p^\nu p^\mu\right]^{-1} \tag{1.192}$$

Once again the Green's function can be written as in Equation 1.180, but in this case both a and b will not vanish. Hence, after a little algebra one finds

$$a = -\frac{1}{p^2} \qquad\qquad b = -\frac{\xi - 1}{p^4} \tag{1.193}$$

This sets our Green's function to be

$$\tilde{G}_{\mu\nu}(p) = \frac{-\eta_{\mu\nu} - (\xi - 1)\frac{p_\mu p_\nu}{p^2}}{p^2} \tag{1.194}$$

Thus, the causal propagator is then

$$G_{\mu\nu}(p) = \frac{-\eta_{\mu\nu} - (\xi - 1)\frac{p_\mu p_\nu}{p^2}}{p^2 + i\epsilon} \tag{1.195}$$

As the propagator is properly defined, we have solved one of the three problems mentioned before. However, it seems that all our physical quantities depend on the gauge fixing parameter, making our formulation a little bit odd. To solve this, one can choose any value of ξ. The most popular choices are

$$\begin{aligned}\xi = 0 &\rightarrow \text{Landau gauge} \\ \xi = 1 &\rightarrow \text{Feynman gauge} \\ \xi = 3 &\rightarrow \text{Yennie gauge}\end{aligned}$$

In particle physics one usually works in Feynman gauge, as it eliminates the projections over the momentum states, whereas both Landau and Yennie gauges are largely used in condensed matter theory.

In addition to solving the propagator problem, the introduction of a gauge fixing Lagrangian

also gives rise to non vanishing conjugated fields. This can be seen by applying the definition of the conjugated fields to Equation 1.190:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu} \tag{1.196}$$

$$= -\dot{A}^\mu + \left(1 - \frac{1}{\xi}\right)\eta^{\mu 0}\partial_\nu A^\nu \tag{1.197}$$

Hence, π^0 has a non vanishing value and the Hamiltonian formalism can be applied to quantize the field. Nevertheless, it is still not possible to quantize the photon field. This can be seen if we impose canonical commutation relations for the fields

$$[A_\mu(x), \pi_\mu(y)] = i\delta(x-y) \tag{1.198}$$

Now, let us take the derivative of this expression, such that we want to analyze how the Lorenz gauge behaves in an operator level:

$$\begin{aligned} \partial_\mu[A_\mu(x), \pi_\mu(y)] &= i\partial_\mu\delta(x-y) \\ [\partial_\mu A^\mu(x), \pi^\mu(y)] &= i\partial_\mu\delta(x-y) \end{aligned} \tag{1.199}$$

Note that the left hand side of the equation vanishes due to the Lorenz gauge, while the right hand side is non zero, which means that the Lorenz gauge does not hold at operator level. Additionally, this is not the only problem that we will encounter while trying to quantize the field.

From Equation 1.173 we know that there are four polarization vectors on the expansion, which must form a basis for the four vector space. Without loss of generality, let us choose the polarization vectors in an specific frame, where they are given by

$$\epsilon_0^\mu(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \epsilon_i^\mu(p) = \begin{pmatrix} 0 \\ \vec{\epsilon}_i(p) \end{pmatrix} \tag{1.200}$$

Here we defined ϵ_0 to be a normalized timelike vector, which is orthogonal to the other 3 polarizations. From classical electrodynamics, we know that the vector polarizations are orthogonal to each other. Namely, we impose two polarizations to live in the transverse plane and one to be longitudinal to the momentum. In our case, we choose $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ to live in the transverse plane. This means

$$\vec{\epsilon}_{1/2}(p) \cdot \vec{\epsilon}_3(p) = 0 \tag{1.201}$$

As $\vec{\epsilon}_3$ is a longitudinal polarization vector, we choose it to be

$$\vec{\epsilon}_3(p) = \frac{1}{E_p}\vec{p} \tag{1.202}$$

All this definitions imply that the polarization vectors obey the following relations [ref:scheck]

$$\sum_r \zeta_r \epsilon_r^\mu(p)\epsilon_r^\nu(p) = -\eta^{\mu\nu} \qquad\qquad \epsilon_r^{*\mu}\epsilon_{r'\mu} = -\zeta_r\delta_{rr'} \tag{1.203}$$

with $\zeta_0 = -1$ and $\zeta_i = 1$ Choosing $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ to be transverse to the momentum means that they are our physical degrees of freedom. This means that we need to properly eliminate the ϵ_0^μ and ϵ_3^μ, this will give us a proper quantization procedure for the photon field. Now, it is straightforward to realize that the canonical commutation relations, along with the orthonormality conditions for the polarization vector, induce commutation relations for the Fourier coefficients

$$[a_r(p), a_s(p')] = 0 \qquad\qquad \left[a_r(p), a_{r'}^\dagger(p')\right] = \zeta_r T\delta_{rr'}(2\pi)^3\delta(p-p') \tag{1.204}$$

These commutation relations seem to be well defined for all states. Nonetheless, it is easy to find that $r = 0$ generates negatively normed states as

$$\left[a_0(p), a_0^\dagger(p')\right] = \zeta_0(2\pi)^3\delta(p-p') = -(2\pi)^3\delta(p-p') < 0 \tag{1.205}$$

So, apparently trying to quantize the photon field gives us again three problems

1. ϵ_0 and ϵ_3 are unphysical degrees of freedom.
2. ϵ_0 induces negatively normed states.
3. $\partial_\mu A^\mu$ does not hold at operator level.

It seems impossible to quantize the photon field given the problems that keep originating in canonical quantization. However, Gupta and Bleuler [ref:papers gupta and bleuler] found a way to tackle all three problems at once. They first considered that the Hilbert space, on which the field (as operator) acts, can be decomposed as the direct sum of physical and unphysical subspaces,

$$Hil = Hil_{Phys} \oplus Hil_{unphys}, \tag{1.206}$$

such that the Lorenz gauge must be obeyed in Hil_{phys}. This means that the Lorenz gauge is obeyed in all Hilbert space as an *average*

$$\langle\psi|\, \partial_\mu A^\mu \,|\psi\rangle = 0 \tag{1.207}$$

Let us study this condition (known as the Gupta-Bleuler condition) a little further. First, let us decompose the photon field in terms of creation and annihilation operators as

$$A_\mu = A_\mu^{(+)} + A_\mu^{(-)} \tag{1.208}$$

where the first term is associated with the annihilation operator, and the second one with creation

operators. By using this splitting into the Gupta-Bleuler condition, we get

$$\langle\psi|\,\partial_\mu A^\mu\,|\psi\rangle = \langle\psi|\,\partial_\mu A^{(+)\mu}\,|\psi\rangle + \langle\psi|\,\partial_\mu A^{(-)\mu}\,|\psi\rangle = 0 \tag{1.209}$$

Fortunately, both contributions vanish as

$$\partial_\mu A^{(+)\mu}\,|\psi\rangle = 0 \qquad\qquad \langle\psi|\,\partial_\mu A^{(-)\mu} = 0 \tag{1.210}$$

As one of the terms is the Hermitian conjugate of the other, let us work only with the first one

$$\begin{aligned}\partial_\mu A^{(+)\mu} &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_r \epsilon_r^\mu(p)a_r(p)e^{ipx} \\ &= -i\sum_r \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}p_\mu\epsilon_r^\mu(p)a_r(p)e^{-ipx}\end{aligned} \tag{1.211}$$

Without loss of generality, let us choose p^μ in the Z direction,

$$p^\mu = (E_p, 0, 0, E_p),$$

such that

$$\partial_\mu A^{(+)\mu} = -i\sum_r \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}E_p[a_0(p) - a_3(p)]e^{-ipx} \tag{1.212}$$

This implies that the contributions of unphysical states cancel. This can be seen by noting

$$\begin{aligned}&\langle\psi|\,\partial_\mu A^{(+)\mu}\,|\psi\rangle = 0 \\ &-i\sum_r \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}E_p\,\langle\psi|\,[a_0(p) - a_3(p)]\,|\psi\rangle\,e^{-ipx} = 0 \\ &\langle\psi|\,[a_0(0) - a_3(p)]\,|\psi\rangle = 0\end{aligned} \tag{1.213}$$

As the contributions from unphysical degrees of freedom cancel out, there is no need of eliminating negatively normed states, as the help us to keep the consistency of the theory. We will see later on that this unphysical states can be explicitly introduced in the Lagrangian as ghost fields.

To finish the quantization procedure, we use that the Gupta-Bleuler condition induces that only two matrix elements of the normal-ordered Hamiltonian contribute in average, as the other two cancel

$$\langle\psi|:\mathcal{H}:|\psi\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_p\,\langle\psi|\sum_{r=1}^{2} a_r^\dagger(p)a_r(p)\,|\psi\rangle \tag{1.214}$$

So far we have dealt with quantization procedures for the photon field, as well as scalars and fermions, ignoring that they can interact between each other. However, we have not yet studied a formalism to study interacting fields; this formalism is known as the Dirac picture (or interaction

picture) of quantum mechanics and will be one of the main topics in the next section.

1.3.4 Pictures of Quantum Mechanics.

As previously stated, we have developed a formalism that studies the dynamics of free photons, scalars and fermions without having clear if they act as time independent (or dependent) operators, hence we do not know whether if we are on Schrödinger or Heisenberg picture. However, it is clear for us that these fields interact between each other.

Before studying how we can let fields interact, it is useful for us to recall how Schrödinger and Heisenberg pictures look like, as they are based on two different, but analogous, formulations referred to as wave and matrix mechanics respectively. Their main difference lies on the behavior of both states and operators.

In Schrödinger operators, such as the Hamiltonian, do not evolve with time, whereas states do. This evolution is given by the Schrödinger equation[6]

$$i\partial_t \left|\psi_s(t)\right\rangle = H \left|\psi_s(t)\right\rangle \tag{1.215}$$

According to Schrödinger equation, an initial state evolves with time from an initial state in terms of the unitary time evolution operator $U(t, t_0)$:

$$\left|\psi_s(t)\right\rangle = U(t, t_0) \left|\psi_s(t_0\right\rangle \qquad\qquad U(t, t_0) = \exp\{(-iH(t, t_0)\} \tag{1.216}$$

On the other hand, in Heisenberg's picture operators take the role states and become dynamical, while states will remain constant with time. These dynamics is given by the Heisenberg equation

$$i\partial A_h(t) = [A_h(t), H] \qquad\qquad \left|\psi_h\right\rangle = \left|\psi_s(t_0)\right\rangle \tag{1.217}$$

As stated above, both pictures are analogous and are in fact related by the means of unitary transformations. This can be seen by noting that the expectation value of a given operator, A, is picture independent. To check that both pictures are related by unitary transformations let us consider the expectation value:

$$\begin{aligned} \langle A_s\rangle &= \langle\psi_s(t)|A_s|\psi_s(t)\rangle \\ &= \langle\psi_s(t_0)|\, U^\dagger(t, t_0) A_s U(t, t_0) \left|\psi_s(t_0)\right\rangle \\ &= \langle\psi_h|\, U^\dagger(t, t_0) A_s U(t, t_0) \left|\psi_h\right\rangle \end{aligned} \tag{1.218}$$

As we require $\langle\psi_s(t)|A_s|\psi_s(t)\rangle = \langle\psi_h|A_h(t)|\psi_h\rangle$, we then find that the operators in both pictures

[6] In this section we use the subscripts s, h, and i to denote if the states or operatos are in Schrödinger, Heisenberg or Interaction picture.

are related via

$$A = U^{\dagger}(t, t_0) A_s U(t, t_0) \tag{1.219}$$

Now, we have studied both Lagrangians and Hamiltonians of free particles, whose spectra is well known, without considering their interactions, which one can introduce via a correction

$$H = H_0 + H' \tag{1.220}$$

The introduction of these corrections resembles the structure of perturbation theory. However, corrections associated to interactions are not small enough to apply perturbation theory, hence a new formalism is needed.

This new formalism is known as the interaction picture of quantum mechanics, and was developed by Dirac during his studies on electrodynamics. So far, we know that the free hamiltonian, H_0, gives us time evolution for states in the Schrödinger picture

$$U_0(t, t_0) = \exp\{[-iH_0(t, t_0]\} \rightarrow i\partial_t U_0(t, t_0) = H_0 U(t, t_0) \tag{1.221}$$

Now, let us define the interaction picture states as

$$|\psi_i(t)\rangle = U_0^{\dagger}(t, t_0) \, |\psi_s(t)\rangle \tag{1.222}$$

Hence, if we take the time derivative we get

$$\begin{aligned} i\partial_t \, |\psi_s(t)\rangle =& i\partial_t [U_0(t, t_0) \, |\psi_i(t)\rangle] \\ =& i\left[\frac{\partial U_0}{\partial t} \, |\psi_i(t)\rangle + iU_0(t, t_0)\partial_t \, |\psi_i(t)\rangle\right] \\ =& H_0 U_0(t, t_0) \, |\psi_i(t)\rangle + iU_0 \partial_t \, |\psi_i(t)\rangle \end{aligned} \tag{1.223}$$

On the other hand, we know that the total time evolution of Schrödinger states is given by

$$i\partial_t \, |\psi_s(t)\rangle = (H_0 + H') \, |\psi_s(t)\rangle \tag{1.224}$$

Hence, if we replace this expression into Equation 1.223, we obtain

$$iU_0(t, t_0)\partial_t \, |\psi_i(t)\rangle = H_i(t) U_0(t, t_0) \, |\psi_i(t)\rangle \tag{1.225}$$

If we now multiply by $U_0^{\dagger}$ on the left

$$\begin{aligned} iU_0^{\dagger} U_0 \partial_t \, |\partial_i(t)\rangle &= U_0^{\dagger} H_i U_0 \, |\psi_i(t)\rangle \\ i\partial_t \, |\psi_i(t)\rangle &= U_0^{\dagger} H_i U_0 \, |\psi_i(t)\rangle \end{aligned} \tag{1.226}$$

Originally, the interaction Hamiltonian can be time independent. However, $U^{\dagger} H U$ looks like a

time dependent operator, just like in Heisenberg picture

$$H_i(t) = U_0^\dagger(t, t_0) H_i U_0(t, t_0) \tag{1.227}$$

Hence, states in the interaction picture also obey the Schrödinger picture

$$i\partial_t \left|\psi_i(t)\right\rangle = H_i(t) \left|\psi_i(t)\right\rangle \tag{1.228}$$

Note that this implies that both operators and states evolve with time, which makes the interaction picture a mixture between the two previous ones. Now, if we set proper boundary conditions, we can find an integral equation for the interacting states

$$\left|\psi_i(t)\right\rangle = \left|\psi_i(t_0)\right\rangle + \frac{1}{i}\int_{t_0}^{t} \mathrm{d}t' H_i(t') \left|\psi_i(t')\right\rangle \tag{1.229}$$

Some references like [ref:peskin, lahiri, kaku] use this same procedure, but find an iterative solution for the time evolution operator. However, we can arrive to their formula by noting

$$\left|\psi_i(t)\right\rangle = U(t, t_0) \left|\psi_i(t_0)\right\rangle \to U(t) = U(t_0) + \frac{1}{i}\int_{t_0}^{t} \mathrm{d}t' H_i(t') U(t') \tag{1.230}$$

Initial times are usually unknown, so we set that at $t_0 = -\infty$ particles are free so

$$U(t) = 1 + \frac{1}{i}\int_{-\infty}^{t} \mathrm{d}t' H_i(t') U(t') \tag{1.231}$$

If we now consider an infinite number of iterations, we obtain

$$U(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \cdots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n H_i(t_1) \cdots H_i(t_n) \tag{1.232}$$

Note that we can rewrite the integrals as

$$\begin{aligned} &\sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \cdots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n H_i(t_1) \cdots H_i(t_n) = \cdots \\ &\cdots \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} \int_{\infty}^{t} \cdots \int_{-\infty}^{t} \mathrm{d}t_1 \cdots \mathrm{d}t_n T\{H_i(t_1) \cdots H_i(t_n)\} \end{aligned} \tag{1.233}$$

In other words the unitary time evolution operators can be written in terms of a Dyson series,

$$U(t) = 1 + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} \int_{\infty}^{t} \cdots \int_{-\infty}^{t} \mathrm{d}t_1 \cdots \mathrm{d}t_n T\{H_i(t_1) \cdots H_i(t_n)\}, \tag{1.234}$$

which is nothing but the exponential series. Therefore

$$U(t) = \exp\left(-i \int_{-\infty}^{t} \mathrm{d}t' H_i(t') \right) \tag{1.235}$$

In addition to this, one usually does not know the final times, so we can study the limit $t \to \infty$. In this limit, the unitary time evolution operator becomes the so called S-matrix

$$S = S(t) = \lim_{t\to\infty} U(t) = \exp\left(-i \int_{-\infty}^{\infty} \mathrm{d}t H_i(t) \right) \tag{1.236}$$

In terms of the Hamiltonian densities, the S-matrix can be rewritten as

$$S(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4x_1 \cdots \int \mathrm{d}^4x_n T\{\mathcal{H}_i(t_1) \cdots \mathcal{H}_i(t_n)\} \tag{1.237}$$

This being said, we can study how this operator behaves between initial and final states. In other words, our duty relies on calculating the matrix elements associated to time ordered products of the Hamiltonians.

However, we still do not know how to tackle these products. Fortunately for us, there is a method known as the Wick expansion,based purely on combinatorics. We will study if further on the next section.

1.3.5 Wick's Theorem

On our previous section we developed a formalism, on which it is possible to find how fields interact with each other, referred to as the interaction picture of quantum mechanics. In this picture, we found that the non trivial part of the S-Matrix depends on time ordered products of the Hamiltonian densities.[7]

Our first encounter with the time ordered product was made while studying the causal propagator

$$iD_F(x,y) = \langle \Omega | \, T[\phi(x)\phi(y)] \, | \Omega \rangle \tag{1.238}$$

However, we are considering our fields to be Heisenberg operators. As we want to use interacting fields, we transform them via

$$\phi = U^{-1}\phi_I U \tag{1.239}$$

[7]Without loss of generality, we use scalar fields for our calculation.

where U is the time evolution operator. This gives us:

$$\begin{aligned}\langle\Omega|\, T[\phi(x)\phi(y)]\,|\Omega\rangle &= \langle 0|\, U^{-1}(t_1)\phi_I(x)U(t_1)U^{-1}(t_2)\phi_I(y)U(t_2)\,|\Omega\rangle \\ &= \langle\Omega|\, U^{-1}(t_1)\phi_I(x)U(t_1,t_2)\phi_I(y)U(t_2)\,|\Omega\rangle\end{aligned} \tag{1.240}$$

Now, we make an insertion of the unit matrix in terms of the unitary time evolution operator at an arbitrary time t [ref: Kaku]

$$\langle\Omega|\, U^{-1}(t_1)\phi_I(x)U(t_1,t_2)\phi_I(y)U(t_2)\,|\Omega\rangle = \langle\Omega|\, U^{-1}(t)U(t,t_1)\phi_I(x)U(t_1,t_2)\phi_I(y)U(t_2,-t)U(-t)\,|\Omega\rangle \tag{1.241}$$

Note that this expression can be written in terms of the time ordered product

$$\langle\Omega|\, U^{-1}(t)U(t,t_1)\phi_I(x)U(t_1,t_2)\phi_I(y)U(t_2,-t)U(-t)\,|\Omega\rangle = \langle\Omega|\, U^{-1}(t)T[\phi_I(x)\phi_I(y)U(t,-t)]U(-t)\,|\Omega\rangle \tag{1.242}$$

Additionally, we know that $U(-t)\,|0\rangle = 0$. As we require the vacuum to be stable, after an infinite time we obtain

$$\lim_{t\to\infty}\langle\Omega|\, U^{-1}(t) \propto \langle\Omega| \tag{1.243}$$

Thus

$$\lim_{t\to\infty}\langle\Omega|\, U^{-1}(t) = \kappa\,\langle\Omega| \tag{1.244}$$

As we assume the ground state to be properly normalized, we for the κ phase:

$$\kappa = \lim_{t\to\infty}\langle\Omega|\, U^{-1}(t)\,|0\rangle = \Big[\,\langle 0|\, U(t)\,|0\rangle\,\Big]^{-1} \tag{1.245}$$

Therefore, the causal propagator is

$$iD_F(x-y) = \frac{\langle\Omega|\, T[\phi_I(x)\phi_I(y)\exp\Big(i\int \mathrm{d}^4x\mathcal{H}_I(\phi_I)\Big)]\,|\Omega\rangle}{\langle\Omega|\, T\Big\{\exp i\int \mathrm{d}^4x\mathcal{H}(\phi_I)\Big\}\,|\Omega\rangle} \tag{1.246}$$

Now, let us take the argument of the time ordered product in the denominator, and expand it in terms of creation and annihilation operators

$$\phi(x)\phi(y) = \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y) \tag{1.247}$$

Note that this is related to the normal ordered product by

$$:\phi(x)\phi(y): = \phi(x)\phi(y) - \big[\phi^{(+)}(x), \phi^{(-)}(y)\big] \tag{1.248}$$

Hence

$$T[\phi(x)\phi(y)] =: \phi(x)\phi(y) : + \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] \tag{1.249}$$

In our discussion about quantization we saw that

$$\left[\phi^{(+)}(x), \phi^{(-)}(y)\right] = \langle\Omega| T[\phi(x)\phi(y)] |\Omega\rangle = iD_F(x-y) \tag{1.250}$$

This gives us the following expression for the time ordered product

$$T[\phi(x)\phi(y)] =: \phi(x)\phi(y) : + \overbrace{\phi(x)\phi}(y) \tag{1.251}$$

where we defined the contraction

$$\overbrace{\phi(x)\phi}(y) = iD_F(x-y) \tag{1.252}$$

This form of the time ordered product is known as the Wick expansion of the time ordered product. In the case of n fields, one obtains ref[gifted, kaku, peskin, lahiri]:

$$\begin{aligned} T[\phi_1(x_1)\cdots\phi_n(x_n)] = &: \phi_1\cdots\phi_n : + [: \overbrace{\phi_1\phi}_2\phi_3\cdots\phi_n : + \text{permutations}] \\ &+ [: \overbrace{\phi_1\phi}_2\phi_3\phi_4\cdots : + \text{permutations}] + \cdots \end{aligned} \tag{1.253}$$

For vector fields we have

$$\overbrace{A_\mu(x)A}_\nu(y) = iD_{\mu\nu}(x-y) \tag{1.254}$$

And for fermions

$$\overbrace{\psi(x)\bar{\psi}}(y) = iS_F(x-y) \tag{1.255}$$

Now that we know how to expand time ordered products by the means of Wick contractions, it is our duty to start studying theories involving interactions. The simplest of them is known as Quantum Electrodynamics (QED), and will be studied in the next section.

1.4 The Standard Model of Particle Physics

1.4.1 Quantum Electrodynamics

Based on the last two sections we are now able to calculate all elements for the S-Matrix, at a perturbative level, in terms of the time ordered product of the interaction Lagrangian or Hamiltonian. However, we still have Lagrangians associated for free fields.

It is now natural to wonder how to introduce interactions in QFT. The answer lies on the gauge invariance principle. In QFT the gauge invariance principle is based on the symmetry transformations of the Lagrangian under certain Lie groups, such that the EOM remain invariant. So far, we have the Lagrangian associated to fermion fields

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \tag{1.256}$$

Note that both $\bar{\psi}$ and ψ have two internal degrees of freedom. Hence, it is possible to apply a global $SO(2)$ transformation such that it leaves the Lagrangian density unchanged.[8]However, it is possible to extend the isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ to $SO(2) \cong U(1)$. Hence, we can map[9]

$$\begin{aligned} \psi \to \psi' =& e^{i\theta}\psi \\ \bar{\psi} \to \bar{\psi}' =& e^{-i\theta}\bar{\psi} \end{aligned} \tag{1.257}$$

It is straightforward to check that if θ is space-time independent, the Lagrangian will be invariant. If we now consider infinitesimal global transformations, we obtain the U(1) Noether current

$$J^{\mu} = ie\bar{\psi}\gamma^{\mu}\psi \tag{1.258}$$

By analyzing the structure of this current, we can see that it is some kind of interaction between our fields at a certain point, but we still do not have a mediator of it, so we still can not add this current to our Lagrangian.

Nevertheless, the interaction term can be introduced by considering local gauge transformations. In other words, we now consider the maps

$$\begin{aligned} \psi \to e^{i\theta(x)}\psi =& (1 + i\theta(x))\psi \\ \bar{\psi} \to e^{-i\theta(x)}\bar{\psi} =& (1 - i\theta(x))\bar{\psi} \end{aligned} \tag{1.259}$$

As we consider θ to be sufficiently small, the mass term will be invariant under local U(1) transformations. Unfortunately, the kinetic term does not remain invariant

$$\partial_{\mu}\psi' = e^{i\theta(x)}(i\partial_{\mu}\theta)\psi + e^{i\theta}\partial_{\mu}\psi \tag{1.260}$$

$$\neq e^{i\theta} \tag{1.261}$$

In order to eliminate the extra term, we need to introduce the U(1) covariant derivative,[10]

$$D_{\mu} = \partial_{\mu} - eiA_{\mu}, \tag{1.262}$$

[8]A global transformation is such that it does not depends on the space-time coordinates.

[9]This implies that our fermion fields are in the fundamental representation.

[10]Note that the gauge field A_{μ} is an affine connection on the Lie group

where the gauge field transforms in the adjoint representation

$$A_\mu \to A_\mu - e\partial_\mu \theta(x) \tag{1.263}$$

This being said, our U(1) gauge invariant Lagriangian is

$$\begin{aligned} \mathcal{L} =& i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi & (1.264)\\ =& i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^\mu \psi A_\mu & (1.265) \end{aligned}$$

Note that the Noether current is directly obtained from the structure of the local transformation. Additionally, we obtain that the mediator of the interaction between our fermion fields is the electromagnetic field. In other words, as $\bar{\psi}$ and ψ usually are electrically charged fields, their interactions are mediated by photons. This confirms what we know from basic electrodynamics: Both electric and magnetic fields, which mediate interactions between charges, are made out of photons.

Hence, we obtained a relativistic quantum theory of electrodynamics, now referred to as Quantum Electrodynamics (QED). The total Lagrangian is then

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\gamma^\mu \psi A_\mu \tag{1.266}$$

We obtained an interaction Lagrangian (Hamiltonian), such that we can now use the Wick theorem to obtain all possible S-Matrix elements [11]

$$\mathcal{H}_I = e\bar{\psi}\gamma^\mu A_\mu \psi = e\bar{\psi}\not{A}\psi \tag{1.267}$$

The S-Matrix is then

$$S^{(1)} = \exp\left[-ie \int \mathrm{d}^4 x T(\bar{\psi}\not{A}\psi)\right] \tag{1.268}$$

So, we need to calculate each order in S to obtain every possible interaction between electrically charged particles. First, let us consider first order processes

$$S^{(1)} = -ie \int \mathrm{d}^4 x T(:\bar{\psi}(x_1)\not{\ell}(x_1)\psi(x_1):) \tag{1.269}$$

Note that as we only have one possible space-time point, no contraction is possible. Thus, we expand all fields in terms of creation and annihilation operators

$$\begin{aligned} S^{(1)} =& -ie \int \mathrm{d}^4 x [\bar{\psi}^{(+)}\not{A}^{(+)}\psi^{(+)} + \bar{\psi}^{(-)}\not{A}^{(+)}\psi^{(+)} + \bar{\psi}^{(-)}\not{A}^{(-)}\psi^{(+)} + \bar{\psi}^{(-)}\not{A}^{(-)}\psi^{(-)} \\ &+ A_\mu^{(-)}\bar{\psi}^{(+)}\gamma^\mu\psi^{(+)} - \not{A}^{(-)}\psi^{(-)}\bar{\psi}^{(+)} + \bar{\psi}^{(-)}\gamma^\mu\psi^{(-)}A_\mu^{(+)} - \psi^{(-)}\bar{\psi}^{(+)}\not{A}^{(+)}] \end{aligned} \tag{1.270}$$

[11] we recover the electric charge in order to be consistent with all our references

As we want to analyze each of these terms, let us consider first $[\bar{\psi}^{(+)}\not{A}^{(+)}\psi^{(+)}$. If we note that all particles are being annihilated, then we can take the initial state to be

$$|i\rangle = |p,s\rangle \otimes |q,t\rangle \otimes |k,r\rangle \tag{1.271}$$

where p, q and k are the positron, electron and photon 3-momenta, while s,t and r their spins and polarizations respectively. As final state we take $|f\rangle = |\Omega\rangle$.

Then, the matrix element of this first order process is

$$\begin{aligned} S_{fi}^{(1)} = &- ie \int \mathrm{d}^4x \, \langle\Omega| \sum_{s'} \int \frac{\mathrm{d}^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E'_p}} d_{s'}(p') \bar{v}_{s'}(p') \sum_{t'} \frac{\mathrm{d}^3q'}{(2\pi)^3} \frac{1}{\sqrt{2E'_q}} c_{t'}(q') u_{t'}(p') \cdots \\ &\cdots \sum_{r'} \frac{\mathrm{d}^3k'}{(2\pi)^3} \frac{1}{\sqrt{2E'_k}} a'_r(k') \not{\epsilon}'_r(k') \sqrt{2E_p}\sqrt{2E_q}\sqrt{2E_k} d_s^\dagger(p) c_t^\dagger(q) a_r^\dagger(k) \, |\Omega\rangle \end{aligned} \tag{1.272}$$

If we now use the cannonical commutation (anticommutation) relations for the creation and annihilation operators, we obtain

$$S_{fi}^{(1)} = -ie \int \mathrm{d}^4x e^{-i(p+q+k)} \bar{v}_s \not{p} \not{\epsilon}_r(k) u_t(q) \tag{1.273}$$

After performing the integral, an expression which only depends on the momenta can be obtained

$$S_{fi}^{(1)} = -ie(2\pi)^4 \delta(p+q+k) \bar{v}_s(p) \not{\epsilon}_r(k) u_t(q) \tag{1.274}$$

Note that the delta function ensures momentum conservation. In other words, as there are no particles in the final state this matrix element vanishes, as well as the rest of the first order processes. However, we obtained some interesting results. Namely, we obtained that we can assign momentum space spinors for electrons and positrons, as well as the vertex function. We will also introduce a diagramatic notation for each of these spinors, known as Feynman diagrams. The set of both assignations and diagrams is known as the Feynman rules, and help us to build all of our possible matrix elements for the S-matrix. For basic processes involving no particle propagation, these rules are given by

- For every incoming fermion assign a spinor $u_s(p)$

Figure 1.8: Incoming fermion.

- For every incoming antifermion assign a spinor $\bar{v}_s(p)$

Figure 1.9: Incoming antifermion.

- For every incoming boson assign a polarization vector $\epsilon_r^\mu(p)$

Figure 1.10: Incoming boson.

- For every outgoing fermion assing a spinor $\bar{u}_s(p)$

Figure 1.11: Outgoing fermion.

- For every outgoing antifermion assign a spinor $v_s(p)$

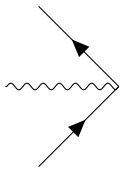
Figure 1.12: Outgoing antifermion.

- For every outgoing boson assign a polarization vector $\epsilon_{\mu r}^*(p)$

Figure 1.13: Outgoing boson.

- All particles connect into a vertex. This vertex has a momentum space function given by $-ie\gamma^\mu$

According to these rules, the first order process that we were studying can be diagrammatically established by



$$\sim \quad \overline{\psi}^{(+)} \not{A}^{(+)} \psi^{(+)} \quad \propto \quad \overline{v}_s(p) \epsilon_\gamma^\mu(k) \gamma_\mu u_t(q)$$

As mentioned above, all matrix elements associated to first order processes vanish due to momentum conservation. One can then check that it also happens to second order processes with no contractions. Hence, we will focus on the one and two contractions regime, as they will give us the final set of Feynman rules for QED.

For second order processes, the S-Matrix is given by

$$S^{(2)} = -\frac{e^2}{2} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 T[: \bar{\psi}(x_1) \not{A}(x_1) \psi(x_1) \bar{\psi}(x_2) \not{A}(x_2) \psi(x_2) :] \tag{1.275}$$

Note that if we consider now processes involving one contraction, it is straightforward to see that only three contractions arise, two associated to fermion fields and one from a photon.

Hence, the matrix element associated to fermion contractions is given by

$$\begin{aligned} S_{1c}^{(2)} = &- \frac{e^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 : \bar{\psi}(x_1) \not{A}(x_1) \overbrace{\psi(x_1)\bar{\psi}}(x_2) \not{A}(x_2) \psi(x_2) : - \cdots \\ &\cdots \frac{e^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 : \overbrace{\psi(x_2)\bar{\psi}}(x_1) \not{A}(x_1) \psi(x_1) \bar{\psi}(x_2) \not{A}(x_2) : \end{aligned} \tag{1.276}$$

Note that we can change the order of the non contracting spinors, as well as the indices, such that the two integrals add

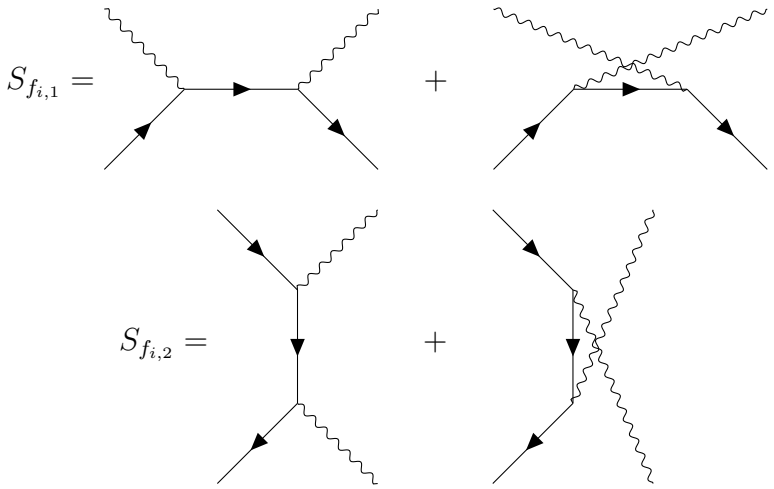
$$S_1^{(2)} = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 i S_F(x_1 - x_2) : \bar{\psi}(x_1) \not{A}(x_1) \not{A}(x_2) \psi(x_2) : \tag{1.277}$$

This corresponds to the following momentum space terms

$$\begin{aligned} S_{f,i1}^{(2)} = &- e^2 (2\pi)^4 \delta(p + k - q - m) [\bar{u}_t(p) \not{\epsilon}_x^*(k) i S_F(q_1) \not{\epsilon}_y(m) u_r(q) \\ &+ \bar{u}_t(p) \not{\epsilon}_y(m) i S_F(q_2) \not{\epsilon}_x^*(k) u_r(q)] \end{aligned} \tag{1.278}$$

$$\begin{aligned} S_{f,i2}^{(2)} = &- e^2 (2\pi)^4 \delta(k_1 + k_2 - p - q) [\bar{v}_s(q) \not{\epsilon}_r^*(k_1) i S_F(q_1) \not{\epsilon}_\ell^*(k_2) u_t(p) \\ &+ \bar{v}_s(q) \not{\epsilon}_\ell^*(k_2) i S_F(q_1) \not{\epsilon}_r^*(k_1) u_t(p)] \end{aligned} \tag{1.279}$$

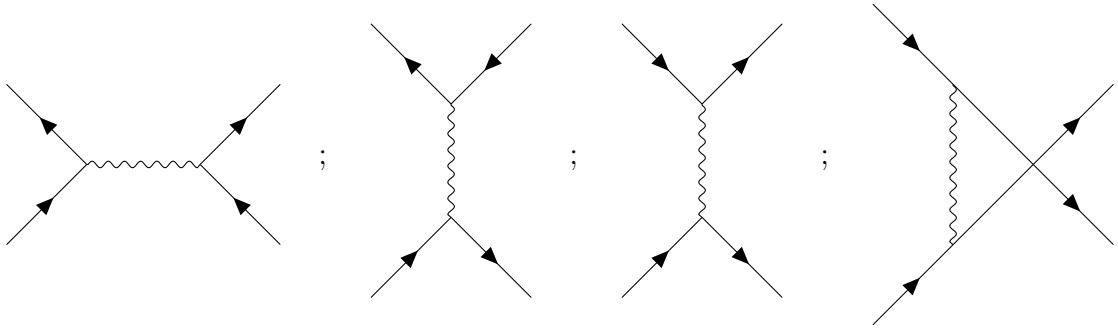
They correspond to the diagrams shown below. It is useful to note that the first two diagrams are associated to Compton scattering of an electron, while the remaining ones correspond to the annihilation of an electron-positron pair into two photons



There are more terms arising from these contractions, such as the ones associated to the Compton scattering of a positron, as well as the Breit-Wheeler process [ref:mandl].

To finish with the 1 contraction contributions, we proceed with the photon contraction. One

can easily check that the non vanishing contributions can be associated to the following diagrams



From all these one contraction contributions we can then give the following Feynman rules, which arise from internal propagating lines

- For each internal fermion line assign a propagator $iS_F(q)$

Figure 1.14: Fermion internal line

- For each internal photon line assign a propagator $iD_{\mu\nu}(q)$. For simplicity we choose the Feynman gauge, such that the second term in the propagator vanishes.

Figure 1.15: Photon internal line

Now, to finish with second order processes, we study the case of two contractions. Let us first consider the case, where a photon contraction, as well as a fermion one, are present. The matrix element is

$$S_3^{(2)} = -\frac{e^2}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2[:\bar{\psi}(x_1)\overbrace{\not{A}(\underbrace{x_1}\psi(x_1)\bar{\psi}(x_2)\not{A}(\underbrace{x_2}}\psi(x_2):+:\bar{\psi}(x_1)\not{A}(\underbrace{x_1}\psi(\overbrace{x_1}\bar{\psi}(x_2)\not{A}(\underbrace{x_2}}\psi(x_2):] \tag{1.280}$$

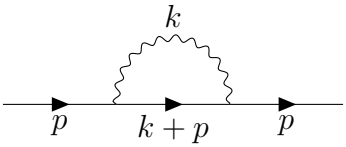
This reduces to

$$S_3^{(2)} = -e^2\int \mathrm{d}^4x_1\mathrm{d}^4x_2[S_F(x_2-x_1)D_{\mu\nu}(x_1-x_2)\gamma^\mu\gamma^\nu:\psi(x_1)\bar{\psi}(x_2):] \tag{1.281}$$

In momentum space, the matrix element corresponds to

$$S_{f,i3}^{(2)} = -e^2(2\pi)^4\delta(p'-p)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\eta_{\mu\nu}}{k^2+i\epsilon}\bar{u}_s(p')\gamma^\mu\frac{\not{k}+\not{p}+m}{(k+p)^2-m^2+i\epsilon}\gamma^\nu u_r(p) \tag{1.282}$$

By using our previously known Feynman rules, we obtain the following diagram



These type of diagrams are known as loop terms, and are considered to be corrections to certain physical parameters such as the vertex function and the propagator. In this case, this diagram is known as the self energy of the electron. Note that the existence of loop terms gives us one final rule:

- For each loop present in a diagram, integrate over the momenta of the additional particle $\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$.

Note that the value of the self energy of the electron depends on how one calculates the integral. As it has poles, one can easily check that this integral diverges linearly with k [ref:kaku]. These kind of divergences are known as Ultra-Violet (UV) as they happen at high values of k. Fortunately, this linear divergence vanishes due to the symmetry of the integral.

However, there is still an extra logarithmic divergence at low values of k. This integral can be finite made if one adds one term, known as a regulator, to the photon propagator

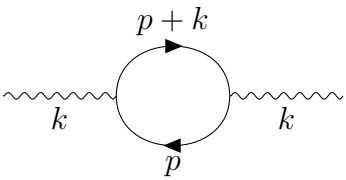
$$D_{\mu\nu} = \frac{-\eta_{\mu\nu}}{k^2 - \mu^2 + i\epsilon} \tag{1.283}$$

The procedure of adding these terms to the integral is known as regularization, and implies our lack of knowledge at certain physical scales[12]. It is worth noticing that this addition causes the photon to be virtual, i.e massive.

If we keep calculating all second order contractions, we find that there is one extra pair of them. This contractions reduce to a single expression which in position space is given by

$$S^{(2)}_{f,i4} = -\frac{e^2}{2} \int \mathrm{d}x_1 \mathrm{d}x_2 i S_F(x_1 - x_2) i S_F(x_2 - x_1) : \not{A}(x_1) \not{A}(x_2) : \tag{1.284}$$

Therefore, its Feynman diagram is



Note that we have obtained a loop term once again. In momentum space, this loop term takes the

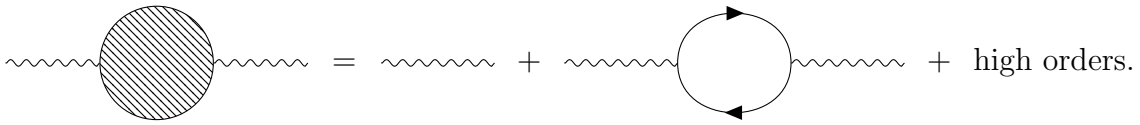
[12]For more information in regularization see [Ref:srednicki, schwartz]

form

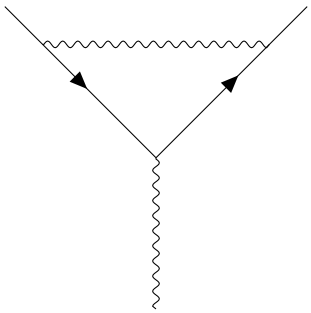
$$S^{(2)}_{f,i4} = -e^2(2\pi)^4\delta(k'-k)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\not{p}+m}{p^2-m^2+i\epsilon}\gamma^\mu\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\epsilon}\gamma^\nu\right]\epsilon^*_\mu(k')\epsilon_\nu(k) \quad (1.285)$$

This integral also diverges, but in a different way than the previous one. Let us count dimensions to realize this: d^4p has dimension 4, while the propagators have dimension -1. Hence, the integral will have a quadratic divergence when integrating over the fermion momentum p. To tackle these divergencies one needs to use renormalization procedures, see [ref: kaku, srednicki] for more information.

Even though the existence of these divergencies complicates our calculations, QED arose as the first renormalizable QFT. In other words, all divergencies can be taken out, such that all loop terms give small contributions to our physical parameters. As a matter of fact, our previous loop diagram acts like a correction to the photon propagator



If we keep calculating higher order contractions on the scattering matrix, we find that loop diagrams also modify quantities like the magnetic moment for the electron. More specifically, diagrams like



predict that the magnetic moment of the electron, g, obtains a correction given by [ref: peskin]

$$\left(\frac{g-2}{2}\right)_{theo} = 0.001159652411 \quad (1.286)$$

This matches remarkably with the experimental results obtained by using Penning traps, up to twelve significant figures [ref:PDG]

$$\left(\frac{g-2}{2}\right)_{exp} = 0.001159652209 \quad (1.287)$$

All the results and rules that we obtained in this section allow us to properly calculate the matrix elements of the scattering matrix. However, it is worth noticing that in all our cases these

elements take the form,

$$S = (2\pi)^4 \delta(\text{final momenta} - \text{initial momenta}) \times -i\mathcal{M}, \tag{1.288}$$

whew $\mathcal{M}$ is known as the Feynman, or invariant, amplitude. In the next section we will see how this amplitude contains all the physical information regarding cross sections, decay widths and decay times.

1.4.2 Interlude 1: Cross Sections and Fermi's Golden Rule

Before heading to relativistic cross sections and decay widths, let us recall some basic concepts of time independent perturbation theory. First, let us consider a system whose dynamics are governed by a Hamiltonian H_0. Hence, the energies associated to any eigenstate can be calculated by the means of the Schrödinger equation

$$H_0 \left|\psi\right\rangle = E_0 \left|\psi\right\rangle . \tag{1.289}$$

However, if the system is perturbed these energy values will change depending on the behavior of the exerted perturbation. Say that one can describe this pertubation by the means of a Hamiltonian H', then the total Hamiltonian will take the form [ref: griffithsqm],

$$H = H_0 + \lambda H',$$

where we consider λ to be small. If we write the total energies as a power series in λ we find that the energy corrections, between two states, are non trivial as they take the form (up to second order)

$$E_{fi}^{(1)} = \left\langle f\right| H' \left|i\right\rangle \tag{1.290}$$

$$E_{fi}^{(2)} = \sum_{j\neq i} \frac{\left\langle f\right| H' \left|j\right\rangle \left\langle j\right| H' \left|i\right\rangle}{E_i - E_j} \tag{1.291}$$

Thus, the total corrections can be collected in the transition matrix [ref: thomson]

$$T_{fi} = \left\langle f\right| H' \left|i\right\rangle + \sum_{j\neq i} \frac{\left\langle f\right| H' \left|j\right\rangle \left\langle j\right| H' \left|i\right\rangle}{E_i - E_j} + \text{higher orders} \tag{1.292}$$

This being said, one can calculate the transition rate between two states by the means of

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i) \tag{1.293}$$

Here, $\rho(E)$ is known as the density of states and gives us information about the states that are accessible given a transition matrix. This last expression is known as the Fermi Golden Rule

(FGR).

So, all that we need is to calculate the density of states, which is formally defined by

$$\rho(E_i) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E=E_i} \tag{1.294}$$

where n is the number of states accessible at a given energy. It is worth noticing that this number of states usually gives us information about the manifold formed by all states. Now, let us rewrite the density as

$$\rho(E_i) = \int \mathrm{d}E\delta(E - E_i)\frac{\mathrm{d}n}{\mathrm{d}E} \tag{1.295}$$

Hence

$$\Gamma_{fi} = 2\pi \int \mathrm{d}E\delta(E - E_i)|T_{fi}|^2 \tag{1.296}$$

If the number of states depends explicitly on the momentum, we can use the chain rule such that

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}p}\left|\frac{\mathrm{d}p}{\mathrm{d}E}\right| \tag{1.297}$$

This relation allows us to relate dn to the phase space occupied by a particle with energies between E and E+dE. For the non relativistic case, one can set periodic boundary conditions such that

$$\vec{n} = \frac{L}{2\pi}\vec{P} \rightarrow \mathrm{d}^3p = \left(\frac{2\pi}{L}\right)^3 \mathrm{d}^3n \tag{1.298}$$

Note that this last relation implies

$$\mathrm{d}^3n = \frac{L^3}{(2\pi)^3} \times 4\pi p^2 \mathrm{d}p \tag{1.299}$$

Without loss of generality one can set L=1 such that

$$\mathrm{d}n = \frac{\mathrm{d}^3p}{(2\pi)^3} \tag{1.300}$$

On the other hand, in the case of relativistic particles one has to ensure that all energy states available follow the mass-shell condition. This obligates the phase space element to be

$$\mathrm{d}n = \frac{\mathrm{d}^4p}{(2\pi)^4}\delta(p_0 - m^2)\theta(p_0) \tag{1.301}$$

$$= \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_P} \tag{1.302}$$

So, for relativistic particles the transition rate, or decay width, becomes

$$\Gamma_{fi} = 2\pi \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_P} \delta(E_i - E)|T_{fi}|^2 \tag{1.303}$$

If there are more than one final state momenta, we have to introduce a momentum conservation function, as well as integrals over all final states

$$\Gamma_{fi} = 2\pi \int \Pi_{i=1}^{N} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{2E_P} (2\pi)^3 \delta(p_{in} - \sum_{i=1}^{N} p_i) \delta(E_i - E)|T_{fi}|^2 = \quad (2\pi)^4 \int \Pi_{i=1}^{N} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \delta^{(4)}(p_{in} - p_{out})|T_{fi}|^2 \tag{1.304}$$

Now, let us note that we require our transition matrix to be Lorentz invariant, as quantities such as cross sections and decay widths should be scalars. To do so, we introduce the invariant amplitude as

$$\mathcal{M}_{fi} = \Pi_{in} \sqrt{2E_{in}} \Pi_{out} \sqrt{2E_{out}} T_{fi} \tag{1.305}$$

By making this insertion we get that for a $1 \to 2$ process the decay width becomes

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_{in}} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \frac{\mathrm{d}^3 p_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} \delta(p_{in} - p_1 - p_2)|\mathcal{M}_{fi}|^2 \tag{1.306}$$

It is worth noticing that the only parameter, that contains all the dynamical information about the physical system is the invariant matrix element $\mathcal{M}_{fi}$, and it is the same matrix that we obtain through the usage of Feynman rules.

In the Centre Of Mass Frame (COM), the decay width can be written as [ref:thomson]

$$\Gamma_{fi} = \frac{f(m_0, m_1, m_1)}{32\pi^2 m_0^2} \int \mathrm{d}\Omega |\mathcal{M}_{fi}|^2 \tag{1.307}$$

where the function f is given by

$$f(m_0, m_1, m_2) \frac{1}{m_0} \sqrt{[m_0^2 - (m_1 + m_2)^2][m_0^2 - (m_1 - m_2)^2]} \tag{1.308}$$

Given a total decay rate, one can also obtain the lifetime of a given particle or state as the inverse of the total decay width

$$\tau = \frac{1}{\Gamma_{Total}} = \frac{1}{\sum_i \Gamma_i} \tag{1.309}$$

Finally, there is one more kinematic variable that can be obtained, the so-called interaction cross section, σ. This quantity basically tells us about the probability, per unit area, of a process to

happen. For a $1 \to 2$ process, in the COM, the cross section is given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{COM} = \frac{1}{64\pi^2(E_1+E_2)^2}\left(\frac{|\vec{p}_f|}{|\vec{p}_i|}\right)_{COM}\Pi_{fermions}(2m_f)|\mathcal{M}_{fi}|^2 \tag{1.310}$$

where p_f is the momentum, in the COM, of one of the final state particles.

In the following sections we will study two more fundamental theories, namely the those regarding the strong and weak interactions, and how their Feynman rules arise from the theory.

1.4.3 The Strong Interaction

In previous sections we studied how to obtain the scattering amplitudes from a given interaction Lagrangian, taking QED as an example. Nevertheless, by the time QED was developed, there were two extra interactions that required the use of quantum mechanics to be fully understood. These interactions originated from the study of the atomic nucleus and are known as the Strong and Weak Interactions. In this section we will make a brief summary of how the Strong Interaction (SI) introduced the so called particle zoo, as well as the concept of quarks.

By the end of the 1940's QED was the first successful QFT to leading order, before the development of renormalization. However, it does not fully explain how the atomic nucleus is binds together, in spite of the electromagnetic repulsion between protons. It wasn't until Werner Heisenberg and Pascual Jordan, among other physicists, that a full theoretical study of the quantum behavior of the SI was made. Nonetheless, some empirical based progress was first made by Max Planck.

Planck realized, based on experimental observations, that the mass of an atomic nucleus with Z protons and N nucleons (protons and neutrons) is less than the sum of the individual masses of these constituents [ref:basdevant]

$$m(N,Z) < Zm_p + (N-Z)m_n \tag{1.311}$$

Planck associated this mass deficit to the binding energy that holds both protons and neutrons together [ref:basdevant]

$$E_B = Zm_p + (N-Z)m_n - m(N,Z) \tag{1.312}$$

This binding energy would be later explained in terms of empirical constants, by Weizsäcker and Bethe, through the semi-empirical formula

$$E_B = \alpha N - \beta N^{2/3} - \gamma Z(Z-1)N^{-1/3} - \rho\frac{(N-2Z)^2}{N} + \delta \tag{1.313}$$

They associated each of these constants to how the nucleons are distributed inside the nucleus

[ref:kirsonpaper]

- $\alpha \rightarrow$ Volume term : E_B must be proportional to the number of nucleons in the bulk.
- $\beta \rightarrow$ Surface term: Is a counterterm to the volume one as there are less superficial nucleons than those in the bulk.
- $\gamma \rightarrow$ Coulomb term: Term arising from considering the proton a uniform charge distribution. Can be easily obtained from the Coulomb formula of electric repulsion.
- $\rho \rightarrow$ Pauli term: Extra energy required to satisfy Pauli exclusion principle. Note that more energy is required for a nucleus with large numbers of protons, making it unstable.
- δ Spin term: Energy associated to spin coupling between nucleons.

Note that volume and surface terms do not depend on the number of protons or neutrons, but on the number of nucleons. Additionally, Pauli term tells us that atoms tend to have an equal number of protons and neutrons. In general, the Weizsäcker-Bethe formula tells us that the SI does not distinguish between protons and neutrons.

Inspired by this fact, and due to the nearly equal values of the proton and neutron masses, Heisenberg considered that the a nucleon could be explained as a superposition of neutron and proton states, living in a 2D Hilbert space [ref: basdevant]

$$|N\rangle = a\,|p\rangle + b\,|n\rangle \tag{1.314}$$

It is worth noticing, that the set of all symmetry transformations of this Hilbert space is described by the SU(2) group. However, Heisenberg realized that even though this same symmetry describes spin, this was different and called it's associated quantum number isospin.

This being said, one can define an isospin operator as

$$\vec{T} = T_i \hat{x}_i; i = 1, 2, 3. \tag{1.315}$$

where the T_i are the generators of the SU(2) Lie algebra and are related to Pauli matrices via[13]

$$T_i = \frac{1}{2}\tau_i \tag{1.316}$$

Now, after choosing T_3 and T^2 to be diagonal, one can assign isospin values to protons and neutrons

$$T_3\,|p\rangle = \frac{1}{2}\,|p\rangle \qquad\qquad T_3\,|n\rangle = -\frac{1}{2}\,|n\rangle \tag{1.317}$$

with this assignation, we can infer that protons and neutrons are part of the isospin doublet.

At the same time of Heisenberg and Pascual development of a quantum theory of the SI, Hideki Yukawa realized that the strong interaction is short ranged, and based on QED asserted that this

[13]Here we use τ instead of the usual spin related matrices σ to distinguish between spin and isospin algebras.

interaction needed to be mediated by a massive particle. He called this particle a meson and predicted a mass of $M \approx 200m_e$ [ref: schwartz].

Around 1937 a new particle was discovered by Carl Anderson and Seth Neddermeyer. This new particle, now called the muon, was discovered with a mass $m_\mu \approx 207m_e$ while studying cosmic rays. As the muon mass was in agreement with Yukawa's prediction for the meson, many physicists thought that this new particle was indeed a meson. Nevertheless, it was later proved that the muon interacts weakly with protons and neutrons, implying it was not Yukawa's meson.

Even though the muon had been proved to be a different particle than the meson, there was a new particle discovery in 1941. This discovery was made by Cecil Powell after he used emulsions to see charged tracks from cosmic rays. He saw an unpredicted charged track [ref:powellpaper], which is now associated to charged pions in the decay modes

$$\pi^+ \to \mu^+ + \nu_\mu \qquad\qquad \pi^- \to \mu^- + \bar{\nu}_\mu \tag{1.318}$$

They also studied the properties of these new pions, and found that they interacted strongly with nucleons and had a mass $m_{\pi^\pm} = 139.6MeV$ in total agreement with Yukawa's prediction [ref:pdg]. At low energies, the quantum theory of SI predicts that mesons, like the pion, mediate the strong interactions [ref:greinerqcd]. Around 1950 a new pion, in this case a neutral one with mass $m_{\pi^0} = 134.9MeV$, was discovered in the channel [ref:neutralpion]

$$\pi^0 \to \gamma\gamma \tag{1.319}$$

As mesons participate in the SI, there was a need to fit them within the isospin scheme, more precisely into an isospin triplet. This could be done as the masses of all pions are almost equal. In terms of isospin states, these particles have the assignation [14][ref:griffiths]

$$\left|\pi^0\right\rangle = |1,0\rangle \qquad\qquad \left|\pi^+\right\rangle = |1,1\rangle \qquad\qquad \left|\pi^-\right\rangle = |1,-1\rangle \tag{1.320}$$

According to experimental measurements, mesons differ from protons and neutrons not only in their isospin, but also in their spin values. This allowed physicists to divide particles in the isospin scheme into two different families: Baryons and Mesons. Baryons, like the proton and neutron are characterized by having half-integer spin values, contrary to the integer values of meson spin.

Treating isospin as an exact symmetry of the SI implies that process like,

$$\pi^+ + p \to \pi^+ + p \qquad\qquad \pi^- + p \to \pi^- + p$$

must be observed in nature. In fact, these scattering processes were observed but had a particularity, they presented resonances at 1.2 GeV [ref:fermipapercross]. Isospin conservation then tells

[14] Here the isospin states are noted by $|T, T_3\rangle$

us that these resonances can be associated with new physical states

$$\left|\Delta^{++}\right\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle \qquad \left|\Delta^{0}\right\rangle = \left|\frac{3}{2}, -\frac{1}{2}\right\rangle \tag{1.321}$$

respectively. Note that these states live in the isospin quadruplet, hence two more resonances needed to be discovered in nature, namely

$$\left|\Delta^{+}\right\rangle = \left|\frac{3}{2}, \frac{1}{2}\right\rangle \qquad \left|\Delta^{-}\right\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle \tag{1.322}$$

where found in 1956 [ref:ashkinpionproton].

The discovery of the Δ resonances proved that the isospin model worked at the energies that could be reached by mid-1950's. However, it did not explained why baryonic and mesonic-like resonances, like kaons and Λ baryons existed. As more energetic processes became reachable, due to improvements on technology and resources, particles like Σ and Ξ also did not couple to the isospin scheme. Thus, Murray Gell-Mann, Abraham Pais and Kazuhiko Nishijima introduced a new quantum number to explain all these new resonances and their behavior, the strangeness S.

Pais, Gell-Mann and Nishijima assigned the following strangeness values to all known baryons and mesons

$$\begin{aligned}
&p, n, \pi^{+}, \pi^{-}, \pi^{0}, \Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-} \rightarrow S = 0 \\
&k^{+}, k^{0}, k^{-} \rightarrow S = 1 \\
&\Lambda, \Sigma \rightarrow S = -1 \\
&\Xi \rightarrow S = -2
\end{aligned}$$

Additionally, Gell-Mann and Nishijima introduced another quantum number to distinguish between baryons and mesons, the baryon number B. They made the assignation

$$\begin{aligned}
&p, n, \Delta, \Lambda, \Sigma, \Xi \rightarrow B = 1 \\
&\pi^{+}, \pi^{-}, \pi^{0}, k^{+}, k^{0}, k- \rightarrow B = 0
\end{aligned}$$

Then, George Zweig, doctoral student to Gell-Mann, unified both quantum numbers into the so-called hypercharge via

$$Y = B + S \tag{1.323}$$

which is then associated with the electric charge via the Gell-Mann Nishijima formula

$$Q = T_3 + \frac{1}{2}Y \tag{1.324}$$

Note that this formula tells us that the electric charge is not a fundamental quantity, whose role is then taken by the isospin and hypercharge. Given all these quantum numbers, one can then

classify baryons and mesons in the so called multiplets [15][ref:griffiths]. For spin-$\frac{1}{2}$ baryons we have

Particle	T	T_3	B	Y	S
p	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0
n	$\frac{1}{2}$	$-\frac{1}{2}$	1	1	0
Λ^0	0	0	1	0	-1
Σ^+	1	1	1	0	-1
Σ^0	1	0	1	0	-1
Ξ^0	$\frac{1}{2}$	$\frac{1}{2}$	1	-1	-2
Ξ^-	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	-2

The geometric representation of this baryon octet is given by

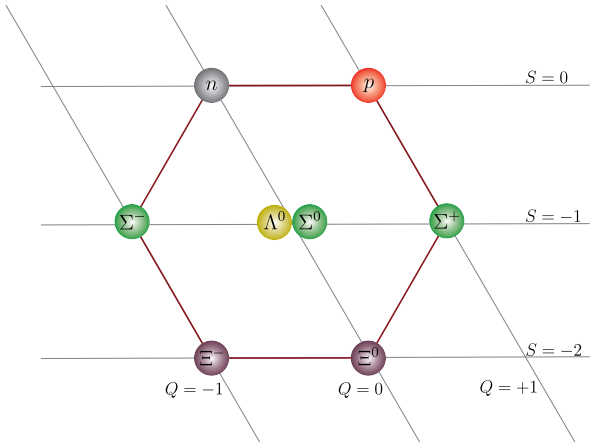


Figure 1.16: Geometrical representation of the baryon octet

For mesons we also have an octet[16]

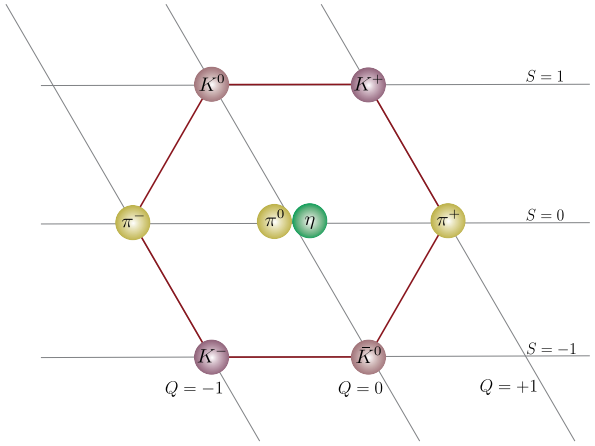


Figure 1.17: Geometrical representation of the meson octet

[15]Loosely speaking, a multiplet is a geometric figure or table whose elements can transform into another via group transformations

[16]There are also decouplets for both baryons and mesons, for more information see[ref:griffiths]

Gell-Mann and Nishijima then studied the set of all possible transformations that leave these multiplets invariant, arriving to SU(3)[17]. As SU(3) is the group of all unitary transformations of a 3D Hilbert Space, it has a basis formed by three root vectors that are known as the up, down and strange quarks, that carry $\frac{1}{2}$ spin

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.325}$$

These root vectors form the fundamental representation of SU(3), noted by 3, which is geometrically given by

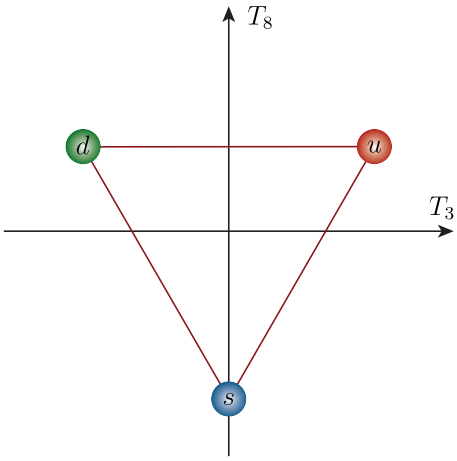


Figure 1.18: Geometrical representation of the fundamental representation of SU(3)

Note that the anti-fundamental representation, which is given by the conjugated vectors[18] is also possible, and is noted by $\bar{3}$.

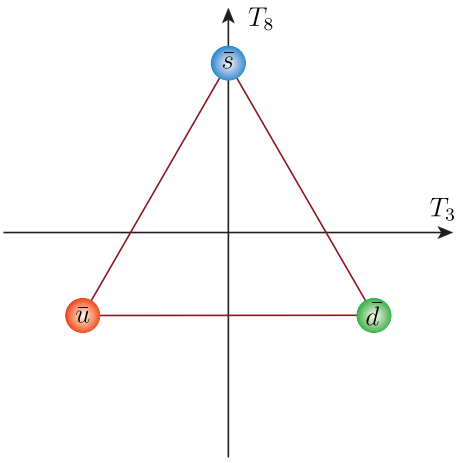


Figure 1.19: Geometrical representation of the anti-fundamental representation of SU(3)

[17] In this case this SU(3) is related to the flavor symmetry
[18] $|\bar{u}\rangle, |\bar{d}\rangle, |\bar{s}\rangle$

This means that baryons and mesons are composite particles which agrees with the experimental observations of deep inelastic scattering events [ref:thomson]. More specifically, baryons are particles composed by three quarks $|q_1 q_2 q_3\rangle$, while mesons are made out of a $|q\bar{q}\rangle$ pair.

However, the particle content of some baryons and mesons violates Pauli exclusion principle as the theory predicts $|\Delta^{++}\rangle = |uuu\rangle$ and $|sss\rangle$. As all three quarks have the same quantum numbers, these resonances should not exist on nature. Another problem arose from the three quark model as the GIM mechanism required the existence of a fourth quark, the charm [ref:gim]. This problem was first resolved by considering a 4D Hilbert space as state space and therefore a flavor SU(4) symmetry.

The new SU(4) scheme predicted the existence of new baryonic and mesonic resonances, that were later discovered, such as the J/ψ resonance. Nevertheless, the energies at which these particles were discovered suggested that the charm quark was more massive than the other three. This implies that the SU(4) symmetry is not exact and therefore would only work to predict quark content. Some years after this finding, a new exact symmetry would be postulated by Yoshiro Nambu.

Nambu considered the addition of a new quantum number that would resolve the problem regarding Pauli exclusion principle in hadrons, called the color charge. With this quantum number, all fundamental particles that participate in the SI would carry different colors, or linear combinations of them, such that the final hadron states would be color singlets. This means that quark states have the form

$$|\psi\rangle = C_i\,|\psi(p)\rangle\,; \quad C_1 = r = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad C_2 = g = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad C_3 = b = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.326}$$

Note that the Hilbert space associated with the color vectors has SU(3) as symmetry group, thus we need to build a QFT for color invariant Lagrangians.

1.4.4 Quantum Chromodynamics and SU(3)

In the last section we made a brief review of how the need of an $SU(3)_C$ invariant theory arises from a mixture between the quark model and the exclusion principle[19]. This section will act as a brief reminder of some basic concepts behind SU(3) representation theory, as well as will help us to build the basic Feynman rules for QCD without introducing a proper quantization procedure [20].

So far, we have only developed a QFT for the electromagnetic interaction, QED, and it is based on the invariance under the U(1) gauge group, which is an abelian Lie group. In our case for QCD

[19] The subscript C denotes color charge symmetry.

[20] As we saw with the photon field, quantizing vector fields becomes rather problematic. This problems can be easily solved using path integrals and introducing ghost fields. However, we are making a brief study on SM physics not on QFT. See [ref:Schwartz, Peskin, Weinberg] for more information on this topic.

we have SU(3), which is a non abelian Lie group with Lie algebra

$$[T_i, T_j] = i f_{ijk} T_k \tag{1.327}$$

where f_{ijk} are the skew-symmetric structure constants and T_i are the 8 generators of the algebra[21]. In the fundamental representation these generators are proportional to the Gell-Mann matrices via $T_i = \frac{1}{2}\lambda_i$. These matrices are given in Equation 1.328.

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{1.328}$$

Thus, any element of the group can be obtained from the generators via the exponential map given in Equation 1.329. Now, we can ask ourselves about the states on which these new symmetry transformations can be performed. This can be easily seen by noting that both T_3 and T_8 are diagonal, hence they share the same eigenstates given by $|t_3, Y\rangle$. Here, the Hypercharge operator is defined by $Y = \frac{2}{\sqrt{3}} T_8$.

$$g \in SU(3) = \exp(i\alpha_a T_a); \quad a = 1, \cdots, 8 \tag{1.329}$$

Given this set of eigenstates, one can also define SU(3) ladder operators by [22]

$$I_\pm = T_1 \pm iT_2 \quad V_\pm = T_4 \pm iT_5 \quad U_\pm = T_6 \pm iT_7 \tag{1.330}$$

After defining $2V_3 = \sqrt{3}T_8 + T_3$, and $2U_3 = \sqrt{3}T_8 - T_3$, one can easily check that we have three overlaping copies of SU(2) as all these operators obey

$$[I_+, I_-] = 2T_3 \quad [V_+, V_-] = 2V_3 \quad [U_+, U_-] = 2U_3 \tag{1.331}$$

The presence of SU(2) inside SU(3) is reason for the Isospin symmetry to make sense at low energies, but the overlaping tells us that it is not enough to describe all possible states interacting in the SI, hence the need of color charge in a group theory perspective.

As we know, quarks are the particles interacting under the SI and have spin $\frac{1}{2}$, which implies that their dynamics are governed by the Dirac Lagrangian [23]

$$\mathcal{L} = i\bar{\psi}_c^f \gamma^\mu \partial_\mu \psi_c^f - m\bar{\psi}_c^f \psi_c^f \tag{1.332}$$

[21] The structure constants take the role of the Levi-Civita density for SU(2).

[22] These ladder operators are the ones allowing all particles in a multiplet to transform between each other

[23] Here we sum over flavor (f) and color (c) indices.

For the sake of simplicity let us rewrite the spinors as

$$\bar{\psi}^f = (\bar{\psi}_r, \bar{\psi}_g, \bar{\psi}_b), \tag{1.333}$$

such that we reduce the number of indices in the Lagrangian

$$\mathcal{L} = i\bar{\psi}_f \gamma^\mu \partial_\mu \psi_f - m\bar{\psi}_f \psi_f \tag{1.334}$$

Now, we have to check if this Lagrangian is invariant under SU(3) global transformations. Thus, we map

$$\psi_f \to e^{i\alpha_a T_a}\psi_f \qquad\qquad \bar{\psi}_f \to \bar{\psi}_f e^{-i\alpha_a T_a} \tag{1.335}$$

As the global parameter α_a are space-time independent, the Lagrangian remains invariant. Hence, by using Noether's theorem we obtain eight conserved currents given by

$$J^\mu_a = \bar{\psi}_f \gamma^\mu T_a \psi_f \tag{1.336}$$

After checking that the Lagrangian is invariant under SU(3) global transformations, we can promote all transformations to a local setting. This means that our transformations now take the form

$$\psi_f \to \exp(ig_s\alpha_a(x)T_a)\psi_f \qquad\qquad \bar{\psi}_f \to \bar{\psi}_f \exp(-ig_s\alpha(x)T_a) \tag{1.337}$$

Local gauge transformations imply that the derivative term does not transform properly which, as in QED, obligates us to introduce an SU(3) covariant derivative

$$D_\mu = \partial_\mu + ig_s T_a A_{a\mu}, \tag{1.338}$$

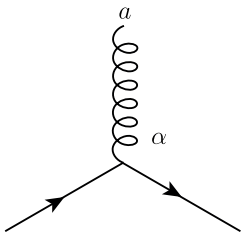
where A^μ_a are the eight gluon fields. To preserve gauge invariance in the theory, the gluon field must transform in the adjoint representation via [ref:schwartz]

$$A_{a\mu} \to A_{a\mu} - \partial_\mu \alpha_a - g_s f_{abc}\alpha_b A_{c\mu} \tag{1.339}$$

Hence, our fermionic gauge invariant Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}_f \gamma^\mu \partial_\mu \psi_f - m\bar{\psi}_f \psi_f - g_s J^\mu_a A_{a\mu} \tag{1.340}$$

Note that from the last term we can extract one of the three fundamental QCD vertices, namely the quark-gluon vertex



$$= ig_s\lambda^a\gamma^\alpha$$

Furthermore, we need to add all possible gluon dynamic terms. This can be done by introducing the non abelian field strength tensor defined as [ref:greinerqcd]

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_s f_{abc} A_b^\mu A_c^\nu \tag{1.341}$$

As vector field dynamics are governed by Lagrangians similar to those given in Equation1.95, the total QCD Lagrangian reads

$$\mathcal{L} = i\bar{\psi}_f\gamma^\mu\partial_\mu\psi_f - m\bar{\psi}_f\psi_f - g_s J_a^\mu A_{a\mu} - \frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a \tag{1.342}$$

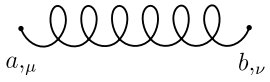
This Lagrangian describes all possible dynamics for SI processes. However, we need to quantize the theory this can be a very complicated procedure, due to the non abelian behavior of the gluon field. Hence, one needs to introduce functional integral quantization and Faddeev-Popov ghost fields. For more insight on these topics see [ref:amateur,srednicki, schwartz, kleinert].

With this Lagrangian, one can also deduce the behavior of g_s after renormalization procedures, where it goes to lower values as the momentum grows [ref:peskin]. Additionally, one can use Lattice Field Theory (LFT) to note that the theory is confining, which means that the potential takes the form

$$V(r) = ar + \frac{b}{r} \tag{1.343}$$

which explains why the SI is a close ranged interaction. Finally, one can use extra terms in the Lagrangian as part of the interaction Hamiltonian and use Wick's theorem to obtain all possible QCD Feynman rules. Basic rules are given by

- Gluon propagator

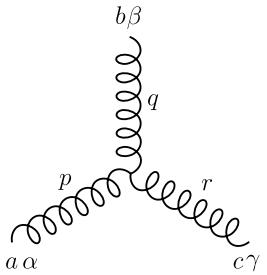


$$-i\frac{\eta_{\mu\nu}\delta_{ab}}{p^2 + i\epsilon}$$

- Quark propagator

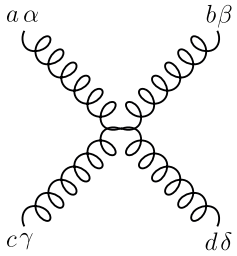
$$i\frac{\delta_{ab}}{\not{p} - m + i\epsilon}$$

- 3 Gluon vertex factor



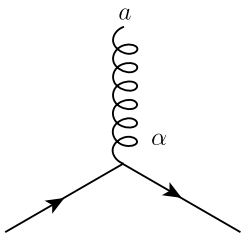
$$g_s f_{abc}[\eta^{\alpha\beta}(p-q)^\gamma + \eta^{\beta\gamma}(q-r)^\alpha + \eta^{\gamma\alpha}(r-p)^\beta]$$

- 4 Gluon vertex



$$\begin{aligned} &- ig_s^2 f_{xac} f_{xbd}(\eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha\delta}\eta^{\beta\gamma}) \\ &- ig_s^2 f_{xad} f_{xbc}(\eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha\gamma}\eta^{\beta\delta}) \\ &- ig_s^2 f_{xab} f_{xcd}(\eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\delta}\eta^{\beta\gamma}) \end{aligned}$$

- Quark-Gluon vertex factor



$$= ig_s \lambda^a \gamma^\alpha$$

1.4.5 The Weak Interaction and The Higgs Mechanism

So far we have analyzed how electromagnetic phenomena, and the SI, arise from requiring gauge invariant theories for fermions. However both of these theories do not give a theoretical explanation, at a quantum level, of nuclear decays or gravity. In this section we will focus on the first of them as quantum gravity is still an open problem in physics, which is tackled by more complicated theories such as String Theory and Loop Quantum Gravity.

In the 1900's after the discovery of radioactivity by Henri Becquerel, both physicists and chemists started to understand how atomic nuclei decay into lighter isotopes. This understanding was then broadened by Pierre and Marie Curie, as they studied different chemical isotopes, and their decays, and discovered that radioactive phenomena occur in three different types

- α decays: The atom radiates Helium nuclei and a lighter isotope.
- β decays: There is a change in the atomic number of the nuclei and an electron is emmited.

- γ radiation: Emission of highly energetic light beams.

The Curies found that β decays to be the most interesting type of radiation as it can be understood as the transmutation of a nucleus with atomic number Z. This transmutation can nowadays be understood as

$$Z \to Z \pm 1 + e^{\mp} \tag{1.344}$$

Thus, the decay of the neutron could be understood within this formalism with reaction given by

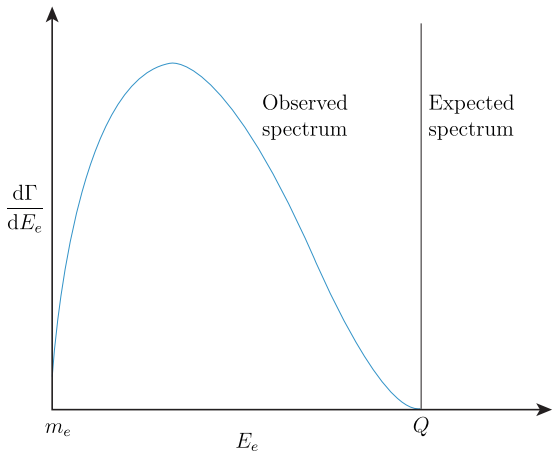
$$n \to p + e^{-} \tag{1.345}$$

In their studies of the β decay Pierre and Marie Curie found that the spectrum of the electron energy on decay of the tritium, ${}^3H \to {}^3 He + e^-$, was not what they expected. To understand this fact let us consider the tritium at rest. This will then imply, due to momentum conservation, that ${}^3He$ and the electron will go into opposite directions. This momentum conservation will then fix the electron energy to be [ref:haxton]

$$Q = M({}^3H) - M({}^3He) + \delta \tag{1.346}$$

where we considered that the energy of the nuclei is approximately their rest mass, accordingly with the initial considerations, and δ is the recoil energy of the ${}^3He$ nuclei (can be neglected). Hence, this unique value of the electron energy would give a monochromatic spectrum when measured in an experiment.

Nevertheless, different experiments showed that the energy distribution of the electrons was continuous within a range between m_e and the Q-value as seen in Figure **??**. A couple of years later a second problem was found, the decay did not respect angular momentum conservation.



To explain these phenomena various hypotheses were considered, the most polemical was the one given by Niels Böhr. Böhr's idea consisted in considering energy and angular momentum conservation to be statistical phenomena, such as entropy and enthalpy in statistical mechanics

[ref: tesispreg]. However, the most brilliant, and correct, idea was given by Wolfgang Pauli in 1933, where he considered that the β decay includes a massless spin-$\frac{1}{2}$ particle, which he called the neutrino. He realized this fact by noting that the angular momentum rule

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \tag{1.347}$$

does describe how the neutron can be produced from three different half integer spin particles. However Pauli was reluctant with his formulation as the neutrino, due to the lack of mass, would be almost impossible to detect at the time [24]

One of Pauli students, Enrico Fermi, was present in the conference where his advisor postulated the existence of the neutrino. Based on this proposition and the recently discovered quantum theory of electromagnetism, now QED, Fermi proposed that the β decay could be explained by the means of two currents that interact in a single point as seen in Figure 1.20

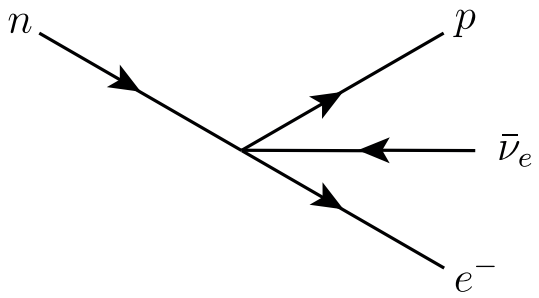


Figure 1.20: Graphical representation of Fermi's point-like interaction.

However, experimental phenomena proved that the interaction was not point-like and had to be mediated through a massive vector boson, contrary to QED. Around the 1950's Gell-Mann and Nishijima formulated the quark model, based on Isospin and hypercharge conservation. This allowed physicists to understand the β decay as the process

$$d \to u + e^- + \bar{\nu}_e \tag{1.348}$$

Inspired by the isospin model, which was not exact for quarks, Fermi attempted to build an SU(2) gauge invariant theory for the weak interaction, where β decay is just one possible manifestation of it. He argumented that quarks and leptons could be arranged into isospin doublets as $mu \approx m_d$ and $m_e << m_u$ (thus can be taken as negligible). These doublets are given by

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \quad \ell = \begin{pmatrix} \nu_e \\ e \end{pmatrix} \tag{1.349}$$

[24] It was indeed discovered by analyzing the energy spectra of nuclear radiation in 1956.

The isospin assignation for each element of the doublets is then [25]

$$T_3 |u\rangle = \frac{1}{2} |u\rangle \quad T_3 |\nu_e\rangle = \frac{1}{2} |\nu_e\rangle \quad T_3 |d\rangle = \frac{-1}{2} |d\rangle \quad T_3 \left|e^-\right\rangle = \frac{-1}{2} \left|e^-\right\rangle \tag{1.350}$$

Now, we need an SU(2) gauge invariant Lagrangian. Based on Fermi's point-like interacting currents, a possible Lagrangian can be taken as

$$\mathcal{L} = i\bar{q}\gamma^\mu \partial_\mu q + i\bar{\ell}\gamma^\mu \partial_\mu \ell - m_q \bar{q} q - m_\ell \bar{\ell}\ell + \frac{G_F}{\sqrt{2}} u^\dagger d e^\dagger \nu_e, \tag{1.351}$$

where G_F is the Fermi coupling constant [26]

Nevertheless, in 1956 Yang and Lee proposed that as there was no experimental evidence showing the weak interaction to be invariant under parity transformations, a parity violating theory must not be discarded [27] [ref:yangleeparity]. Their idea, though polemical and disregarded by physicists like Pauli, was proved right by Wu in 1957 after studying the decay of ^{6}0Co isotopes under a magnetic field [ref:wu]. She observed an excess of decay events on one specific direction of the applied magnetic field, contrary to the expected symmetrical distribution, implying that the weak interaction does not conserve parity.

The preference of an specific direction of decay in the weak interaction caused many physicists to look for solutions. Perhaps the most successful one was proposed by Feynman, where he considered that the theory could be explained via the introduction of vector-axial, or (V-A), currents as they are the only ones matching the experimental data given by Wu. The proposed currents take the form [ref:haxton]

$$J^\mu = \frac{G_F}{\sqrt{2}} \bar{\psi}\gamma^\mu P_L \psi; \quad P_L = \frac{1}{2}(1 - \gamma_5) \tag{1.352}$$

Note that using these (V-A) currents, obligates us to use a new representation of the gamma matrices, namely the Weyl (or chiral) representation, where they are given by [ref:gifted]

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{1.353}$$

As γ_5 is diagonal, one can decompose a Dirac spinor[28] in terms of the eigenstates of this matrix, known as left-handed and right-handed chiralities

$$\psi = \psi_L + \psi_R \tag{1.354}$$

[25] This assignation was later extended to up-like quarks (u, s, t), down-like quarks (d, c, b), electron-like leptons (e, μ, τ) and their corresponding neutrinos (ν_e, ν_μ, ν_τ).

[26] This constant was later understood as the weak coupling constant of an effective theory at low energies.

[27] We understand a parity transformation as mapping $\vec{x} \to -\vec{x}$ and $\vec{p} \to -\vec{p}$.

[28] A Dirac spinor is the 4-component solution of the Dirac equation.

As these eigenstates form a basis of the spinor space, one can define projectors as

$$P_L = \frac{1}{2}(1-\gamma_5)P_R = \frac{1}{2}(1+\gamma_5) \rightarrow P_L P_R = 0 \qquad\qquad P^2_{L/R} = P_{L/R} \tag{1.355}$$

Based on this projections, one can check that (V-A) currents imply that only left-handed chiralities participate on the weak interaction[29]. Thus, the doublets previously defined in Equations 1.349 and 1.350 are only formed by these left-handed states. This being said, if we replace Dirac spinors in terms of Equation 1.354 we obtain that spinor dynamics is given now by

$$\mathcal{L} = i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R - m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R \tag{1.356}$$

Note that an SU(2) transformation like

$$\psi_L \rightarrow \psi'_L \propto \exp(i\alpha\gamma_5)\psi_L$$

will not leave the mass terms invariant, we say then that mass terms break gauge invariance. Additionally, note that as neutrinos only participate in weak processes, they can only posses left-handed chirality which means that they are massless.

As mass terms break gauge invariance, physicists like Weinberg, Salam, Feynman, Rarita and Glashow started studying an analogous theory of the one originally proposed by Fermi, without including mass terms. In other words, they considered Lagrangians like

$$\mathcal{L} = i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R \tag{1.357}$$

where the left-handed chiralities are $SU(2)_L$ doublets [30] and the right-handed states are singlets. Hence, based on Equation 1.351 quark and lepton dynamics are given by

$$\mathcal{L} = i\bar{q}_L\gamma^\mu\partial_\mu q_L + i\bar{\ell}_L\gamma^\mu\partial_\mu\ell_L + i\bar{u}_R\gamma^\mu\partial_\mu u_R + i\bar{d}_R\gamma^\mu\partial_\mu d_R + i\bar{e}_R\gamma^\mu\partial_\mu e_R \tag{1.358}$$

where u_R, d_R and e_R are right-handed singlets for the up-like quarks, down-like quarks and charged leptons respectively.

As our doublets are the ones interacting via weak processes, a global $SU(2)_L$ transformation, for a general doublet ψ_L, takes the form [ref:goldberg]

$$\psi_L \rightarrow \exp\left(-ig_w\theta a\frac{\tau_a}{2}\right)\psi_L \quad \bar{\psi}_L \rightarrow \bar{\psi_L}\exp\left(ig_w\theta a\frac{\tau_a}{2}\right) \tag{1.359}$$

where $a = 1, 2, 3$, g_w is the weak coupling constant and τ_a are the Pauli matrices a. If the θ_a are space-time independent, the Lagrangian will be gauge invariant. Hence, we use Noether's theorem

[29] In other words, SU(2) will only act on left-handed states

[30] The subscript L denotes the chirality on which this group acts

to obtain three conserved currents which are given by

$$J_a^\mu = g_w \bar{\psi}_L \gamma^\mu \frac{\tau_a}{2} \psi_L \tag{1.360}$$

To see the structure of these currents let us, without loss of generality, take ψ_L to be a lepton doublet, such that

$$J_1^\mu = \frac{g_w}{2}(\bar{\nu}_{eL}\gamma^\mu e_L + \bar{e}_L \gamma^\mu \nu_{eL}) \tag{1.361}$$

$$J_2^\mu = -i\frac{g_w}{2}(\bar{\nu}_{eL}\gamma^\mu e_L - \bar{e}_L \gamma^\mu \nu_{eL}) \tag{1.362}$$

$$J_3^\mu = \frac{g_w}{2}(\bar{\nu}_{eL}\gamma^\mu \nu_{eL} - \bar{e}_L \gamma^\mu e_L) \tag{1.363}$$

Note that J_3 takes a very familiar form, namely that of a neutral current like the one we obtained from QED [31]:

$$J_{EM}^\mu = Q\bar{e}\gamma^\mu e \tag{1.364}$$

Now that we obtained a theory which is globally invariant, we consider local gauge transformations. I.e we map

$$\psi_L \to \exp(-ig_w \theta a(x) T_a)\psi_L \quad \bar{\psi}_L \to \bar{\psi}_L \exp(ig_w \theta a(x) T_a); \quad T_a = \frac{1}{2}\tau_a \tag{1.365}$$

Note that making θ_a to be space-time dependent will break gauge invariance as

$$\partial_\mu \psi_L \to -ig_w(\partial_\mu \theta_a) T_a e^{-ig_w \theta_a(x) T_a}\psi_L + e^{-ig_w\theta_a(x)T_a}\partial_\mu \psi_L$$

Thus, inspired by our QED results, we introduce the $\mathrm{SU}(2)_L$ covariant derivative

$$D_\mu = \partial_\mu + ig_w T_a W_a^\mu, \tag{1.366}$$

where the W_a are the three vector fields that mediate weak interactions. Additionally, these bosons transform in the adjoint representation [ref:schwartz]

$$W_a^\mu \to W_a^\mu + \partial^\mu \theta_a - g_w \sum_{b,c} \epsilon_{abc} W_b^\mu \theta_c(x) \tag{1.367}$$

It is worth noticing that there is an extra term in Equation 1.367 with respect to Equation 1.263, this is associated to the fact that $SU(2)_L$ is a non abelian group, as the commutator

$$[T_a, T_b] = i\epsilon_a bc T_c, \tag{1.368}$$

[31] This fact would inspire Weinberg, Glashow and Salam to think of electromagnetism as a consequence of another phenomenon.

does not vanish. In a similar way as for gluons, we introduce the field strength tensor as

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_w \sum_{b,c} \epsilon^{bcd} W^c_\mu W^d_\nu \tag{1.369}$$

With this definition, the total $SU(2)_L$ gauge invariant Lagrangian is given by

$$\mathcal{L} = i\bar{\psi}_L \gamma^\mu D_\mu \psi_L - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \tag{1.370}$$

Note that the kinetic term for the W bosons will introduce 4-point and 3-point diagrams like those given in Figures 1.21 and 1.22

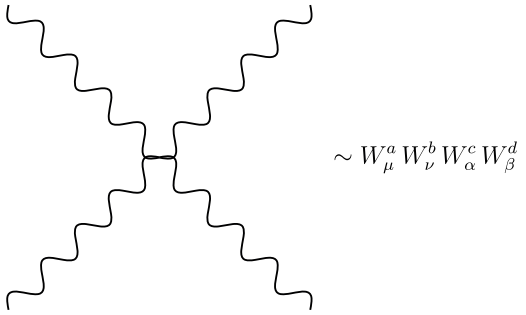


Figure 1.21: 4-Point diagram arising from W boson Lagrangian

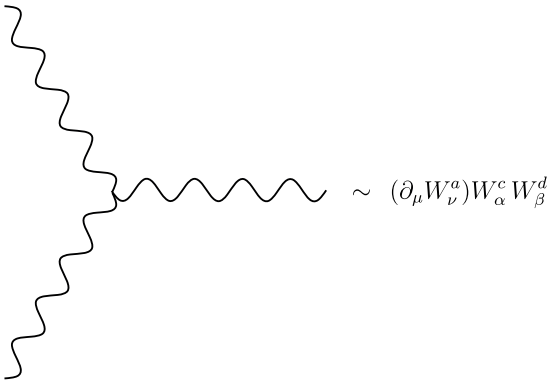


Figure 1.22: 3-Point diagram arising from W boson Lagrangian

As we chose T^2 and T_3 to be diagonal, see Equation 1.350, we can define ladder operators in analogy SU(2)-based spin in non relativistic quantum mechanics [32] [ref:griffithsqm,thomson]

$$T_\pm = T_1 \pm iT_2 \tag{1.371}$$

This will then induce the construction of gauge bosons associated to these new degrees of freedom,

[32] This is also known as the spherical basis of SU(2)

namely [ref:goldberg]

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \tag{1.372}$$

as well as new charged currents given by

$$J_\pm^\mu = \frac{1}{2}(J_1^\mu \pm iJ_2^\mu) \tag{1.373}$$

Expanding for the lepton doublets, these currents take form [33]

$$J_+^\mu = g_w \bar{\nu}_L \gamma^\mu e_L \quad J_-^\mu = g_w \bar{e}_L \gamma^\mu \nu_L \tag{1.374}$$

As we obtained a $SU(2)_L$ invariant Lagrangian, we could then try to get the Feynman rules at tree level for the weak interaction. However, the theory possess an additional symmetry which arises from the Dirac behavior of the spinors, just like they do in QED. In other words, by taking our spinors to be Dirac-like we are obtaining an extra U(1) symmetry, referred as the hypercharge symmetry [34]

This new $U(1)_Y$ symmetry, when taken as global, maps the left-handed and right-handed states via Equations 1.375 and 1.376

$$\psi_L \to \exp\left(-\frac{i}{2}g' Y_L \theta\right)\psi_L \tag{1.375}$$

$$\psi_R \to \exp\left(\frac{i}{2}g' Y_R \theta\right)\psi_R \tag{1.376}$$

where g' is the $U(1)_Y$ coupling constant. Note that both left and right handed spinors are not forced to have the same hypercharge values, this is due to their doublet and singlet behavior under $SU(2)_L$. As the Lagrangian is gauge invariant under these new global transformations, we obtain two different Noether currents, one from each chirality, namely

$$J_\mu^{Y_L L/R} = \frac{g'}{2} Y_{L/R} \bar{\psi}_{L/R} \gamma^\mu \psi_{L/R}, \tag{1.377}$$

which for left-handed leptons becomes

$$J_\mu^{Y_L} = \frac{g'}{2} Y_L (\bar{\nu}_L \gamma_\mu \nu_L + \bar{e}_L \gamma_\mu e_L) \tag{1.378}$$

Following the same procedure as in QED we find that the derivative term breaks gauge invariance under local transformations, such that we need to introduce the covariant derivatives (one for each

[33] Experiments have shown us that these linear combinations are the physical currents and gauge bosons.

[34] Thus, we add a subscript Y to the U(1) group to distinguish from QED.

chirality)

$$D_\mu^{L/R} = \partial_\mu + ig' Y_{L/R} B_\mu, \tag{1.379}$$

such that the gauge field B_μ transforms just like in Equation 1.263 as $U(1)_Y$ is abelian[35]

The presence of the new $U(1)_Y$ gauge symmetry, together with $SU(2)_L$, in the weak interaction means that the total gauge group is $SU(2)_L \times U(1)_Y$. By acting simultaneously with both transformations, we obtain that the interacting Lagrangian is given by [ref:griffiths][36]

$$\mathcal{L} \supset -J_\mu^{Y_R} B^\mu - J_\mu^{Y_L} B^\mu - J_3^\mu W_\mu^3 - J_+^\mu W_\mu^+ - J_-^\mu W_\mu^- \tag{1.380}$$

For left-handed leptons, the neutral part of the Lagrangian reads

$$\mathcal{L} \supset \frac{-1}{2}[(g' Y_L B^\mu + g_w W_3^\mu)\bar{\nu}_L \gamma^\mu \nu_L + (g' Y_L B^\mu - g_w W_3^\mu)\bar{e}_L \gamma_\mu e_L] \tag{1.381}$$

Note that the terms between parenthesis add in a similar way as the result of a matrix multiplication with a vector. In other words, it takes the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_{11}a + a_{12}b \\ a_{21}a + a_{22}b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

This inspired Steven Weinberg, Glashow and Salam to realize that the B and W_3 fields were not the physical fields, but rather a superposition of them. Weinberg noticed that the structure of the interaction take a form similar to that of QED, namely $QA^\mu J_\mu$, inspiring his colleagues to confirm that the electromagnetic interaction, as well as the weak interaction, arise from a unified electroweak interaction. Additionally, he noticed that the matrix that takes the gauge fields to the physical fields must be a rotation to preserve unitarity in the weak interaction. Weinberg's rotation matrix is defined by

$$\begin{pmatrix} B^\mu \\ W_3^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix}, \tag{1.382}$$

where Z^μ is a new vector boson that mediates neutral currents and A^μ is the photon field. The weak mixing angle or Weinberg angle is then defined by [37]

$$-\frac{g'}{g_w} = \tan\theta_w \tag{1.383}$$

[35] From now on we will only focus in the interacting part of the Lagrangian.
[36] The $\supset$ indicates that the terms on the right-hand side are part of the total Lagrangian.
[37] Experimental measurements have shown that $\sin^2\theta_w = 0.231$ [ref:pdg]

If we now replace Equation 1.382 into 1.381, the neutral interaction Lagrangian will read

$$\mathcal{L} \supset -\left[Y_L \frac{g'}{2}\cos\theta_w - \frac{g_w}{2}\sin\theta_w\right]\bar{\nu}_L\gamma^\mu\nu_L A^\mu - \left[Y_L\frac{g'}{2}\cos\theta_w + \frac{g_w}{2}\sin\theta_w\right]\bar{e}_L\gamma^\mu e_L A_\mu - \left[Y_L\frac{g'}{2}\sin\theta_w + \frac{g_w}{2}\cos\theta_w\right]\bar{\nu} \tag{1.384}$$

From the currents coupled to the photon, we then find that the electric charge can be written in terms of the weak coupling constants as

$$Q = Y_L\frac{g'}{2}\cos\theta_w + \frac{g_w}{2}\sin\theta_w \tag{1.385}$$

As neutrinos are postulated as neutral particles, its coupling with the photon must vanish. This means

$$Y_L\frac{g'}{2}\cos\theta_w = \frac{g_w}{2}\sin\theta_w, \tag{1.386}$$

which reduces to $Y_L = -1$ if we use Equation 1.383[38]. Additionally, we notice that $Y_L = -1$ along with Equation 1.383 imply that $Q = g_w \sin\theta_w$.

After replacing the electric charge and the hypercharge the neutral Lagrangian becomes

$$\mathcal{L} = -Q\bar{e}_L\gamma^\mu e_L A_\mu - \frac{g_w}{2\cos\theta_w}\bar{\nu_L}\gamma^\mu\nu_L Z_\mu + \frac{g_w}{2\cos\theta_w}\cos(2\theta_w)\bar{e_L}\gamma^\mu e_L Z_\mu \tag{1.387}$$

If we read this Lagrangian, we note that it tells us that neutrinos and leptons behave differently when interacting with the Z boson. This fact was indeed seen at experiments such as LEP[ref:lep] and TEVATRON[ref:tevatron], where the branching fraction of the Z boson to leptons and neutrinos are approximately 10% and 20% respectively [ref:pdg]. Furthermore, note that the definition of the electric charge in terms of both hypercharge and couplings to the W_3 boson imply that the Gell-Mann-Nishijima formula that we saw in QED still holds in the electroweak interaction due to the $SU(2)\otimes U(1)$ structure of the gauge group, namely

$$Q = T_3 + \frac{1}{2}Y_{L/R}$$

If we now follow the same procedure with the right-handed states we also find $Y_R(\nu_R) = 0$, $Y_R(e_R) = -2$, $Y_R(u_R) = \frac{4}{3}$ and $Y_R(d_R) = -\frac{2}{3}$. Using Gell-Mann-Nishijima formula, we can rewrite the Z boson Lagrangian, for a generic spinor ψ, as

$$\mathcal{L} \supset \frac{-g_w}{2\cos\theta_w}[T_3 - Q\sin^2\theta_w]\bar{\psi}\gamma^\mu\psi Z_\mu \tag{1.388}$$

As the Z boson is a superposition of both B and W_3, it couples differently to each component of the (V-A) currents that participate in the weak interaction. Hence, we define the vector and axial

[38]This value only holds for leptons. For quarks one obtains $Y_L(u) = \frac{1}{3}$ and $Y_L(d) = \frac{1}{3}$[ref:griffiths].

couplings to the Z boson as [ref:goldberg]

$$c_v = T_{3L} - Q\sin^2\theta_w; \quad c_a = T_{3L} \tag{1.389}$$

Now, using the left-handed projection operator given in Equation 1.355 we obtain that the interacting Z boson Lagrangian reads

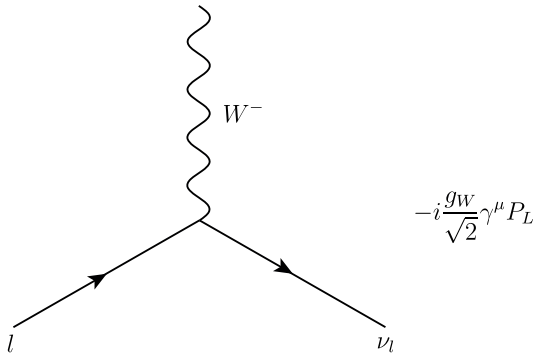
$$\mathcal{L} = -\frac{g_w}{2\cos\theta_w}\bar{\psi}\gamma^\mu(c_v - c_a\gamma_5)\psi Z_\mu \tag{1.390}$$

Hence, the total Lagrangian, up to ghost field terms, will now be

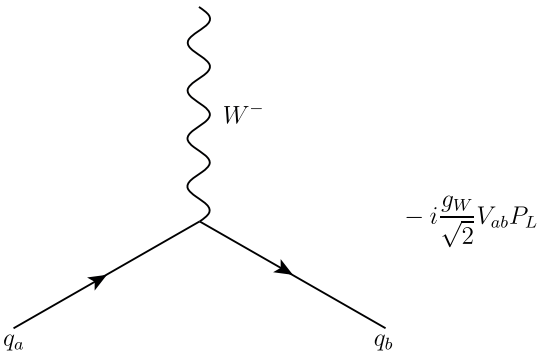
$$\mathcal{L} = i\psi\gamma^\mu\partial_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}W_{\mu\nu}W^{\mu\nu} - J^\mu_\pm W^\pm_\mu - Q\bar{\psi}\gamma^\mu\psi A_\mu - \frac{g_w}{2\cos\theta_w}\bar{\psi}\gamma^\mu(c_v - c_a\gamma_5)\psi Z_\mu \tag{1.391}$$

If we expand this Lagrangian and its currents, we can extract the tree-level Feynman rules, which are given by [ref:griffiths]:

- Lepton-Neutrino coupling with W boson

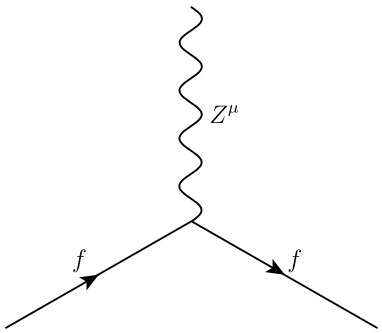


- Quark coupling with W boson [39]



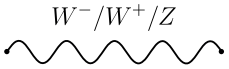
- Fermion coupling with the Z boson

[39] Here the V_{ab} is the CKM quark mixing matrix, which arises from the experimental observation of CP violation in Kaon decay [ref:thomson].



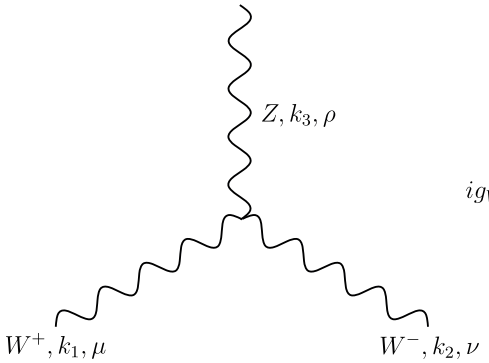
$$-i\frac{g_W}{2\cos\theta_W}\gamma^\mu(C_v - C_a\gamma_5)$$

- Vector boson propagator



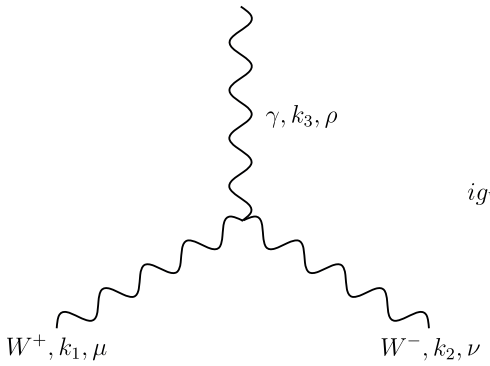
$$iD_{\mu\nu}(p) = -i\frac{(\eta_{\mu\nu} - p_\mu p_\nu/m^2)}{p^2 - m^2 + i\epsilon}$$

- W boson coupling with Z boson



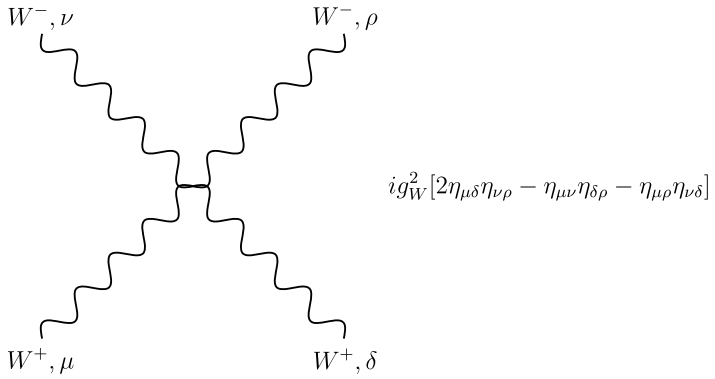
$$ig_W\cos\theta_W[\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu + \eta_{\rho\mu}(k_3 - k_1)_\nu]$$

- W boson coupling with the photon

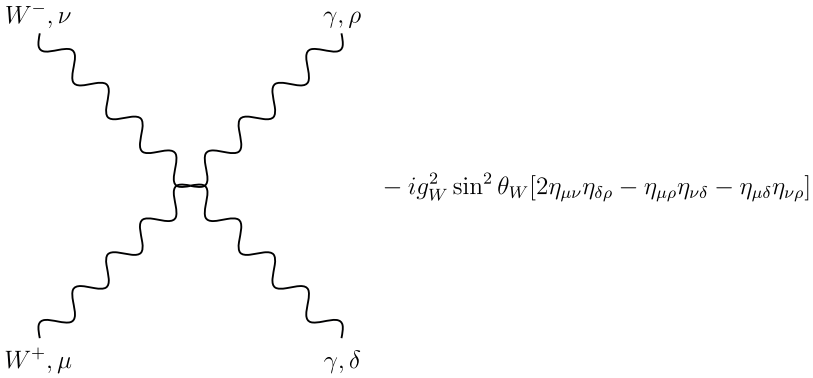


$$ig_W\sin\theta_W[\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_2 - k_3)_\mu + \eta_{\rho\mu}(k_3 - k_1)_\nu]$$

- 4 vector boson coupling



- W boson-Photon 4 point interaction



For the remaining Feynman rules see [ref:griffiths, goldberg, thomson].

So far, we have studied a $SU(2)_L \otimes U(1)_Y$ gauge theory of fermions and proved that the weak and electromagnetic interaction unify into the electroweak interaction. However, we have not yet included mass terms for both fermions and gauge bosons. To do so, we have to note once again that mass terms violate, and therefore break, gauge invariance in an spontaneous way as we will see later on.

The mass problem in the electroweak interaction was solved by Higgs, Englert, Kibble, Guralnik and other physicists after they analyzed how photons gain mass terms after Spontaneous Symmetry Breaking (SSB) in Ginzburg-Landau theory, giving rise to superconductivity and superfluidity. Based on this theory, they introduced a complex scalar doublet ϕ, whose dynamics are contained in the following Lagrangian [40]

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - V(\phi^* \phi) \tag{1.392}$$

where V is a self-interacting potential.

In order to preserve gauge invariance and renormalizability, the potential must be given by

[40] This doublet has $T_3 = \frac{1}{2}$ and Y=1

[ref:schwartz]

$$V(\phi^*\phi) = \mu^2\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)2 \rightarrow \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{1.393}$$

Note that if $\mu^2 > 0$ the potential will be symmetric, as seen in Figure 1.23, such that the minimum of the potential will be located at $\phi_0 = 0$.

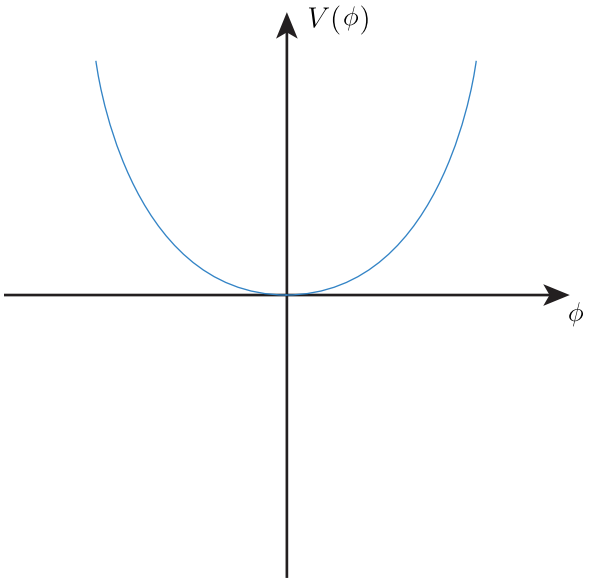


Figure 1.23: Interaction potential if $\mu^2 > 0$.

However, if μ^2 value becomes less than zero, for example due to a decreasing temperature, the symmetry of the potential due to this spontaneous change of sign [41] as seen in Figure **??**

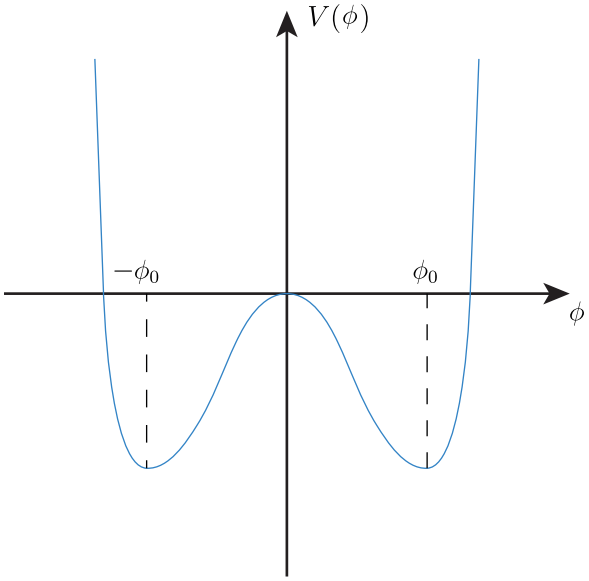


Figure 1.24: Interaction potential if $\mu^2 < 0$.

[41] That is why it is called an SSB. Note that it is also a phase transition.

with minima occurring at $|\phi_0|^2 = \frac{-2\mu^2}{\lambda}$. Nevertheless, note that the minima are not unique and that a phase transition such as,

$$\phi_0 \to e^{-i\alpha}\phi_0 \tag{1.394}$$

will also be a minimum of the potential, leaving the dynamics of the field invariant. Thus, there is an infinite number of available minima, or vacuum expectation values, and that choosing one of them will break the symmetry of the potential.

Suppose now that we choose one of these minima, namely v. As the base of the potential is, topologically speaking, a circle no extra energy is required for moving around it, introducing redundancies known as Goldstone bosons [ref:schwartz][42]. As we are choosing a minimum, we can expand the doublet in terms of angular and radial modes (Goldstone and Higgs fields respectively) as [43]

$$\phi = \begin{pmatrix} G^{\pm} \\ \frac{v+h+iG^0}{\sqrt{2}} \end{pmatrix} \tag{1.395}$$

Now, we need to eliminate the Goldstone modes. This can be done by the means of the so called *Unitary Gauge*, such that the doublet becomes [ref:Weinberg1971]

$$\phi = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \tag{1.396}$$

To preserve the local $SU(2) \times U(1)$ gauge invariance of the doublet Lagrangian, we must introduce the covariant derivative

$$D_\mu = \partial_\mu + ig_w T_a W_\mu^a + i\frac{g'}{2}B_\mu, \tag{1.397}$$

such that the gauge invariant Lagrangian is

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \tag{1.398}$$

Let us focus on the kinetic term of the Lagrangian, considering those terms related to vector bosons alone. This term reads [ref:schwartz]

$$(D_\mu\phi)^*(D^\mu\phi) \supset \left| \left(\partial_\mu + ig_w T_a W_\mu^a + i\frac{g'}{2}B_\mu\right) \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \tag{1.399}$$

$$= \frac{v^2}{8}\left| \left(g_w\tau_a W_\mu^a + g' B_\mu \mathbb{1}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \tag{1.400}$$

[42] Formally speaking, Goldstone bosons are redundant degrees of freedom in a local gauge theory, whereas they become relevant at global symmetry breaking of gauge theories.

[43] This parametrization is known as Higgs-Kibble or Kibble parametrization.

If we expand the sum over a in terms of the Pauli matrices we obtain

$$(D_\mu\phi)^*(D^\mu\phi) \supset \frac{v^2}{8}\left|\begin{pmatrix} g_w W_1^\mu - i g_w W_2^\mu \\ -g_w W_3^\mu + g' B^\mu \end{pmatrix}\right|^2 \tag{1.401}$$

$$= \frac{v^2}{8}[g^2(W_1^\mu W_\mu^1 + W_2^\mu W_\mu^2) + (g_w W_3^\mu - g' B^\mu)^2] \tag{1.402}$$

If we now replace the W bosons in terms of the $\pm$ fields and W_3 and B are replaced in terms of Equation 1.382, the Lagrangian becomes

$$(D_\mu\phi)^*(D^\mu\phi) \supset \frac{1}{2}\left(\frac{g_w v}{2}\right)^2 W_\mu W^\mu + \frac{1}{2}\frac{v^2(g_w^2 + g'^2)}{4} Z_\mu Z^\mu \tag{1.403}$$

$$= \frac{1}{2} m_w^2 W_\mu W^\mu + \frac{1}{2} m_z^2 Z_\mu Z^\mu \tag{1.404}$$

This results is one of the most important in physics as it explains how gauge bosons gain mass due to SSB, thus explaining why the weak interaction is short ranged. Note that there is no mass term for photons, as the terms related to its linear combination cancel out.

We just proved that electroweak gauge bosons gain mass due to SSB of the Higgs potential. Now, we need to construct mass terms for fermions. The easiest way to do this is to introduce couplings (Yukawa couplings) between the scalar doublet and the fermions. This can be done via

$$\mathcal{L}_{Yuk} = \Gamma^u_{ij} \bar{q}_{Li} \tilde{\phi} u_{Rj} + \Gamma^d_{ij} \bar{q}_{Li} \phi d_{Rj} + \Gamma^e_{ij} \bar{e}_{Li} \phi e_{Rj}; \quad \tilde{\phi} = -i\tau_2 \phi^*, \tag{1.405}$$

where the Γ matrices are the couplings between fermions and the Higgs doublet (or its conjugated version $\tilde{\phi}$). Note that we conveniently constructed the Yukawa couplings to be between left-handed and right-handed states, as mass terms arise as a direct coupling between them according to Equation 1.356.

For simplicity, and without loss of generality, let us consider only one lepton family with Yukawa Lagrangian given by

$$\mathcal{L} \supset f_e \bar{e}_L \phi e_R \tag{1.406}$$

Note that if ϕ gets a vacuum expectation value, we can use parametrize it using Equation 1.395, such that the Lagrangian reads

$$\mathcal{L} \supset f_e \bar{e}_L \left(\frac{v+h}{\sqrt{2}}\right) e_R \tag{1.407}$$

$$= \frac{f_e v}{\sqrt{2}} \bar{e}_L e_R + \frac{f_e}{\sqrt{2}} \bar{e}_L h e_R \tag{1.408}$$

By noticing that the first term is a constant coupling both chiral states, we define the lepton mass

as $m_e = \frac{f_e v}{\sqrt{2}}$, such that the Lagiangian becomes

$$\mathcal{L} \supset m\bar{e}_L e_R + \frac{m}{v}\bar{e}_L h e_R \tag{1.409}$$

So, we have obtained mass terms for fermions based on SSB and couplings between the Higgs boson and them. This being said, let us finish our discussion by noting that one can write a total gauge group for the strong and electroweak interactions, without unifying them yet. To see this recall that quarks interact under the SI and also under the electroweak interaction, acting as color triplets, left-handed doublets and right-handed singlets respectively. Thus, one can say that a total gauge theory for quarks is invariant under the gauge group

$$SU(3)_C \otimes SU(2)_L \otimes U(1)\textbf{??} \tag{1.410}$$

Additionally, this group also describes a gauge theory of leptons as experiments have shown us that leptons do not participate in the SI, therefore they are considered color singlets. Hence, note that we have obtained a local Lorentz invariant gauge theory that describes three of the fundamental forces of nature. Nowadays we call this theory the **Standard Model of particle physics**(SM).

However, this brief summary does not include some topics that live within the formalism of the SM such as ghost fields, quantization of Yang-Mills theories, renormalization or CP violation as they would require us to make a whole set of notes, and become very specific to be just an introduction to the theory. We then refer the reader to [ref:thomson, schwartz, griffiths, langacker] for more information on these topics.

Bibliography

[1] J.R Taylor. *Classical Mechanics*. University Science Books, 2005.

[2] J.L. Safko. H. Goldstein., C.P. Poole Jr. *Classical Mechanics*. Pearson, 2001.

[3] D. V. Schroeder M. E. Peskin. *An Introduction To Quantum Field Theory*. Westview Press, 2016.

[4] T. Lancaster & S. Blundell. *Quantum Field Theory For The Gifted Amateur*. Dover Publications, 2007.