RegressionModelsNotes

Coursera Course by John Hopkins University

INSTRUCTORS: Dr. Brian Caffo, Dr. Roger D. Peng, Dr. Jeff Leek

Contents

Intro

This course covers regression analysis, least squares and inference using regression models. Special cases of the regression model, ANOVA and ANCOVA will be covered as well. Analysis of residuals and variability will be investigated. The course will cover modern thinking on model selection and novel uses of regression models including scatterplot smoothing.

GitHub Link for Lectures

Link to the GitHub for this course

Course Book

Regression Models for Data Science in R, through Leanpub

Further Reading: **Advanced Linear Models for Data Science**

Instructor's Note

"*We believe that the key word in Data Science is 'science'. Our course track is focused on providing you with three things:*
1) *An introduction to the key ideas behind working with data in a scientific way that will produce new and reproducible insight*
2) *An introduction to the tools that will allow you to execute on a data analytic strategy, from raw data in a database to a completed report with interactive graphics*
3) *Giving you plenty of hands on practice so you can learn the techniques for yourself.*

Regression Models represents a both fundamental and foundational component of the series, and it presents the single most practical data analysis toolset. Using only a bare minimum of mathematics, we will attempt to provide you with the fundamentals for the application and practice of regression. We are excited about the opportunity to attempt to scale Data Science education. We intend for the courses to be self-contained, fast-paced, and interactive, and we intend to run them frequently to give people with busy schedules the opportunity to work on material at their own pace.

Brian Caffo and the Data Science Track Team"

Data Science Specialization Community Site

The site is created using GitHub Pages

In addition, Johns Hopkins has **a site on Statistical Methods and Applications for Research in Technology** that Dr. Caffo helps manage.

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Least Squares and Linear Regression

Regression

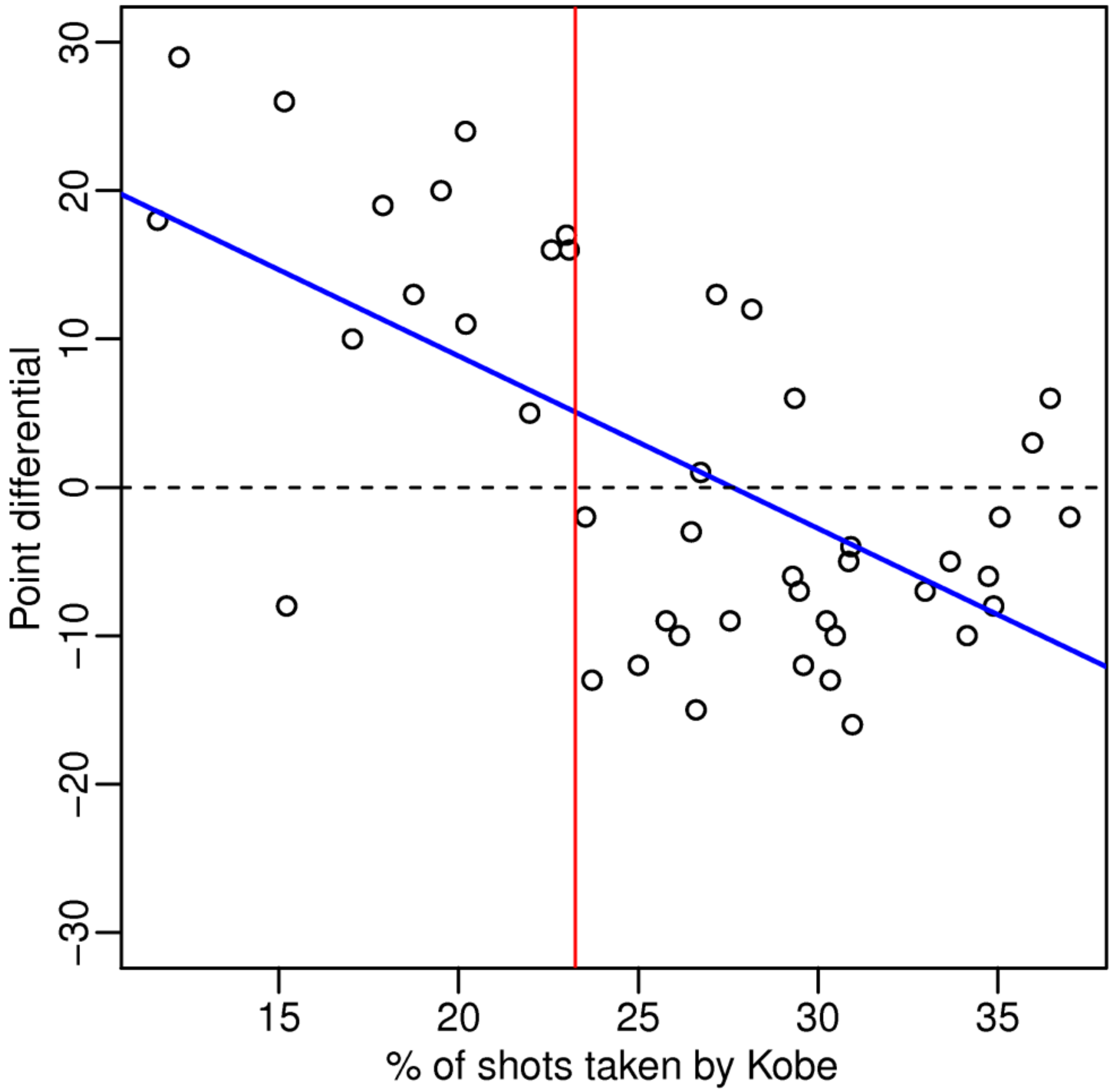
Introduction to Regression

- The simplicity and intrepretability offered by regression models should make them a first tool of choice for any practical problem.

- First discovered by **Francis Galton** who coined most of the terminology we use today.

Relevant Simply Statistics Post

Simply Statistics is a blog by Jeff Leek, Roger Peng and Rafael Irizarry, who wrote this post



- "Data supports claim that if Kobe stops ball hogging the Lakers will win more"
- "Linear regression suggests that an increase of 1% in percent of shots taken by Kobe results in a drop of 1.16 (+/- 0.22) in score differential."
  + Standard error given as "+/- 0.22"

Questions for this Class

In reference to Galton's parent/children height data, which can be accessed from the `galton` dataset in the `UsingR` package.
Consider trying to answer the following kinds of questions:
* To use the parents' heights to predict childrens' heights.

* To try to find a parsimonious (explain the data), easily described mean relationship between parent and children's heights.
* To investigate the variation in childrens' heights that appears unrelated to parents' heights (residual variation).
* To quantify what impact genotype information has beyond parental height in explaining child height.
* To figure out how/whether and what assumptions are needed to generalize findings beyond the data in question.
* Why do children of very tall parents tend to be tall, but a little shorter than their parents and why children of very short parents tend to be short, but a little taller than their parents? (This is a famous question called "Regression to the mean".)

Introduction to Basic Least Squares

- Let's look at the data first used by Francis Galton in 1885.

- Galton was a statistician who invented the term and concepts of regression and correlation, founded the journal Biometrika, and was the cousin of Charles Darwin.

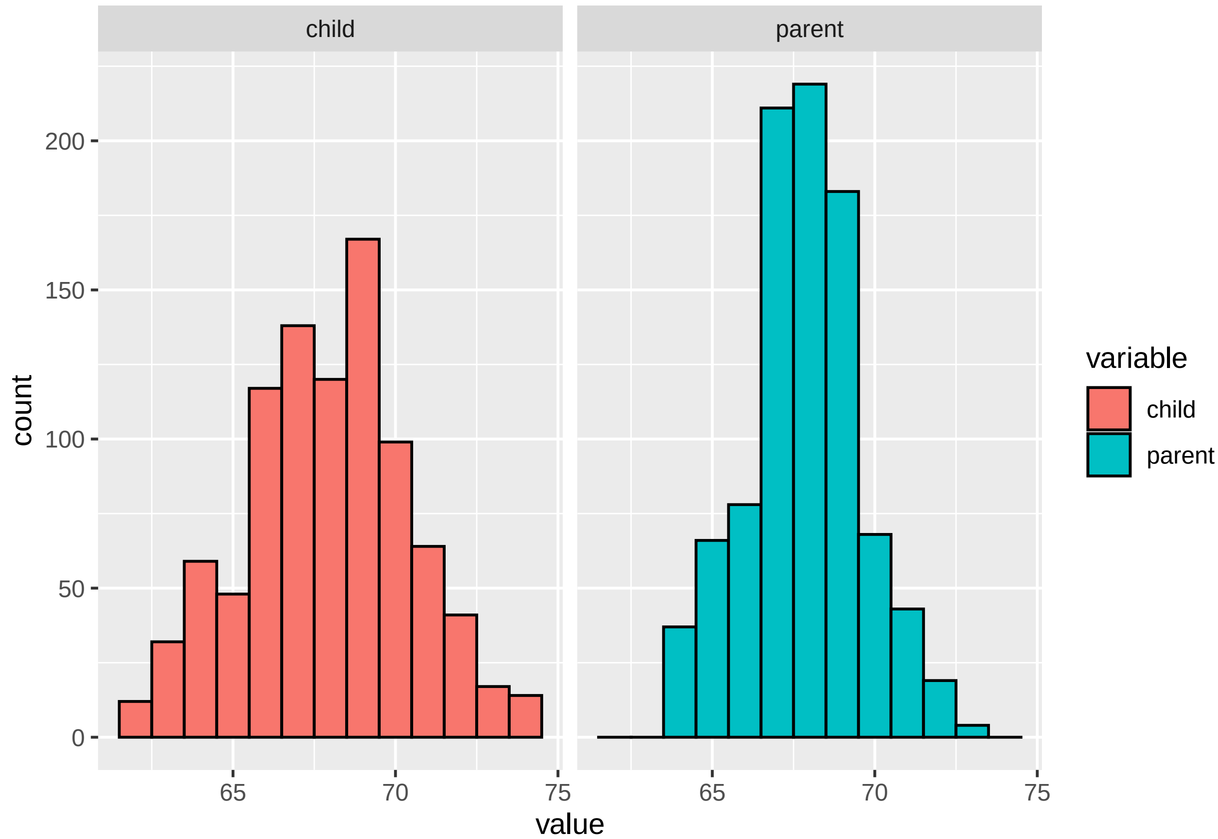
- Let's look at the marginal (parents disregarding children and children disregarding parents) distributions first.
  + Parent distribution is all heterosecual couples.
  + Correction for gender via multiplying female heights by 1.08.
  + Overplotting is an issue from discretization.

```
library(UsingR); data(galton); library(reshape2); library(tidyverse)
```

```
long <- melt(galton)
```

```
## No id variables; using all as measure variables
```

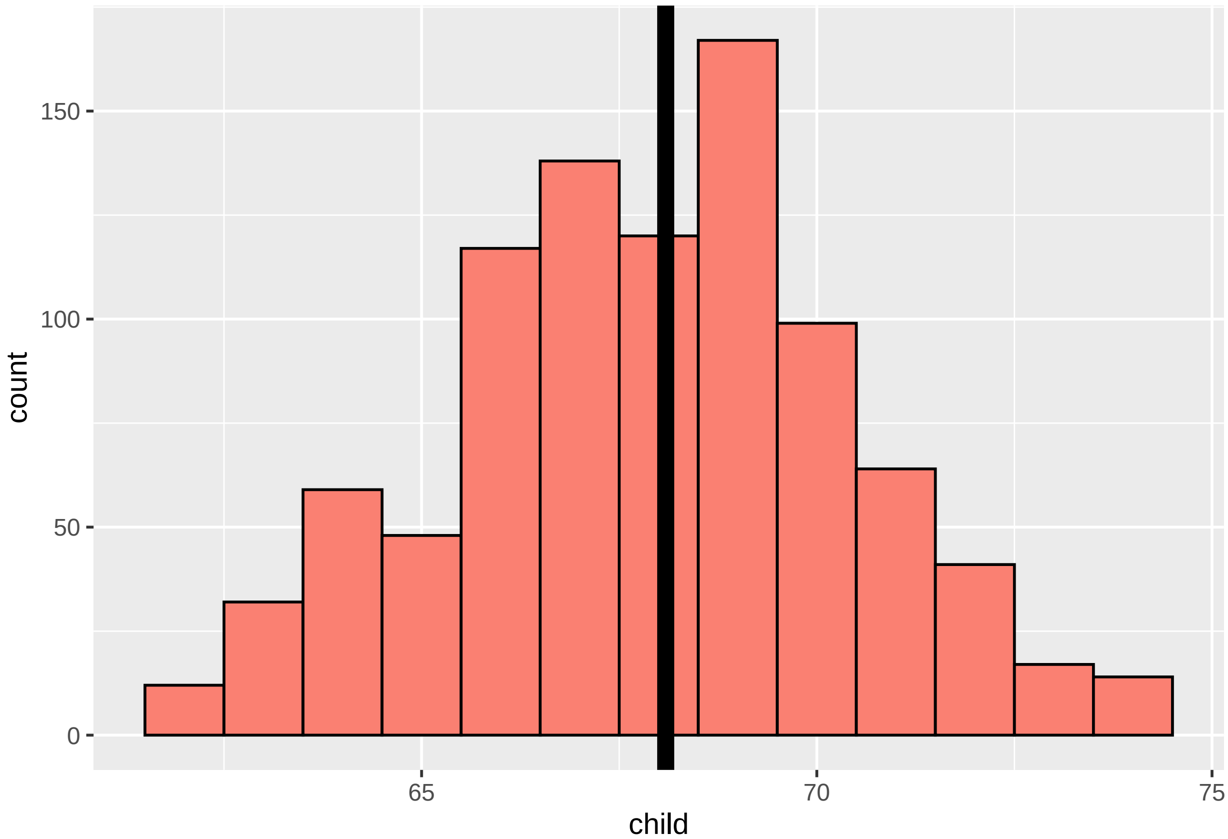
```
plot <- ggplot(long, aes(x = value, fill = variable)) +
        geom_histogram(colour = "#000000", binwidth = 1)
plot + facet_grid(.~variable)
```



Finding the Middle via Least Squares

- Consider only the children's heights
  + How could one describe the "middle"?
  + One definition, let Y_i be the height of child i for $i = 1, ..., n = 928$, then define the middle as the value of μ that minimizes
  $\sum_{i=1}^{n}(Y_i - \mu)^2$
- This is the physical center of mass of the histogram.
- The result of this is that $\mu = \bar{Y}$

```
ggplot(galton, aes(x = child)) +
        geom_histogram(fill = "salmon", colour = "#000000", binwidth = 1) +
        geom_vline(xintercept = mean(galton$child), size = 3)
```



- The above plot of child heights has a mean of 68.0884698

Technical Details

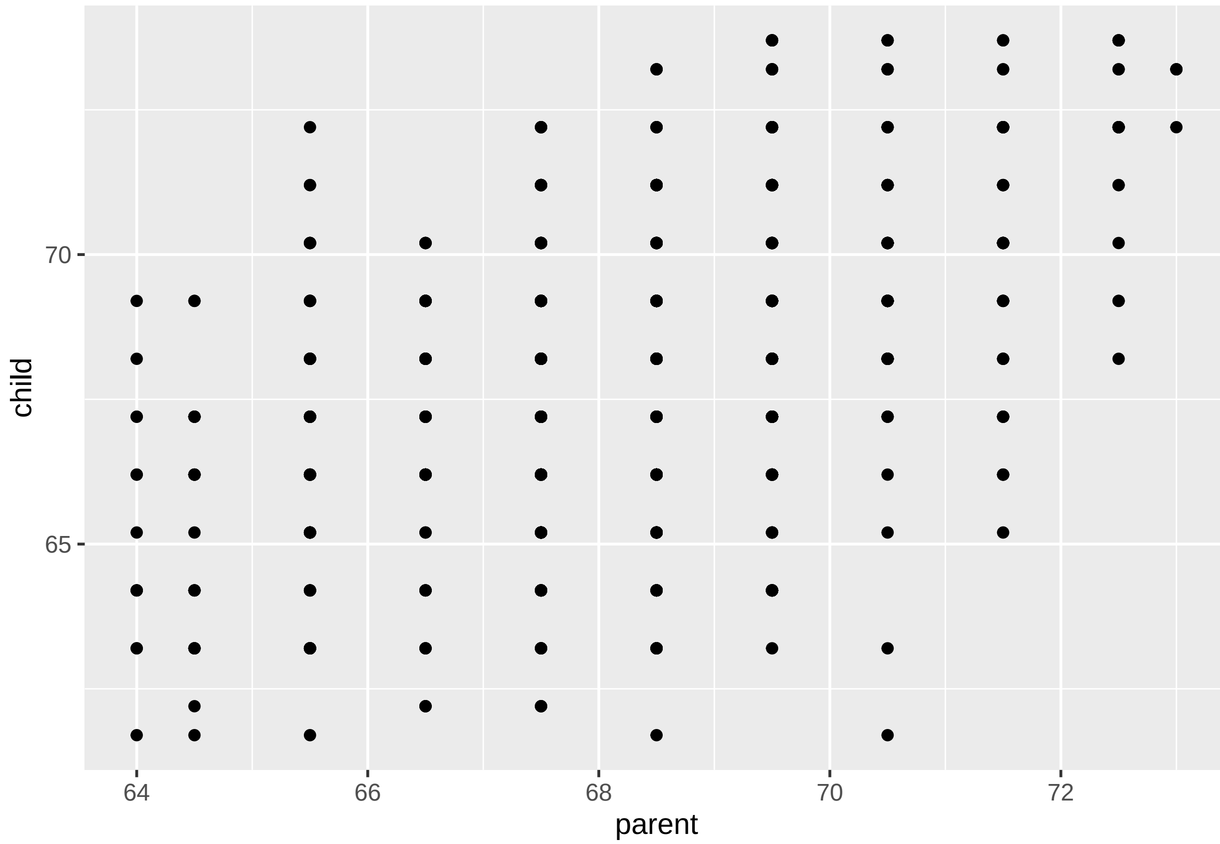
Proof that $\bar{Y}$ is the minimizer for $\sum_{i=1}^n (Y_i - \mu)^2$
$\sum_{i=1}^n (Y_i - \mu)^2 = \sum_{i=1}^n (Y_i - \bar{Y} + \bar{Y} - \mu)^2$
$= \sum_{i=1}^n (Y_i - \bar{Y}^2 + 2\sum_{i=1}^n (Y_i - \bar{Y})(\bar{Y} - \mu) + \sum_{i=1}^n (\bar{Y} - \mu)^2$
$= \sum_{i=1}^n (Y_i - \bar{Y})^2 + 2(\bar{Y} - \mu)\sum_{i=1}^n (Y_i - \bar{Y}) + \sum_{i=1}^n (\bar{Y} - \mu)^2$
$= \sum_{i=1}^n (Y_i - \bar{Y})^2 + 2(\bar{Y} - \mu)(\sum_{i=1}^n Y_i - n\bar{Y}) + \sum_{i=1}^n (\bar{Y} - \mu)^2$
$= \sum_{i=1}^n (Y_i - \bar{Y})^2 + 0 + \sum_{i=1}^n (\bar{Y} - \mu)^2$
$\geq \sum_{i=1}^n (Y_i - \bar{Y})^2$

Therefore, $\sum_{i=1}^n (Y_i - \mu)^2$ is minimized when $\bar{Y} = \mu$

Introductory Data Example

Comparing Childrens' Heights and Their Parents' Heights

```
ggplot(galton, aes(x = parent, y = child)) + geom_point()
```

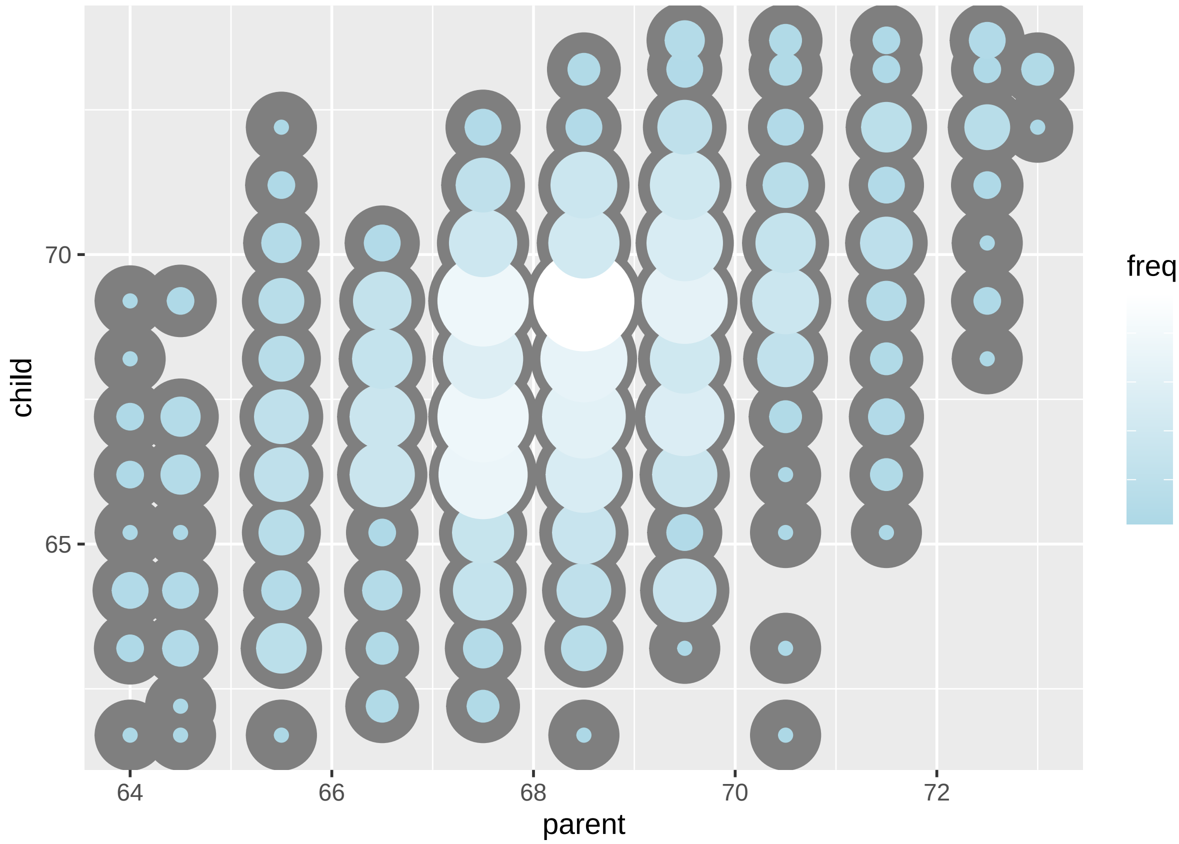


- These points are overplotted, there are multiple overlays at each point, so let's make a better plot

```
freqData <- as.data.frame(table(galton$child, galton$parent))
names(freqData) <- c("child", "parent", "freq")
freqData$child <- as.numeric(as.character(freqData$child))
freqData$parent <- as.numeric(as.character(freqData$parent))
plot <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child)) +
        scale_size(range = c(2, 20), guide = "none") +
        geom_point(colour = "grey50",
                   aes(size = freq + 20, show_guide = FALSE)) +
        geom_point(aes(colour = freq, size = freq)) +
        scale_colour_gradient(low = "lightblue", high = "#FFFFFF")
```

```
## Warning: Ignoring unknown aesthetics: show_guide
```

```
plot
```



Regression Through the Origin

- Suppose that X_i are the parents' heights

- Consider picking the slope β that minimizes $\sum_{i=1}^{n}(Y_i - X_i\beta)^2$

- This is exactly using the orgin as a pivot point picking the line that minimizes the sum of squared vertical distances of the points to the line

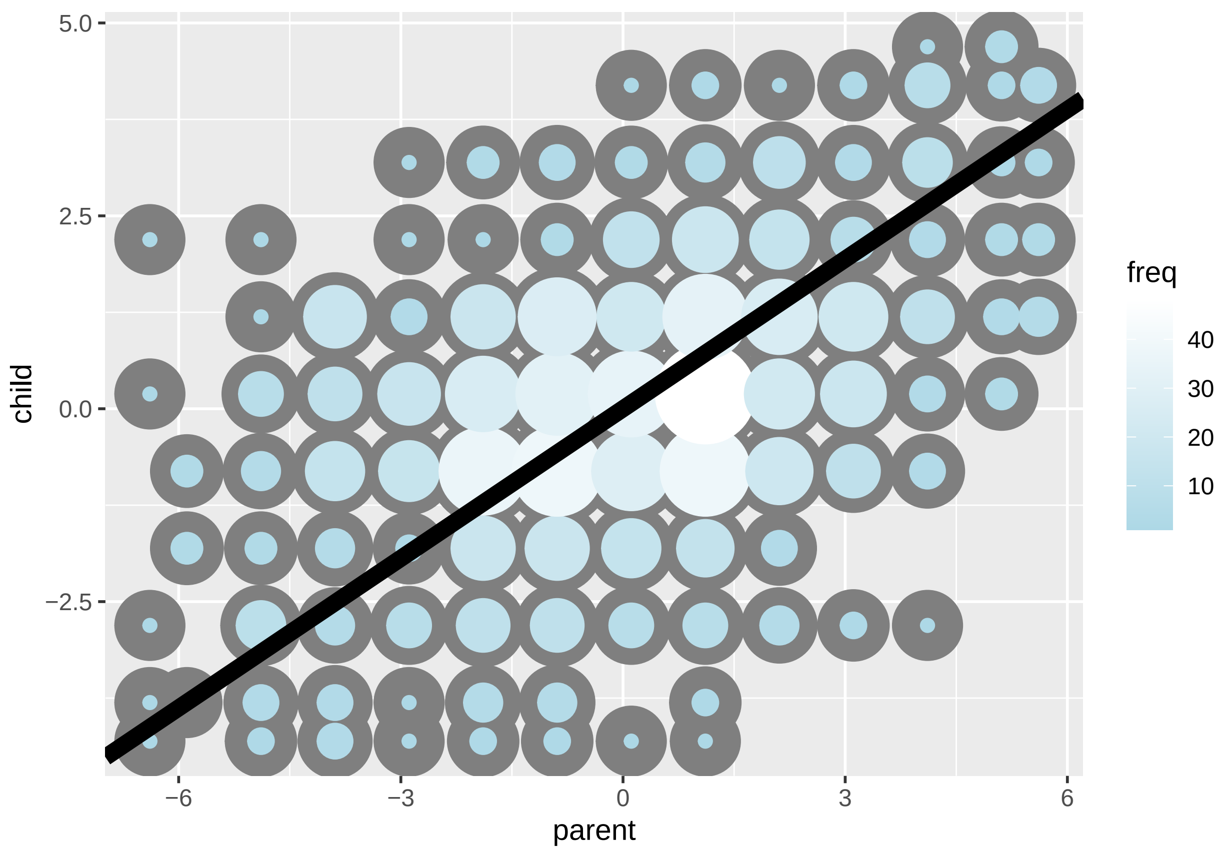
- Subtract the means so that the orgin is the mean of the parent and children's heights + A plot with a regression line going through true (0,0) often doesn't make sense, so subtracting the means realigns the orgin to be in the middle of the data

```
freqData <- as.data.frame(table(galton$parent - mean(galton$parent),
                                galton$child - mean(galton$child)))
names(freqData) <- c("child", "parent", "freq")
freqData$child <- as.numeric(as.character(freqData$child))
freqData$parent <- as.numeric(as.character(freqData$parent))
plot <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child)) +
        scale_size(range = c(2, 20), guide = "none") +
        geom_point(colour = "grey50",
                   aes(size = freq + 20)) +
```

```r
        geom_point(aes(colour = freq, size = freq)) +
        scale_colour_gradient(low = "lightblue", high = "#FFFFFF") +
        geom_abline(intercept = 0,

                slope = lm(
                        I(child - mean(child)) ~
                                I(parent - mean(parent)) - 1,
                        data = galton)$coeff,

                    size = 3)
plot
```



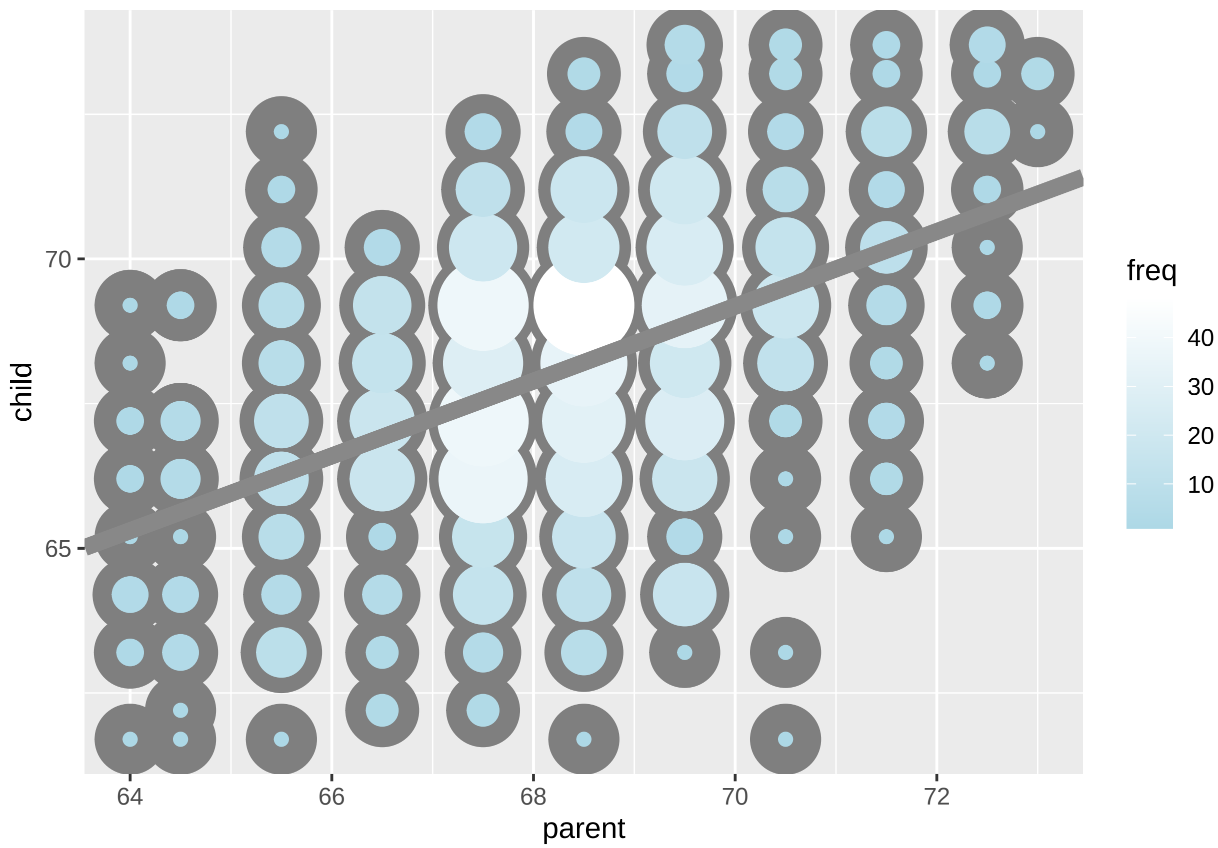
- In the next few lectures we'll talk about why this is the solution

```r
lm(I(child - mean(child)) ~ I(parent - mean(parent)) - 1, data = galton)
```

```
##
## Call:
## lm(formula = I(child - mean(child)) ~ I(parent - mean(parent)) -
##     1, data = galton)
##
## Coefficients:
## I(parent - mean(parent))
##                   0.6463
```

- The `I` function just ignores the intercept, since we already adjusted for that
- We can also fit a line to an un-adjusted model

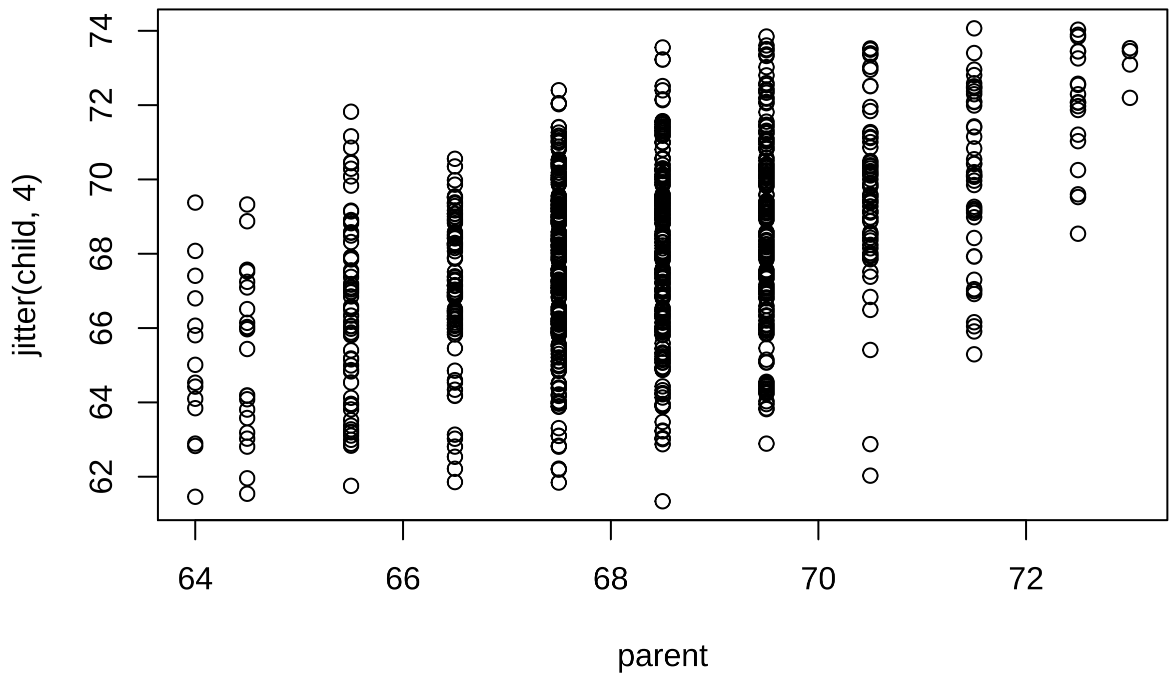
```
freqData <- as.data.frame(table(galton$child, galton$parent))
names(freqData) <- c("child", "parent", "freq")
freqData$child <- as.numeric(as.character(freqData$child))
freqData$parent <- as.numeric(as.character(freqData$parent))
plot <- ggplot(filter(freqData, freq > 0), aes(x = parent, y = child)) +
        scale_size(range = c(2, 20), guide = "none" ) +
        geom_point(colour = "grey50", aes(size = freq + 20)) +
        geom_point(aes(colour = freq, size = freq)) +
        scale_colour_gradient(low = "lightblue", high = "#FFFFFF")
lm1 <- lm(galton$child ~ galton$parent)
plot + geom_abline(intercept = coef(lm1)[1], slope = coef(lm1)[2],
                      size = 3, colour = "#888888")
```



Lesson with `swirl()`: Introduction

- Another way we could have gotten past overlapping plot points is to use the `jitter` function

```
plot(jitter(child,4) ~ parent, galton)
```



Linear Least Squares

- Also called **Ordinary Least Squares (OLS)**; it fits a line through some data.

Notation and Background

Notation

- The empirical mean is defined as
  $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

- If we subtract the mean from data points, we get data that has a mean of 0. That is, if we define:
  $\tilde{X}_i = X_i - \bar{X}$.
  + The mean of $\tilde{X}_i$ is 0

- This process is called "**centering**" the random variables

- Recall from the previous lecture that the mean is the elast squares solution for minimizing $\sum_{i=1}^{n}(X_i - \mu)^2$

The Emprical Standard Deviation adn Variance

- Define the empirical variance as
  $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1}(\sum_{i=1}^n X_i^2 - n\bar{X}^2)$

- The empirical standard deviation is defined as $S = \sqrt{S^2}$.
  + Notice that the standard deviation has the same units as the data.

- The data defined by $\frac{X_i}{s}$ have an empirical standard deviation of 1. + This is called "**scaling**" the data.

Normalization

- The data defined by
  $Z_i = \frac{X_i - \bar{X}}{s}$
  have an empirical mean of 0 and an empirical standard deviation of 1.

- The process of centering then scaling the data is called "**normalizing**" the data.

- Normalized data are centered at 0 and have units equal to standard deviations of the original data.

- For example, a value of 2 from normalized data is saying that data point was two standard deviations larger than the mean.

The Empirical Covariance

- Consider now when we have pairs of data, (X_i, Y_i)

- Their empirical covariance is
  $Cov(X,Y) = \frac{1}{n-1}\sum_{i=1} n(X_i - \bar{X})(Y_i - \bar{Y})$
  $= \frac{1}{n-1}(\sum_{i=1}^n X_i Y_i - n\bar{X}\bar{Y})$

- The correlation is defined as
  $Cor(X,Y) = \frac{Cov(X,Y)}{S_x S_y}$
  + Where S_x and S_y are the estimates of standard deviaitons for the X observations and Y observations, respectively.

Some Facts About Correlation

- $Cor(X,Y) = Cor(Y,X)$

- $-1 \leq Cor(X,Y) \leq 1$

- $Cor(X,Y) = 1$ and $Cor(X,Y) = -1$ only when the X or Y observations fall perfectly on a positive or negative sloped line, repectively.

- $Cor(X, Y)$ measures the strength of the linear relationship between the X and Y data, with stronger relationships as $Cor(X, Y)$ heads towards either -1 or 1 {
- $Cor(X, Y) = 0$ implies no linear relationship

Linear Least Squares

Fitting the Best Line

- Let Y_i be the i^{th} child's height and X_i be the i^{th} (average over the pair of) parents' heights.

- Consider finding the best line
 + Child's Height = β_0 + Parent's Height * β_1
 $\sum_{i=1}^{n} Y_i - (\beta_0 + \beta_1 X_i)^2$
- the least squares model fit to the line $Y = \beta_0 + \beta_1 X$ through the data pairs (X_i, Y_i) with Y_i as the outcome obtains the line $Y = \hat{\beta}_0 + \hat{\beta}_1 X$ where
 $\hat{\beta}_1 = Cor(Y, X)\frac{Sd(Y)}{Sd(X)}$
 $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

- $\hat{\beta}_1$ has the units of Y/X, $\hat{\beta}_0$ has the units of Y.

- The line passes through the point $(\bar{X}, \bar{Y})$

- The slope of the regression line with X as the outcome and Y as the predictor is $\frac{Cor(Y,X)Sd(X)}{Sd(Y)}$

- The slope si the same one you would get if you centered the data, $(X_i - \bar{X}, Y_i - \bar{Y})$, and made a regression through the orgin

- If you normalized the data, $(\frac{X_i - \bar{X}}{Sd(X)}, \frac{Y_i - \bar{Y}}{Sd(Y)})$, the slope is $Cor(Y, X)$

Linear Least Squares Coding Example

```r
y <- galton$child
x <- galton$parent
beta1 <- cor(y,x) * sd(y) / sd(x)
beta0 <- mean(y) - beta1 * mean(x)

#Showing the computations by hand are the same as coef from lm function
rbind(c(beta0, beta1), coef(lm(y~x)))
```

```
##      (Intercept)         x
## [1,]    23.94153 0.6462906
## [2,]    23.94153 0.6462906
```

- `lm` stands for *linear model*

```
#The slope is the same in centered data
yc <- y - mean(y)
xc <- x - mean(x)
beta1 <- sum(yc * xc) / sum(xc^2)
c(beta1, coef(lm(y ~ x))[2])
```

```
##                   x
## 0.6462906 0.6462906
```

```
lm(yc ~ xc - 1)$coef #minus 1 gets rid of intercept
```

```
##        xc
## 0.6462906
```

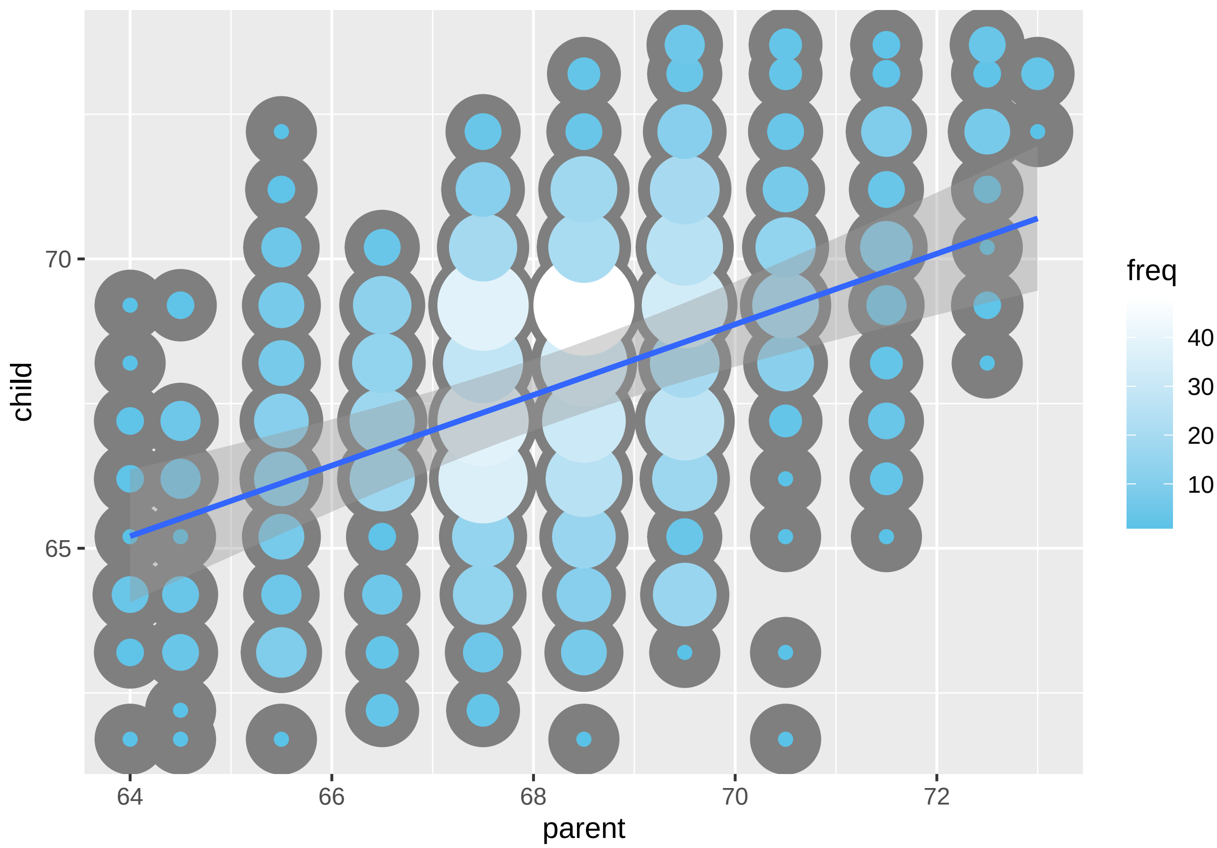
```
#Normalizing variables results in the slope being the correlation
yn <- (y - mean(y))/sd(y)
xn <- (x - mean(x))/sd(x)
results <- cbind(cor(y,x), lm(yn ~ xn)$coef[2], cor(yn, xn))
colnames(results) <- c("cor(y,x)", "Slope(yn ~ xn)", "cor(yn, xn)")
results
```

```
##     cor(y,x) Slope(yn ~ xn) cor(yn, xn)
## xn 0.4587624      0.4587624   0.4587624
```

Adding a Linear Regression to ggplot

```
plot <- ggplot(filter(freqData, freq > 0), aes(parent, child)) +
        scale_size(range = c(2, 20), guide = "none") +
        geom_point(colour = "grey50", aes(size = freq + 20)) +
        geom_point(aes(colour = freq, size = freq)) +
        scale_colour_gradient(low = "#5BC2E7", high = "#FFFFFF")

#Adding smoother
#y ~ x is assumed if not given
plot + geom_smooth(method = "lm", formula = y ~ x)
```



- A confidence interval is also given around the line automaticly

Technical Details

Brian Caffo discusses the proof for least squares regression beta_1 value in this video

Lesson with `swirl()`: Least Squares Estimation

(No new content)

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Regression to the Mean

Regression to the Mean

- $P(Y < x|X = x)$ gets bigger as x tends towards very large values.
  + Similarly $P(Y > x|X = x)$ gets bigger as x tends towards very small values.

- Regression line is like the intrisic part of this relation
  + Unless $Cor(Y, X) = 1$ the intrinsic part isn't perfect

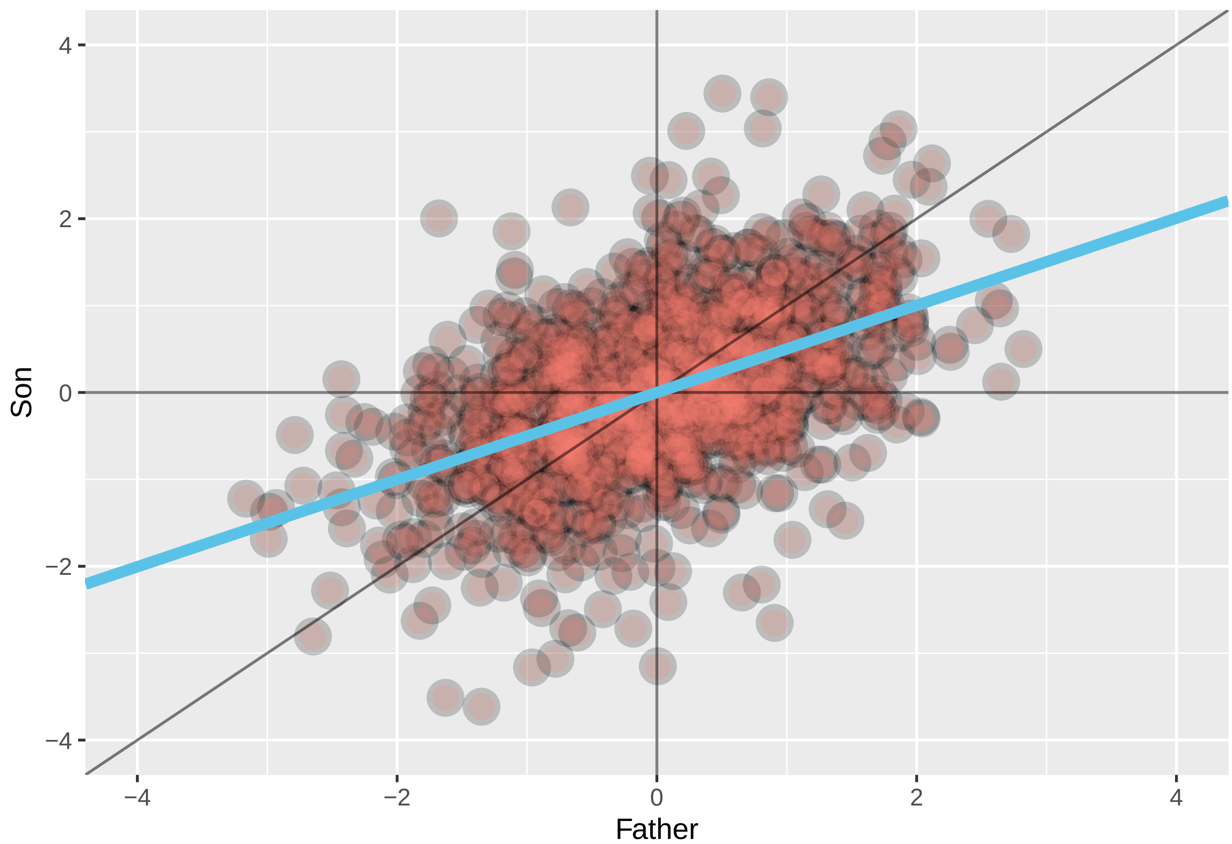
- Suppose we center X (child's hieght) and Y (parent's height) so that they both have a mean of 0
  + Then, recall, our regression line passes through $(0, 0)$

- We then normalize the data points too
  + The slope of the regression line is $Cor(Y, X)$, regardless of which variable is the outcome (since both sds are 1)

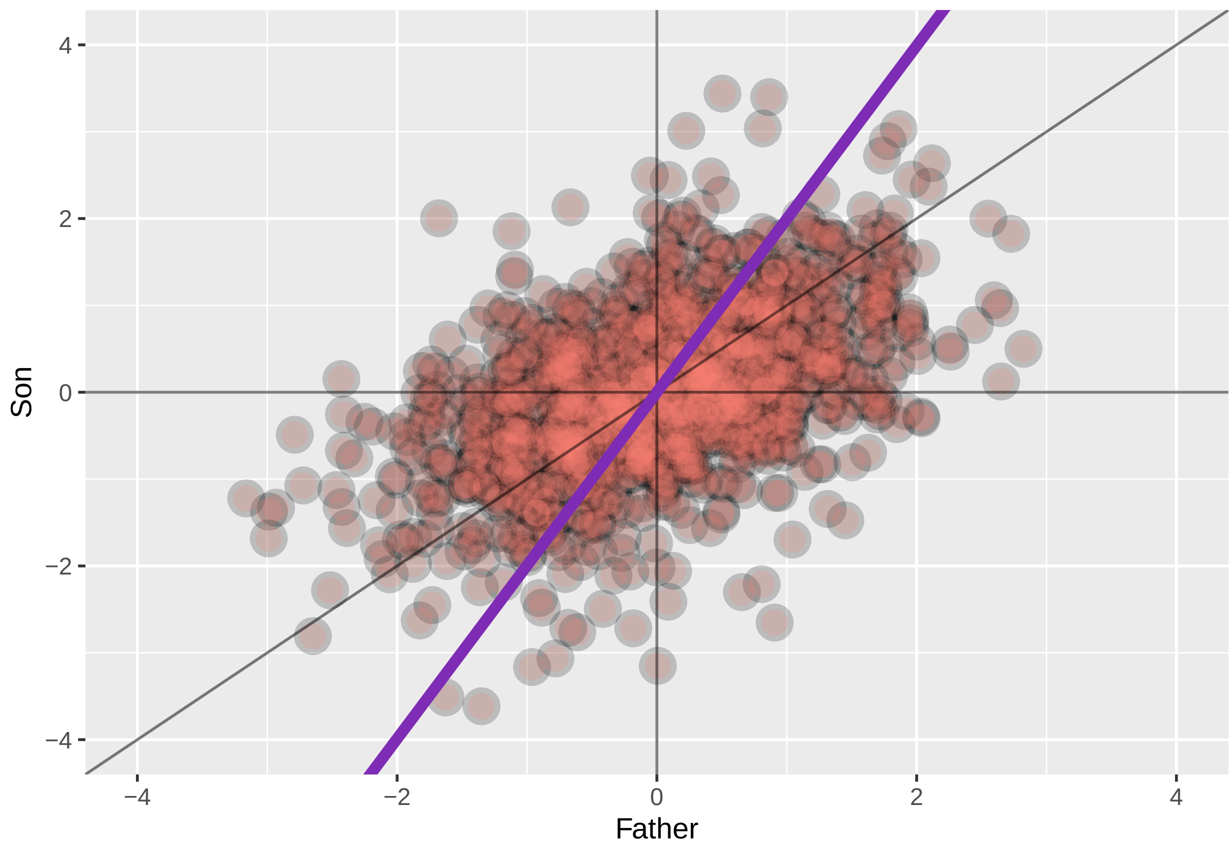
- If the outcome is plotted on the horizontal axis the slope of the least squares line will be $\frac{1}{Cor(Y,X)}$

Plotting the Regression Implicitly

```
library(UsingR); data(father.son)
y <- father.son$sheight
x <- father.son$fheight
y <- (y - mean(y)) / sd(y)
x <- (x - mean(x)) / sd(x)
rho <- cor(x, y) #rho is std greek letter for correlations
plot <- ggplot(data.frame(Father = x, Son = y), aes(Father, Son)) +
        geom_point(size = 6, colour = "#000000", alpha = 0.2) +
        geom_point(size = 4, colour = "salmon", alpha = 0.2) +
        xlim(-4,4) +
        ylim(-4,4) + #Std. norm being +/- 4 is very unlikely
        geom_abline(intercept = 0, slope = 1, alpha = 0.5) +
        geom_vline(xintercept = 0, alpha = 0.5) +
        geom_hline(yintercept = 0, alpha = 0.5)
plot + geom_abline(intercept = 0, slope = rho, size = 2, colour = "#5BC2E7")
```




```
plot + geom_abline(intercept = 0, slope = 1/rho, size = 2, colour = "#7E2CB5")
```



* The blue line is where the Father's height is the predictor and the Son's height is the outcome
* The purple line is where the Son's hieght is the predictor and the Father's height is the outcome (`1/rho` because the outcome is on the horizontal axis)

Lesson with `swirl()`: Residuals

- A residual is the distance between the actual data point and the regression line.
  + I've previously heard it also called the "Unexplained Variation" since the distance form the mean value to data point is the "Total Variation (from the mean)", then the distance from the mean to reg. line is the "Explained Variation".

- You can get some info on a data sets residuals by calling `summary` on the results of `lm` as seen below

```r
summary(lm(child ~ parent, galton))
```

```
##
## Call:
## lm(formula = child ~ parent, data = galton)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -7.8050 -1.3661  0.0487  1.6339  5.9264
##
```

```
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 23.94153    2.81088   8.517   <2e-16 ***
## parent       0.64629    0.04114  15.711   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.239 on 926 degrees of freedom
## Multiple R-squared:  0.2105, Adjusted R-squared:  0.2096
## F-statistic: 246.8 on 1 and 926 DF,  p-value: < 2.2e-16
```

- `est` will return the estimate, $\hat{y}$
- `sqe` will calculate the sum of the squared residuals, also called the Residual Sum of Squares
- var(residuals) = var(data) - var(estimate)
  + As such the variance of residuals is always less than the variance of data
- The residuals shouldn't be correlated to either factor, if it did this may imply a diffrent relationship is present

Quiz 1

1. Given...

```
x <- c(0.18, -1.54, 0.42, 0.95)
w <- c(2, 1, 3, 1)
```

Give the value of μ that minimizes the least squares equation $\sum_{i=1} nw_i(x_i - \mu)^2$

```
sum(w * x) / sum(w)
```

```
## [1] 0.1471429
```

2. Given...

```
x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42)
y <- c(1.39, 0.72, 1.55, 0.48, 1.19, -1.59, 1.23, -0.65, 1.49, 0.05)
```

Fit the regression through the orgin and get the slope treating y as the outsome and x as the regressor.

```
lm(y ~ x - 1)$coef
```

```
##         x
## 0.8262517
```

3. Do `data(mtcars)` from the datasets package and fit the regression model with mpg as the outcome and weight as the predictor. Give the slope coefficient.

```
data(mtcars)
lm(mpg ~ wt, mtcars)$coef
```

```
## (Intercept)          wt
##   37.285126   -5.344472
```

4. Consider data with an outcome (Y) and a predictor (X). The standard deviation of the predictor is one half that of the outcome. The correlation between the two variables is 0.5. What value would the slope coefficient for the regression model with Y as the outcome and X as the predictor?

```r
0.5 * 2/1
```

```
## [1] 1
```

5. Students were given two hard tests and scores were normalized to have empirical mean 0 and variance 1. The correlation between the scores on the two tests was 0.4. What would be the expected score on Quiz 2 for a student who had a normalized score of 1.5 on Quiz 1?

```r
beta1 <- 0.4 * 1/1
beta0 <- 0 - beta1*0
yhat <- beta0 + beta1*1.5
yhat
```

```
## [1] 0.6
```

6. Given...

```r
x <- c(8.58, 10.46, 9.01, 9.64, 8.86)
```

What is the value of the first measurement if x were normalized?

```r
xn <- (x-mean(x))/sd(x)
xn[1]
```

```
## [1] -0.9718658
```

7. Given...

```r
x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42)
y <- c(1.39, 0.72, 1.55, 0.48, 1.19, -1.59, 1.23, -0.65, 1.49, 0.05)
```

What is the intercept for fitting the model with x as the predictor and y as the outcome?

```r
lm(y ~ x)$coef
```

```
## (Intercept)           x
##    1.567461   -1.712846
```

8. You know that both the predictor and response have mean 0. What can be said about the intercept when you fit a linear regression?

- The intercept is the orgin

9. Given...

```r
x <- c(0.8, 0.47, 0.51, 0.73, 0.36, 0.58, 0.57, 0.85, 0.44, 0.42)
```

What value minimizes the sum of the squared distances between these points and itself?

```r
mean(x)
```

```
## [1] 0.573
```

10. Let the slope having fit Y as the outcome and X as the predictor be denoted as β_1. Let the slope from fitting X as the outcome and Y as the predictor be denoted as γ_1. Suppose that you divide β_1 by γ_1 What is this ratio always equal to?

- $\beta_1 = Cor(Y, X)\frac{sd(Y)}{sd(X)}$
- $\gamma_1 = Cor(Y, X)\frac{sd(X)}{sd(Y)}$
- $\frac{\beta_1}{\gamma_1} = \frac{Cor(Y,X)*sd(Y)/sd(X)}{Cor(Y,X)*sd(X)/sd(Y)} = \frac{sd(Y)*sd(Y)}{sd(X)*sd(X)} = \frac{Var(Y)}{Var(X)}$

Linear Regression & Multivariable Regression

Statistical Linear Regression Models

Statistical Linear Regression Models

Basic Regression Model with Additive Gaussian Errors

- Consider developing a probabilistic model for linear regression
  $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$
  + Here the ϵ_i are assumed iid $N(0, \sigma^2)$
  - Can be thought of as accumulated variables that aren't modeled by act on the response as iid gaussian errors + $E[Y_i | X_i = x_i] = \mu_i = \beta_0 + \beta_1 x_i$
  + $Var(Y_i | X_i = x_i) = \sigma^2$

Interpreting Coefficients

Intercept

- β_0 is the expected value of the response when the predictor is 0
  $E[Y | X = 0] = \beta_0 + \beta_1 \times 0 = \beta_0$
  + This isn't always a value of interest, for example when $X = 0$ is impossible (x represents weight) or far outside of the range of data.

- A solution to non-interpretable intercepts is to shift the equation by some value, a then define a new intercept, $\tilde{\beta_0}$.
  $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i = \beta_0 + a\beta_1 + \beta_1(X_i - a) + \epsilon_i = \tilde{\beta_0} + \beta_1(X_i - a) + \epsilon_i$
  + Shifting your X values by value a changes the intercept, but not the slope.
  + Often a is set to $\bar{X}$ so that the intercept is inteerpretted as the expected response at the average X value.

Slope

- β_1 is the expected change in response for a 1 unit change in the predictor

- Consider the impact of changing the units of X.
  $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i = \beta_o + \frac{\beta_1}{a}(X_i a) + \epsilon_i = \beta_0 + \tilde{\beta_1}(X_i a) + \epsilon_i$
  + Since β_1 is in units of Y/X we divide by the factor, a, that we're multiplying with X_i.

- Example: X is height in m and Y is weight in kg. Then β_1 is kg/m. Converting X to cm implies multiplying X by $100cm/m$. To get β_1 in the right units, we have to divide by $100cm/m$ to get it to have the right units.
  $Xm \times \frac{100cm}{m} = (100X)cm$ and $\beta_1 \frac{kg}{m} \times \frac{1m}{100cm} = (\frac{\beta_1}{100})\frac{kg}{cm}$

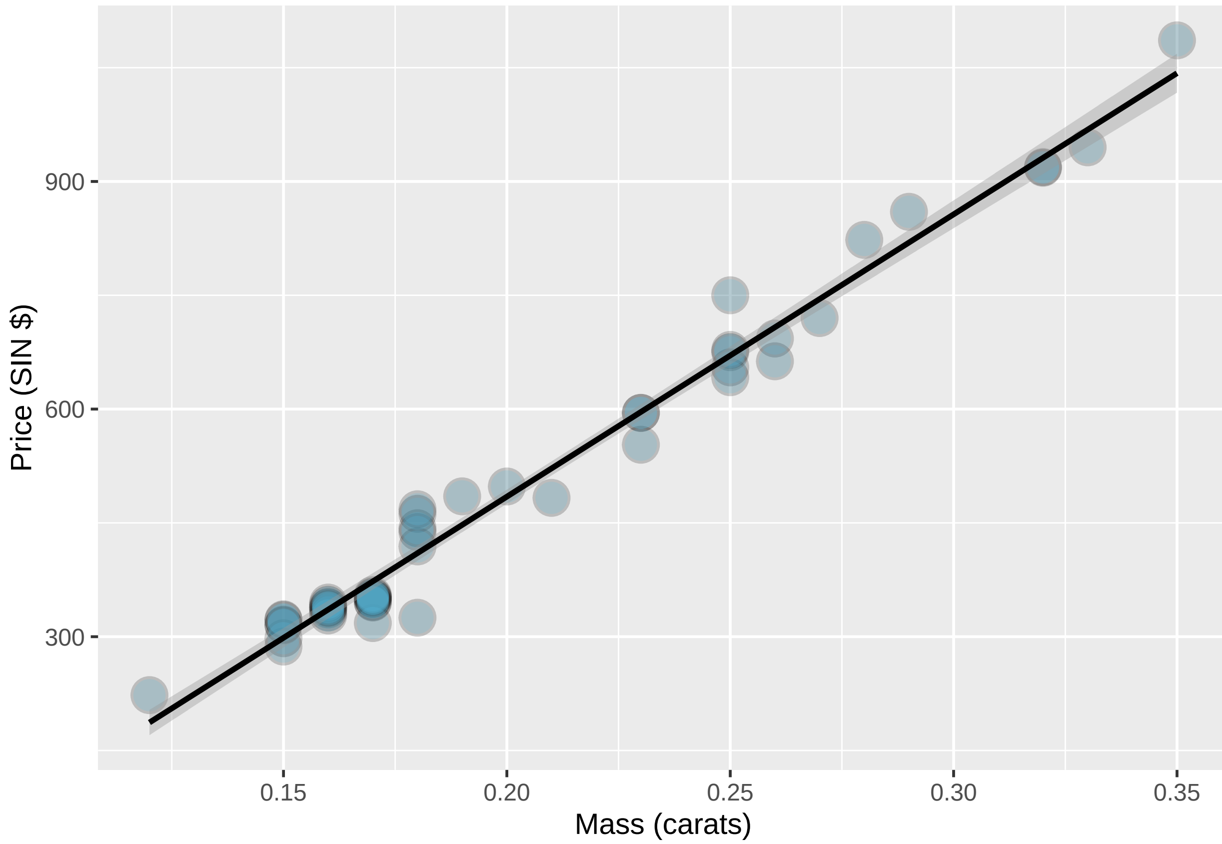
Linear Regression for Prediction

- We can get a prediction for Y, $\hat{y}$ by plugging in the X that we want into our model
  $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$

Example using diamond Data

- The data in this example is diamond prices (in Sigapore dollars) and diamond weight in carats (1 carat = 0.2 g).

```
library(UsingR); data(diamond); library(tidyverse)
plot <- ggplot(diamond, aes(carat, price)) +
        xlab("Mass (carats)") +
        ylab("Price (SIN $)") +
        geom_point(size = 6, colour = "#000000", alpha = 0.2) +
        geom_point(size = 5, colour = "#5BC2E7", alpha = 0.2)
plot + geom_smooth(method = "lm", colour = "#000000", formula = y ~ x)
```



Creating a Model

```
# Fitting the linear regression model
fit <- lm(price ~ carat, data = diamond)
coef(fit)
```

```
## (Intercept)       carat
##   -259.6259   3721.0249
```

- We estimate an expected 3721.02 (SIN) dollar increase in price for every increase of 1 carat in mass of diamonds.
- The intercept, -259.63 is the expected price of a 0 carat diamond, which doesn't make sense to interpret.
  + As such we'll mean center our reg. line
  #### Centering Model on the Mean

```
cfit <- lm(price ~ I(carat - mean(carat)), data = diamond)
cfit$coef
```

```
##            (Intercept) I(carat - mean(carat))
##               500.0833              3721.0249
```

- To do arithmetic operations in the formula in `lm` you have to surround the operation with the `I` function

- The slope has not changed

- The intercept has changed to 500, the expected price for the average sized diamond of the data (0.204 carats).

Changing Units in the Model

- Change unit to 1/10 of a carrat

```
tenthfit <- lm(price ~ I(carat * 10), data = diamond)
coef(tenthfit)
```

```
##   (Intercept) I(carat * 10)
##     -259.6259      372.1025
```

- So now the slope is interpretted as a 372.1 dollar increase for every additional 0.1 carrats of diamond.

Estimating a Value

```
newDiamonds <- c(0.16, 0.27, 0.34)
#Computing manually
fit$coef[1] + fit$coef[2] * newDiamonds
```

```
## [1]  335.7381  745.0508 1005.5225
```

```
#Using predict function
results <- predict(fit, newdata = data.frame(carat = newDiamonds))
names(results) <- as.character(newDiamonds) #renaming not required
results
```

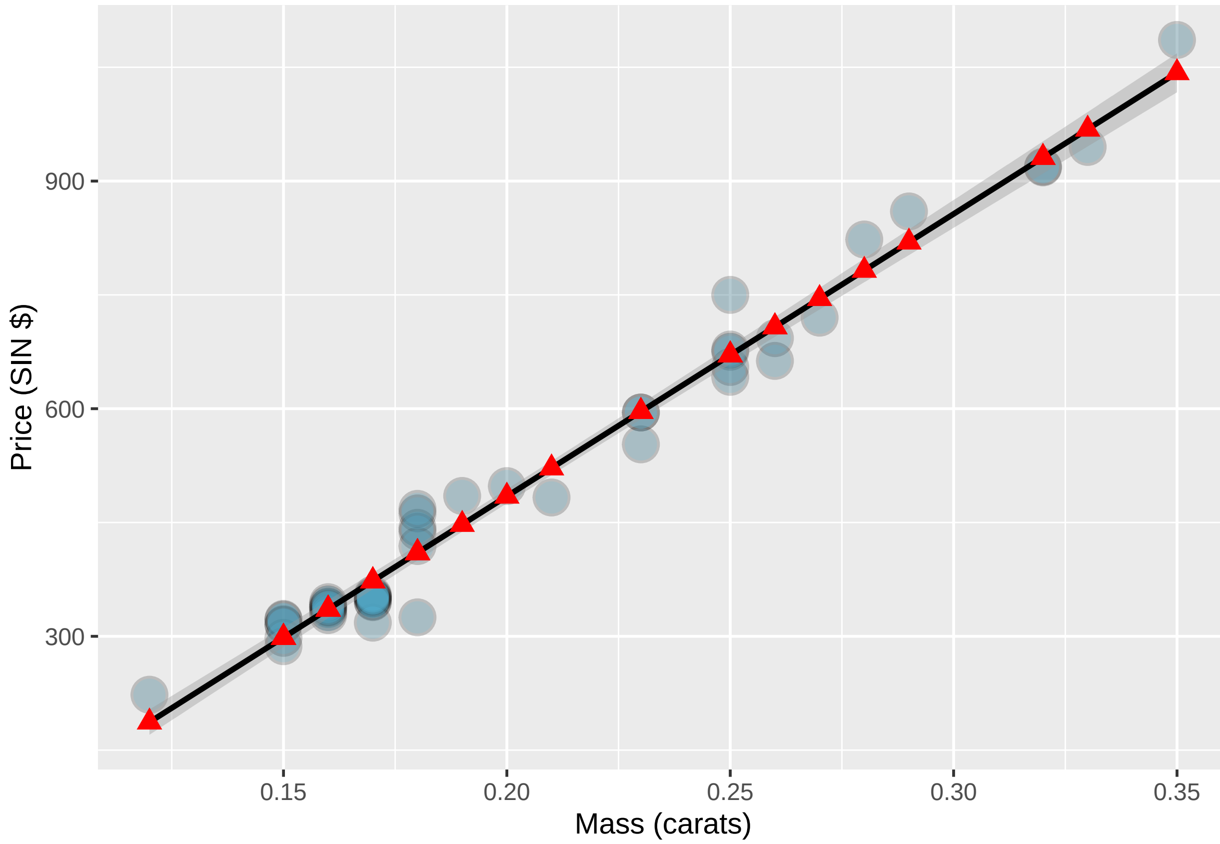
```
##      0.16      0.27      0.34
##  335.7381  745.0508 1005.5225
```

```
#Using predict without `newdata` will return y-hat for given x values
predict(fit)
```

```
##         1         2         3         4         5         6         7         8
##  372.9483  335.7381  372.9483  410.1586  670.6303  335.7381  298.5278  447.3688
##         9        10        11        12        13        14        15        16
##  521.7893  298.5278  410.1586  782.2611  335.7381  484.5791  596.2098  819.4713
##        17        18        19        20        21        22        23        24
##  186.8971  707.8406  670.6303  745.0508  410.1586  335.7381  372.9483  335.7381
##        25        26        27        28        29        30        31        32
##  372.9483  410.1586  372.9483  410.1586  372.9483  298.5278  372.9483  931.1020
##        33        34        35        36        37        38        39        40
##  931.1020  298.5278  335.7381  335.7381  596.2098  596.2098  372.9483  968.3123
##        41        42        43        44        45        46        47        48
##  670.6303 1042.7328  410.1586  670.6303  670.6303  298.5278  707.8406  298.5278
```

```
plot + geom_smooth(method = "lm", colour = "#000000", formula = y ~ x) +
        geom_point(aes(y = as.numeric(predict(fit))),
```

```r
    size = 3, color = "#FF0000", shape = 17)
```



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Residuals

Residuals

- The residuals are the variation from the regression line, that is left unexplained by our model, $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$.
- Observed outcome i is Y_i at predictor value X_i
- Predicted outcome i is $\hat{Y}_i$ at predictor value X_i is $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
- Residual, e_i, is the difference between the observed and predicted outcome: $e_i = Y_i - \hat{Y}_i$. + This is the vertical distanc ebetween the observed data point and the regression line
- Least squares minimizes these residuals, the equation $\sum_{i=1}^{n} e_i^2$
- The e_i can be thought of as estimates of the ϵ_i

Properties of the Residuals

- $E[e_i] = 0$
- If an intercept is included, $\sum_{i=1}^{n} e_i = 0$
- If a regressor variable, X_i, is included in the model $\sum_{i=1}^{n} e_i X_i = 0$
- Residuals are useful for investigating poor model fit
  + Residual plots can highlight these poor fits
- Residuals can be though of as the outcome (Y) with the linear association of the predictor (X) removed.
- One differentiates residual variation (variation after removing the predictor) from systematic variation (variation explained by the regression model).

Residuals, Coding Example

- Using diamond dataset again

```r
data("diamond")
y <- diamond$price
x <- diamond$carat
fit <- lm(y ~ x)

e <- resid(fit) #Getting residuals

yhat <- predict(fit)

# Showing residuals are the same as y - yhat (within a floating point error)
max(abs(e - (y - yhat)))
```

```
## [1] 5.258016e-13
```

```r
# And again, but manually entering the equation for yhat
max(abs(e - (y - (coef(fit)[1] + coef(fit)[2] * x))))
```

```
## [1] 5.258016e-13
```

```r
#Showing sum of resid and resid*x are both 0
sum(e)
```

```
## [1] -3.93019e-14
```

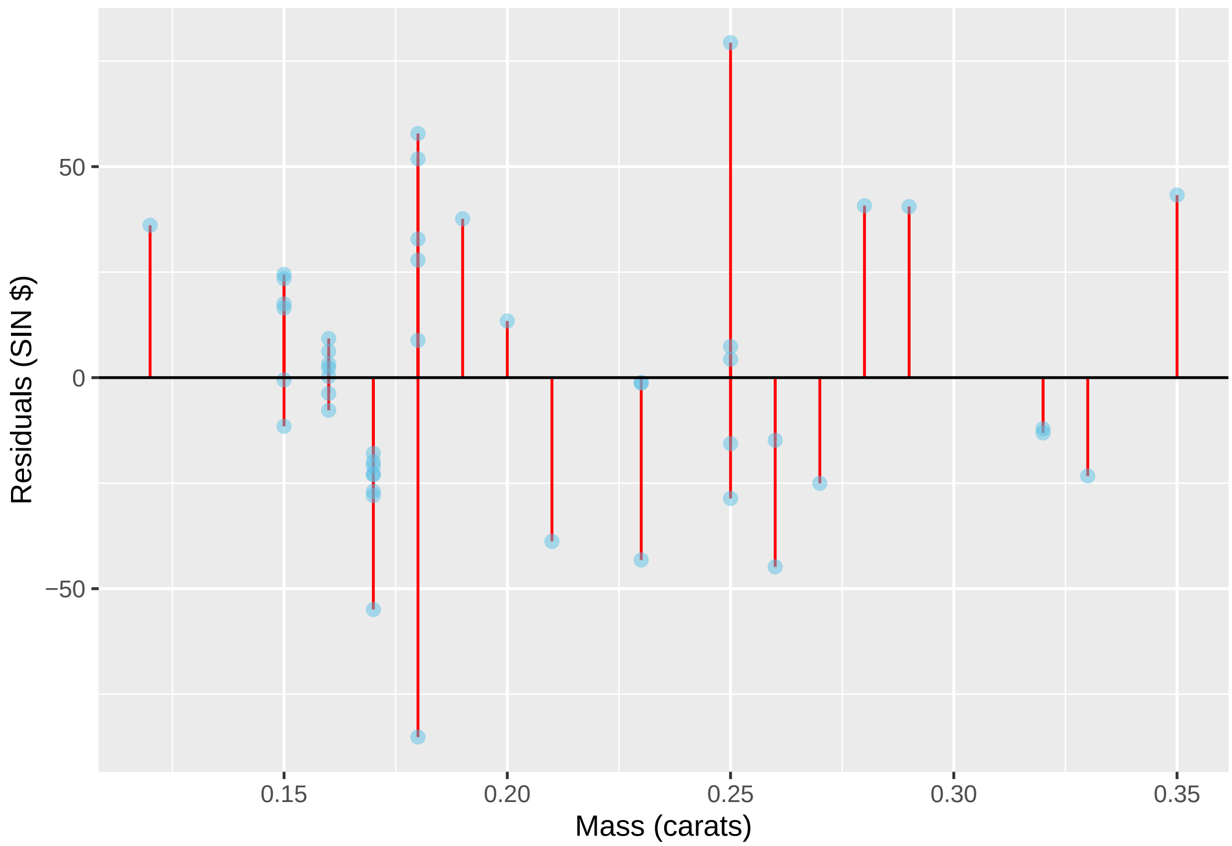
```r
sum(e * x)
```

```
## [1] -1.249001e-15
```

```r
#Plotting the residuals
plot <- ggplot(data.frame(x = x, y = y, resid = e), aes(x, resid)) +
```

```
        geom_segment(aes(xend = x, yend = 0), colour = "#FF0000") +

        geom_point(size = 2, colour = "#5BC2E7", alpha = 0.5) +
        xlab("Mass (carats)") +
        ylab("Residuals (SIN $)") +
        geom_hline(yintercept = 0, color = "#000000")
plot
```



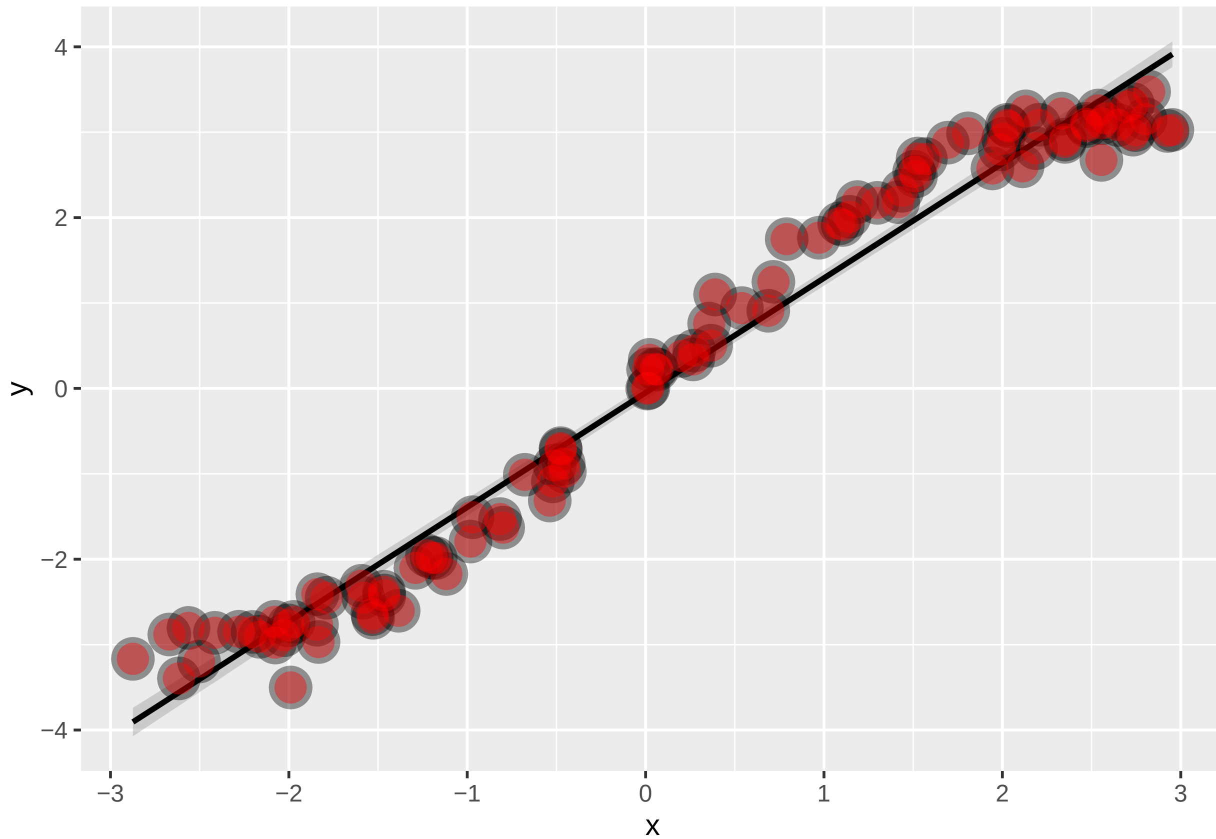
Using Residual Plot to Detect a Poorly Fit Model

- We're going to generate some data that looks linear but actually has an underlying relation to it that will become more apparent after plotting the residuals

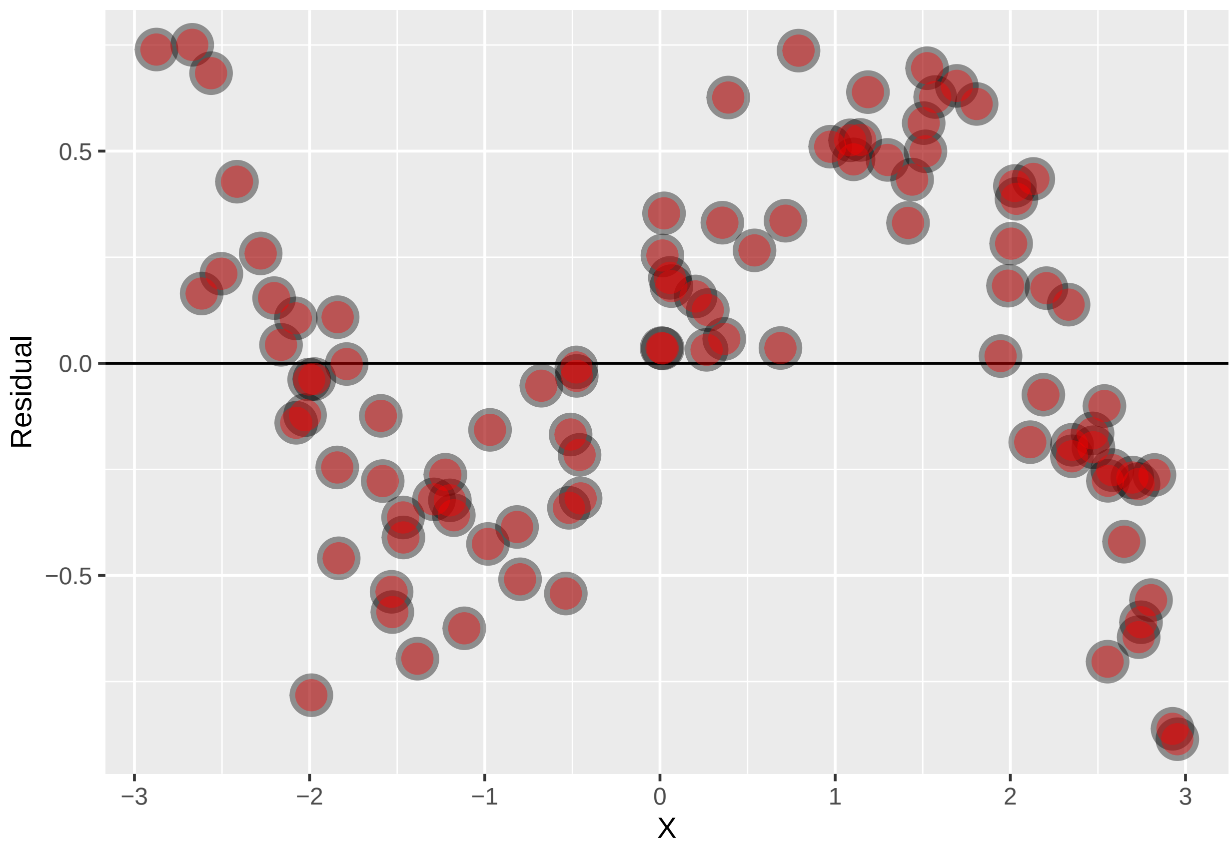
```
set.seed(1618033)
x <- runif(100, -3, 3)
y <- x + sin(x) + #Y is related with sin(x), lm will expose the sin(x) rel.
        rnorm(100, sd = .2) # For noise
plot <- ggplot(data.frame(x = x, y = y), aes(x,y)) +
        geom_smooth(method = "lm", colour = "#000000") +
        geom_point(size = 7, colour = "#000000", alpha = 0.4) +
        geom_point(size = 5, colour = "#FF0000", alpha = 0.4)
```

```
residplot <- ggplot(data.frame(x = x, resid = resid(lm(y ~ x))),
                    aes(x, resid)) +
        geom_hline(yintercept = 0) +
        geom_point(size = 7, colour = "#000000", alpha = 0.4) +
        geom_point(size = 5, colour = "#FF0000", alpha = 0.4) +
        labs(x = "X", y = "Residual")
plot
```

```
## `geom_smooth()` using formula 'y ~ x'
```




```
residplot
```

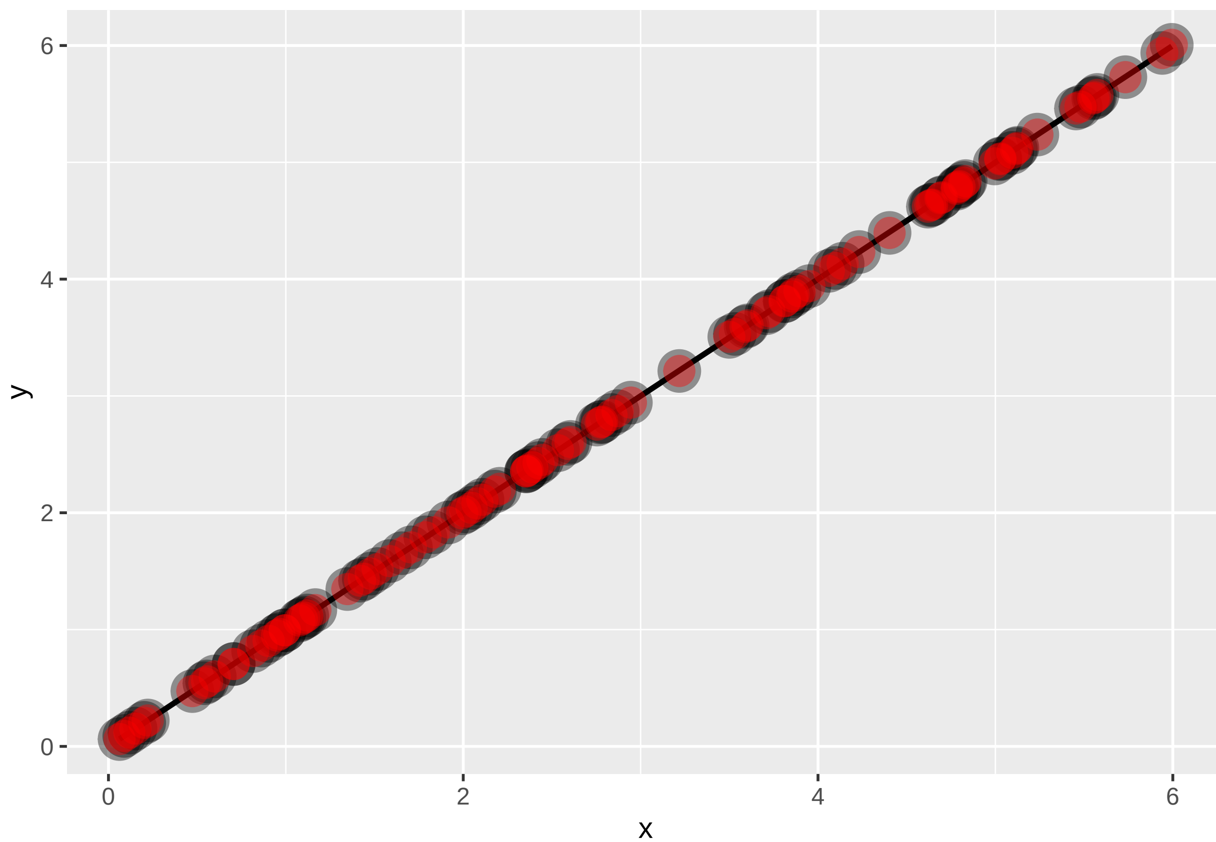


- A secondary pattern can be seen in the residual plot, indicating there might be a better model than a line.

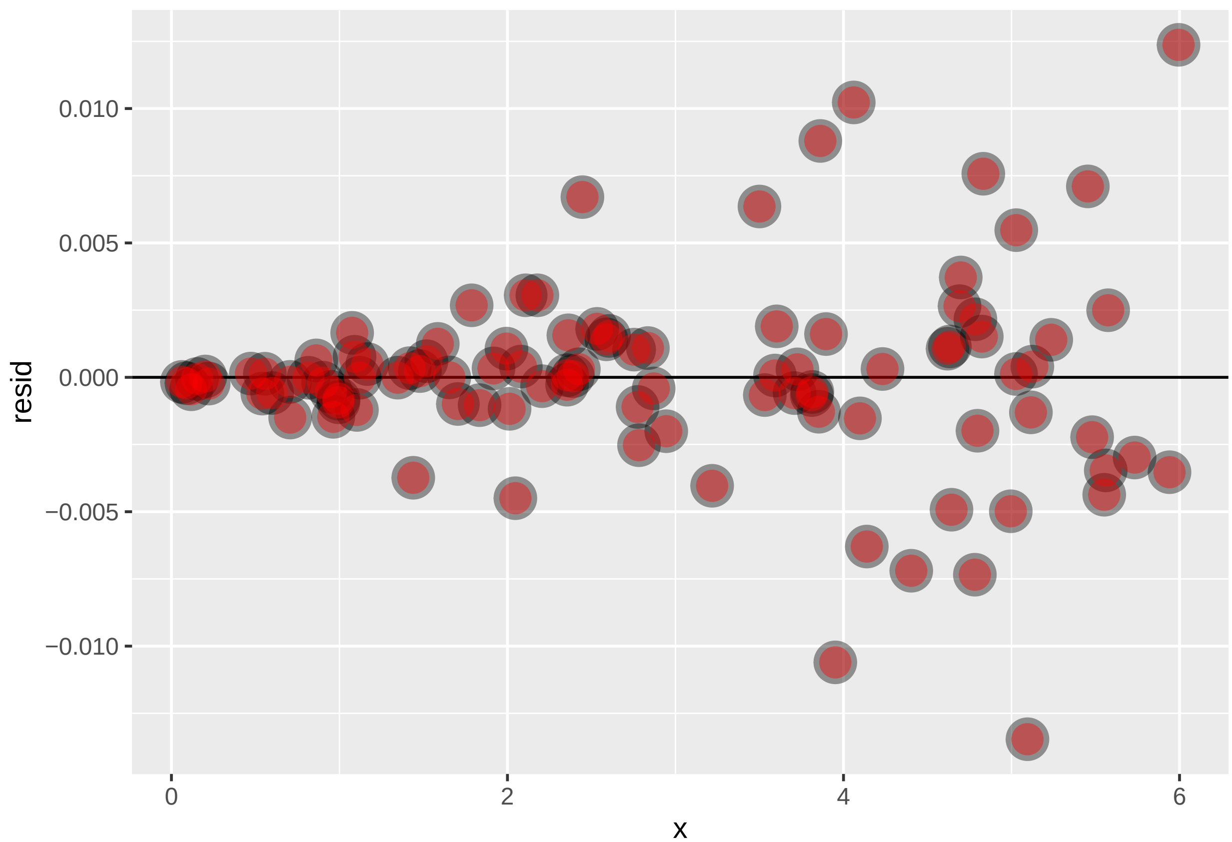
Detecting Heteroskedasticity with a Residual Plot

```
x <- runif(100, 0, 6)
y <- x + rnorm(100, mean = 0, sd = 0.001 * x)#sd increases as x increases
plot <- ggplot(data.frame(x = x, y = y), aes(x,y)) +
        geom_smooth(method = "lm", colour = "black") +
        geom_point(size = 7, colour = "#000000", alpha = 0.4) +
        geom_point(size = 5, colour = "#FF0000", alpha = 0.4)
residplot <- ggplot(data.frame(x = x, resid = resid(lm(y ~ x))),
                    aes(x,resid)) +
        geom_hline(yintercept = 0, colour = "#000000") +
        geom_point(size = 7, colour = "#000000", alpha = 0.4) +
        geom_point(size = 5, colour = "#FF0000", alpha = 0.4)
plot
```

```
## `geom_smooth()` using formula 'y ~ x'
```




```
residplot
```



* The plot looks linear, but plotting the residuals reveals an underlying pattern

Residual Variance

Estimating Residual Variaiton

- Model: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$
- The mean linear estimate of σ^2 is $\frac{1}{n}\sum_{i=1}^n e_i^2$, the average squared residual
- Most people use:
  $\hat\sigma^2 = \frac{1}{n-2}\sum_{i=1}^n e_i^2$
  + with $n-2$ instead of n so that $E[\hat\sigma^2] = \sigma^2$

Diamond Example

```
y <- diamond$price
x <- diamond$carat
n <- length(y)

#Solving resid s.d. implicitly
sqrt(sum(resid(fit)^2) / (n - 2))
```

```
## [1] 31.84052
```

```r
#Getting resid deviation with functions
fit <- lm(y ~ x)
summary(fit)$sigma
```

```
## [1] 31.84052
```

```r
#You can see the value in the summary print out here:
summary(fit)
```

```
##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -85.159 -21.448  -0.869  18.972  79.370
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -259.63      17.32  -14.99   <2e-16 ***
## x            3721.02      81.79   45.50   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 31.84 on 46 degrees of freedom
## Multiple R-squared:  0.9783, Adjusted R-squared:  0.9778
## F-statistic:  2070 on 1 and 46 DF,  p-value: < 2.2e-16
```

Summarizing Variation

- **Total Variability** - the variability around an intercept (mean only regression) + $\sum_{i=1}^{n}(Y_i - \bar{Y})^2$
  + Sum of Regression & Error Variability

- **Regression Variability** - the variability that is explained by adding the predictor
  + $\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$

- **Error Variability** - what's leftover around the regression line
  + $\sum_{i=1}^{n}(Y_i - \hat{Y})^2$

R Squared, the Coefficent of Determination

- R squared is the percentage of the total variability that is explained by the linear relationship with the predictor
  $R^2 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$

- R^2 is the percentage of variation explained by the regression model

- $0 \leq R^2 \leq 1$

- R^2 is the sample correlation squared

- R^2 can bea misleading summary of model fit
  + Deleting data can inflate R^2
  + (For later,) Adding terms to a regression model always increases R^2

- Execute `example(anscombe)` to see the following data:
  + Basically same mean and variance of X and Y
  + Identical correlations (hence the same R^2 value)
  + Same linear regression relationship

Lesson with `swirl()`: Residual Variation

- `deviance` will calculate the sum of the squares of a `lm`

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Inference in Regression

Inference in Regression

Coding Example

Prediction

Lesson with `swirl()`: Introduction to Multivariable Regression

Lesson with `swirl()`: MultiVar Examples

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Quiz 2

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Multivariable Regression, Residuals, & Diagnostics

Multivariable Regression

Multivariable Regression Part 1

Multivariable Regression Part 2

Multivariable Regression Continued

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Multivariable Regression Tips and Tricks

Multivariable Regression Examples Part 1

Multivariable Regression Examples Part 2

Multivariable Regression Examples Part 3

Multivariable Regression Examples Part 4

Lesson with `swirl()`: MultiVar Examples2

Lesson with `swirl()`: MultiVar Examples3

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Adjustment

Adjustment Examples

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Residuals Again

Residuals and Diagnostics Part 1

Residuals and Diagnostics Part 2

Residuals and Diagnostics Part 3

Lesson with `swirl()`: Residuals Diagnostics and Variation

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Model Selection

Model Selection Part 1

Model Selection Part 2

Model Selection Part 3

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Practice Exercise in Regression Modeling

Quiz 3

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Logistic Regression and Poisson Regression

GLMs

Logistic Regression

Logistic Regression Part 1

Logistic Regression Part 2

Logistic Regression Part 3

Lesson with `swirl()`: Variance Inflation Factors

Lesson with `swirl()`: Overfitting and Underfitting

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Poisson Regression

Poisson Regression Part 1

Poisson Regression Part 2

Lesson with `swirl()`: Binary Outcomes

Lesson with `swirl()`: Count Outcomes

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Hodgepodge

Mishmash

Hodgepodge

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Quiz 4

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Course Project

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