We will talk about Watts-Strogatz (SMALL WORLD)
The original article is from 1998 (Nature 393)

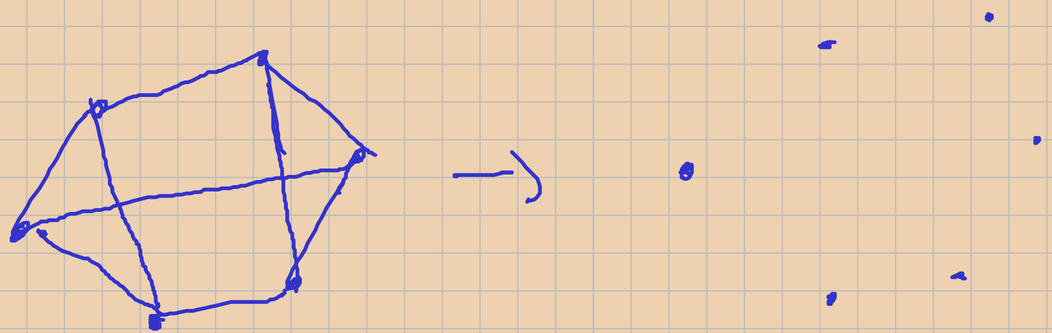
Everything important is in the "for dummies" slide

A regular graph is a graph where all the nodes have the same degree.

We start from a regular graph with fixed k (for example $k=4$)

We associate a probability $0<p<1$, p is the probability that for each node, each link is disconnected and reconnected to a "far away node", a "far away node" is a node that was not linked in the original configuration. This process is called rewiring.

The average degree is the same, but the specific degree of a node could go up or down.



$K=3$

$\frac{NK}{2}$ is the number of links in our regular graph

$p\frac{NK}{2}$ is the average number of links that we rewire

It can be shown that every node, after the process, will have at least K/2 links

$$K_i = \frac{K}{2} + c_i \qquad c_i = c_i^1 + c_i^2$$

$$\begin{cases} c_i^1 \leq \frac{K}{2} & \text{are the edges left in place (with probability 1-p)} \\ c_i^2 = c_i - c_i^1 & \text{are the edges that have been rewired with prob. 1/N (for large N)} \end{cases}$$

By combining these two, we find that the degree distribution depends on p
The average value of k_max will remain the same for different values of p, the width of the distribution gets larger for larger p.

We start with something with a high clustering coefficient, we break a link and put it far away. This is the small world model.
The small world model is a coreperiphery-like model.

The idea of the small world model is born with Stanley Milgram and his idea in 1967. He was a sociologist.

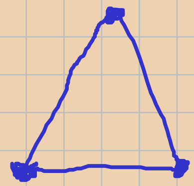
If the network is small world, then the average path lenght goes like ln(N), where N is the number of nodes. We will not demonstrate it.

With ER, if we have higher degree we have a smaller average path lenght (of course, becaus we have more ways to move between the nodes)

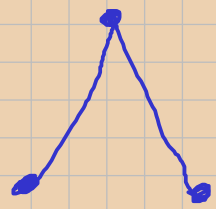
The average path lenght of a regular graph is $APL = \frac{N}{2K}$

In a ER or BA type graph, the APL increases with N, and the local clustering coefficient decreases N.

- In a regular network:



triangle



connected triple

$$C = \frac{3 \cdot \text{total number of triangles}}{\text{total number of connected triples}} \equiv \text{global clustering coefficient}$$

$$C = \frac{3(k-2)}{4(k-1)}$$

we don't show how we get this formula, but it can be shown

- to recap, in a regular graph

$$APL = \frac{N}{2K} \equiv L(0) \qquad C_{coeff} = \frac{3}{4}\frac{(k-2)}{(k-1)} \equiv C(0)$$

- In the case of Watts-Strogatz (small world), we look at $\underbrace{C(p)}_{\text{clustering coefficient after rewiring with p prob.}}/C(0)$ and $\underbrace{L(p)}_{\text{APL after rewiring with p probability.}}/L(0)$

The APL decreases with p, (if we have a high p it's easier to have connections with far away nodes, so the APL decreases, it's a small world!), even with p of just 0.1%, the APL is halved in respect to L(0) (it's from the starting regular graph)

The clustering coefficient decreases with p as well, of course. But way slower, with p = 1% it is basically still the same as C(0), even with p= 10% is still about 70%.

To compare the C or the L of a graph we can't watch the absolute numbers, we can generate a random graph with same N and k for each node but connected random, and calculate the C_rand and L_rand, we say that for that graph L or C is big if they're big when compared to C_rand and L_rand.

- We talk about weak ties.

Granovetter talks about this in an article from 1973 (he was a sociologist)

After we do the rewiring, mutating a regular graph in a small world model, every link that is not changed (and thus remains local) is called a strong tie.
Every link created during the rewiring (so with far away nodes) is called a weak tie.

Strong ties can be seen like a man links with members of his family,
Weak ties can be seen like a husband-wife

Weak ties are STRONG in practice! They are very important in various aspect, for example they lower a lot the APL, they have "cohesive power".

A wedding makes a family much larger!

It's called "weak" because it's very vulnerable, if it's destroyed the graph suffers.

To know if a network is a small world we can do:
We generate C_rand and L_rand, if C is significantly bigger than C_rand and L is significantly smaller (or pretty similar) to L_rand... Then the network is a small world!