We will talk about the Barabasi-Albert model (also called "preferential attachment", or "scale free")

The things we HAVE to know, are in "Barabasi albert for dummies"

The degree distribution is a powerlaw: $p_k \propto \frac{1}{k^3}$

In a log-log distribution, this is a linear function with m=-3, (m = slope of line)

In real world networks, powerlaws are common.

In ER two nodes are connected with a constant probability.
In the BA the probability of a link is different, if we have a network already, the prob. of a new node linked to node i is proportional to the degree of said node i.

$$p_{\to i} = \frac{k_i}{2L}$$

the prob. of j (new node) being connected to node i (already existant) is proportional to the degree of i.

The idea is: I open a new airport, so I add a new node to the airport network, to give a good service to my customers, I prefer to connect it with airports with high degree!

L is a normalizing factor: it's the number of links in the network.

How do we create a barabasi albert model?

At time t=0, I decide how many nodes n_0 I want in this initial network, and how these nodes are connected.

We make a network of fully connected nodes n_0, for the sake of simplicity (not compulsive, just a convention)

At each time step, I want to add a single node, so the number of time steps I need to do is the number of nodes I want my network to have at the end.

ex: $n_0 = 10$ $\quad N = 500$ $\quad \Rightarrow$ 490 time steps

Note: one can also initiate with a ER model, for example.

Another parameter of the model is m. This is the number of links that I want my new nodes to form with existing nodes. This is the same for every step.

This m is the number of links with already existing nodes (so for time steps successive to the first, to all nodes existing, not just n_0)

$$m \leq n_0$$

How do we chose to which nodes to attach the new node?

We do a for cycle iterating on all nodes, and calculate the number P_i for each node

$$P_i = \frac{K_i}{2L}$$

K_i — degree of i

L — total number of connections in the network

Then we do a while (until we get m connections), to exctract random numbers p_i (p_i between 0 and 1), for every node, in order, if P_i is larger than p_i we choose that connection. We iterate until we get m connections, so we can iterate on all the nodes multiple times.

the while must be done on all nodes, but the order of the nodes must be shuffled, so we don't have preferences on some nodes.
If after comparing P and p for each node we don't have m connections, we start again, in the same shuffled order of the beginning.

A new node added usually doesn't remain at degree = m, because new nodes might connect to it.
The last node added will always have degree = m

What is the probability for each node to have its degree increased by one at a specific time step?

$$P_{k \to k+1}(t, t+1) = \frac{k \, p_k(t)}{\ldots} = \frac{k \, p_k(t)}{2m}$$

k — because of "preferential attachment"

normalizzazione

$p_k(t)$ — distribution of degree k

$$\text{mean degree} = \frac{2L}{N}$$

number of links

number of nodes

2 because each link connects 2 nodes

We can then compute the number of nodes that will have a increase of 1 of their degree at a given time step.

$$m_{k \to k+1} = A - B + C$$

$$m_{k \to k+1}(t, t+1) = \cdots = \frac{k\, p_k(t)}{2}$$

We can go from numbers to probabilities

By finding a stationary solution of the equation of the probability with a constrain, we get:

$$p_k(t+1) = p_k(t) = p_k$$

We find a recursive (iterative) solution

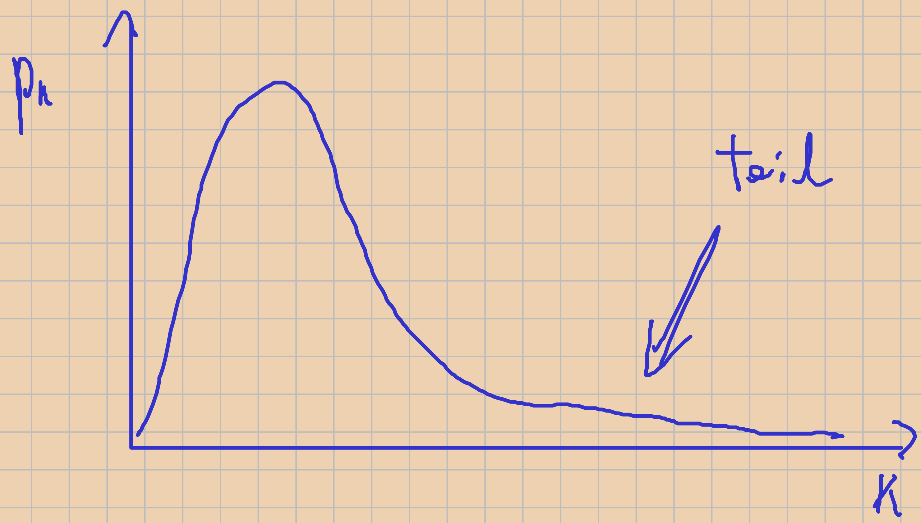
$$p_k = \frac{k-1}{k+2}\, p_{k-1}$$

So we find:

$$\Rightarrow \boxed{p_k \approx \frac{2(m+1)m}{k^3}}$$

This is the distribution of the degree

If we have a model with a degree distribution that has an exponential tail (like ER) going to the higher degrees:



$p_k \propto e^{-\lambda k}$ λ = constant

$$K_{max} = K_{min} + \frac{\ln N}{\lambda}$$

k_max increases with log of number of nodes !

$$\Rightarrow \boxed{K_{max} - K_{min} = \frac{\ln(N)}{\lambda}}$$

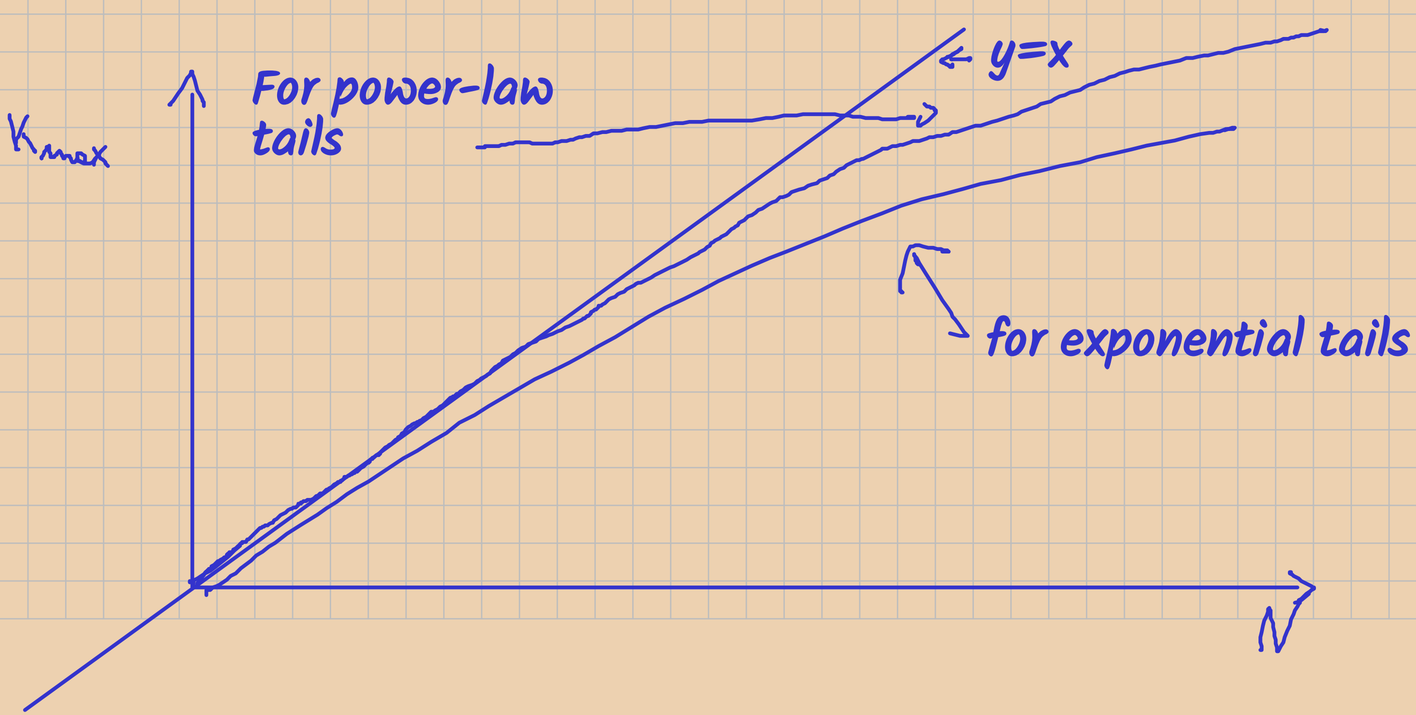
A certain constant, some parameter in the degree distribution.

If we have a distribution of degree with tail power-law

$$p_k \propto k^{-\lambda} \qquad \lambda = \text{constant}$$

$$\Rightarrow \qquad k_{max} = k_{min} \cdot N^{\frac{1}{\gamma-1}} \qquad \gamma = \text{constant}$$

k_max increases with a power-law ! It's a square like $\sqrt[\gamma-1]{N}$



The growth is much bigger than the case for exponential distribution

Scale free networks are networks whose degree distribution's tail follows a power-law.
Like a BA.
The n-th moment of the degree goes like a

$$\langle k^n \rangle \propto N^{\frac{n-\gamma+1}{\gamma-1}}$$

this diverges whenever $n - \gamma + 1 > 0$

this is the exponent of the power-law

$$\begin{cases} \langle k \rangle = \text{Vevence} \\ \langle k^2 \rangle = \text{std. dev.} \end{cases}$$

In more advanced models with preferential attachment, this is not based only on k but

$p \propto \underbrace{\eta}_{} k$ Ginestra Bianconi model

this is a number not easily identified.