

We will analyze 4 complex networks models in this course.

name	mechanism
ER	randomness
BA	preferential attachment
WS	Small World
BE	Core-periphery

Today we look at Erdos-Renyi (ER) which has a randomness mechanism

It was originally published in 1960 (cold war)

All we have to know about ER is condensated in (Erdos-Renyi for dummies slide)

Erdos and Renyi were two mathematicians.

The idea is we have a set of nodes, and each couple of node is connected with a certain probability p .

We fixate the number of nodes, and we generate links in a total random ways.

The probability of the degree is a binomial distribution, that can then be approximated to a Poissonian

The probability of finding a node with high degree is very very low (due to the exponential)

$$p(d) \approx \frac{z^d \cdot e^{-z}}{d!}$$

$$z = p \cdot (n-1)$$

probability of having a link number of nodes - 1

$$\binom{M}{k} = \frac{M!}{k! (M-k)!}$$

We can have two ways of construct an ER model, let's see the type A:

(ER is symmetric, we don't consider direction in the links)

We fixate N (the number of nodes) and K the number of links:

$$M = \frac{N(N-1)}{2} \quad \text{this is the maximum number of links with } N \text{ nodes}$$

$$0 \leq K \leq M$$

We cannot have K larger than M !

We then generate the graph G by generating the K links totally random between the nodes.

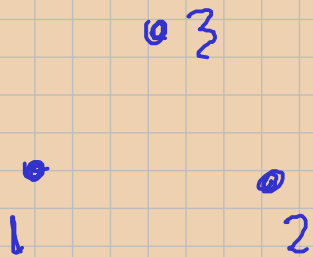
We have N and K , but we can have a certain number of graphs with those characteristics:

$$C_K^M = \binom{M}{K} = \frac{M!}{K! (M-K)!} \quad \text{this is the combination}$$

Computationally: we select the number of nodes N and calculate M , then we compute all the M possible links (NO DIRECTIONALS, (a,b) is the same as (b,a) and only one is considered) in an array long M .

We choose K (less than M), so we choose randomly K values from that array and generate the graph this way!

ex)



$$N=3 \Rightarrow M = \frac{3 \cdot 2}{2} = 3$$

$$\text{If } k=2$$

$$M = \{(1,3), (2,3), (1,2)\}$$

we choose k (so 2) links from that list

By using the combinatorial formula, we can calculate how many different networks we can find

$$\binom{M}{K} = \frac{M!}{K!(M-K)!} = \frac{6}{2(1)} = 3$$

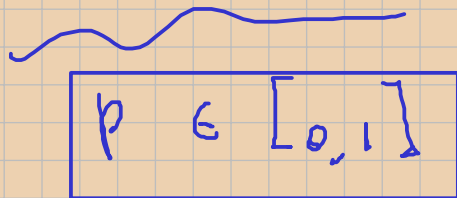
The probability associated to generate a specific graph is:

$$P_G = \frac{1}{\binom{K}{M}} = \frac{1}{3} \text{ in our example.}$$

We can calculate the probability associated to taking a specific link

$$p = \frac{1}{M} = \frac{1}{3} \text{ in our example}$$

In this example P and p are the same, in general they are not.


$$p \in [0, 1]$$

ER model B:

we fixate the number of nodes and select p , the probability to have each link!

p must be between 0 (no connections) and 1 (all nodes connected)

There must be a link (no pun intended) between a ER generated with A model and B model

$$G_{N,K}^{ER}$$

ER model A: we select N and K

$$G_{N,p}^{ER}$$

ER model B: we select N and p

With model B, the probability of extracting one single network from all the possibles is the "bernoulli trial", so the prob. to obtain a network with N nodes and K links is:

$$P_{G,K}^{N,K} = p^K (1-p)^{M-K} \cdot \binom{M}{K}$$

Probability of obtaining a graph with N nodes and K links

The average expected value of links K is

$$K = pM$$

The average of the bernoulli distribution

$$\sigma^2 = p(1-p)M$$

This is the variance associated with the average value K expected

The two models behave similarly when $p = \frac{M}{K}$ check quite formula

$$P = \binom{n}{k} p^k (1-p)^{n-k}$$

this is the probability of having k successes in n tries

p = p = probability of success (to get a certain face in a dice is $1/6$)

k = how many success we want (we want to analyze the prob. of having k succ.)

n = the number of tries we have

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We can write the probability of a specific node i to have degree k

$$P_{k_i=k} = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

here k is how many success we want!

This is the probability that the i node has degree k .

We use $N-1$ (number of nodes -1) because we have as much "tries" as the remaining nodes (we can't have a self link!)

p is the probability of success because it's the prob. of having a link! (link \rightarrow degree)

$$\langle k \rangle = p(N-1)$$

this is the average degree of each node

$$\sigma_k = \sqrt{p(1-p)(N-1)}$$

this is the standard deviation associated with a degree having k degree

Those two are interesting to compute!

- In the limits $N \rightarrow \infty$ and $p \rightarrow 0$ that binomial distribution is approximated to a poissonian!

$$P_k = \frac{e^{-z} z^k}{k!} \quad z = Np$$

$$\Rightarrow \begin{cases} \langle k \rangle = pN \\ \sigma_k = \sqrt{pN} \end{cases}$$

the std. dev. is the square root of the mean z

Because $N \approx N-1$ if $N \rightarrow \infty$

Giant component: it's a large conn. component, is a connected component whose size scales linearly with the number of nodes N : if we have a larger network \rightarrow we have a larger giant component.

If we increase p , more nodes are connected.

Is there a value of p where every node is connected (NB: not every node connected with every node, that is just $p=1$)

- If we have a ER graph generated (model B), we can write the probability of having a link

as: $p = \frac{c}{N}$

$c = \text{constant}$
 $N \rightarrow \text{number of nodes}$

If we want to know the size S_1 of the largest component

$$\begin{cases} \text{if } c < 1 \rightarrow S_1 \propto \ln N \\ \text{if } c = 1 \rightarrow S_1 \propto N^{2/3} \\ \text{if } c > 1 \rightarrow S_1 \propto N \end{cases}$$

if $c > 1$

$$S_2 \propto \ln N$$



the size of the second largest component

- If $c > 1$ the largest component is a GIANT COMPONENT! (the size scales linearly with N), the second largest scales with $\ln(N)$

If $c > 1$ we have a giant component!

It is known that:

$$\langle k \rangle = c$$

The average degree is equal to $c!!!$

So if the average degree is larger than 1, the largest component is a giant component

$$\text{Since } \langle k \rangle = pN \Rightarrow \langle k \rangle > 1 \Rightarrow pN > 1 \Rightarrow$$

$$p > \frac{1}{N}$$

this is the condition on the probability for having the largest component as a giant component.