Conservative Intuitionistic Epsilon Calculus based on Predicate Abstraction and S4 Embedding

It is well known that extending Intuitionistic Predicate logic (**IP**) with the $\varepsilon\tau$-terms of Epsilon Calculus ($\boldsymbol{\varepsilon}\mathbf{P}$) results in an extension which is non-conservative with respect to $\forall$ and $\exists$ intuitionistic quantifiers. Conservative fragments have been proposed in the literature which are based on different restrictions: linguistic (not all $\varepsilon\tau$-terms are accepted as well-formed terms); interpretational ($\varepsilon\tau$-terms may partially denote, and an existence predicate is added); inferential (the 'critical axiom' of $\boldsymbol{\varepsilon}\mathbf{P}$ is restricted).[1]

Despite not acknowledged in the literature, a different strategy is implicitly suggested in Fitting [2]. There, an extension of $\boldsymbol{\varepsilon}\mathbf{P}$ is developed including a predicate abstraction operator λ and **S4** modalities. By λ, the scope of $\varepsilon\tau$-terms naturally disambiguates in an extended $\varepsilon\tau$-translation of quantifiers, for x not occurring in B (Cf. [1], Table 3):

$$
\begin{aligned}
(B \to \exists x\, A) \to \exists x\,(B \to A) &\ \leadsto^{\varepsilon\tau}_{\lambda}\ (B \to [\lambda x\, A](\varepsilon x\, A)) \to [\lambda x\,(B \to A)](\varepsilon x\,(B \to A))\\
\forall x\, \neg\neg A \to \neg\neg \forall x\, A &\ \leadsto^{\varepsilon\tau}_{\lambda}\ [\lambda x\, \neg\neg A](\tau x\, \neg\neg A) \to \neg\neg[\lambda x\, A](\tau x\, A)
\end{aligned}
$$

By **S4** modalities, an extended translation $(\cdot)^{\Box}$ of Intuitionistic formulas extended by λ and $\varepsilon\tau$-terms can be provided:

$$
\begin{aligned}
([\lambda x A](\varepsilon x\, B))^{\Box} &:= [\lambda x (A)^{\Box}](\varepsilon x\,(B)^{\Box})\\
([\lambda x A](\tau x\, B))^{\Box} &:= \Box[\lambda x (A)^{\Box}](\tau x\,(B)^{\Box})
\end{aligned}
$$

In this presentation, I show that these translations induces indeed such a conservative extension of **I** with $\varepsilon\tau$-terms and λ. I prove this by adapting Yasuhara's [5, 6] cut-free complete sequent calculus for the (extensional version of) $\boldsymbol{\varepsilon}\mathbf{P}$ and his completeness proof by countermodel extraction from a failed proof-search. The first system $\boldsymbol{\varepsilon\lambda}\mathbf{S4}$ includes G3-style rules for **S4** and λ abstraction. The second system $\boldsymbol{\varepsilon\lambda}\mathbf{I}$ includes a G3-style multisuccedent rules for intuitionistic connectives and λ abstraction which internalize the $(\cdot)^{\Box}$ translation. Both system are shown sound and complete for their respective semantics by adapting Yasuhara's completeness proofs to Mints' [3] proof-search method for multisuccedent intuitionistic sequent calculi, and the extended $(\cdot)^{\Box}$ embedding of $\boldsymbol{\varepsilon\lambda}\mathbf{I}$ in $\boldsymbol{\varepsilon\lambda}\mathbf{S4}$ proven.

[1] Baaz, M. and Zach, R. (2022). "Epsilon theorems in intermediate logics". *The Journal of Symbolic Logic 87*(2), 682–720.

[2] Fitting, M. (1975). "A Modal Logic ε-Calculus". *Notre Dame Journal of Formal Logic 16*(1), 1–16.

[3] Mints, G. (2000). *A short introduction to intuitionistic logic*. Kluwer Academic Publishers, New York.

[4] Mints, G. (2015). "Intuitionistic Existential Instantiation and Epsilon Symbol". In Wansing, H. (Ed.), *Dag Prawitz on Proofs and Meaning*, Springer, Cham, 225–238.

[5] Yasuhara, M. (1982). "Cut Elimination in ε-Calculi". *Mathematical Logic Quarterly 28*(20-21), 311–316.

[6] Yasuhara, M. (1989). "An Addition to "Cut Elimination in ε-Calculi"". *Mathematical Logic Quarterly 35*(6), 483–484.

[7] Zach, R. (2017). "Semantics and Proof Theory of the Epsilon Calculus". In Ghosh, S., Prasad, S. (Eds.), *Indian Conference on Logic and Its Applications*, Springer, Berlin, 27–47.

[1] Conservativity involving $\varepsilon\tau$-terms are known as 'Epsilon Theorems'. See [7] for an introduction to Epsilon Calculus, and [1, 4] for references on conservative fragments of extensions of Intuitionistic and Intermediate logics with $\varepsilon\tau$-terms.