Quantum Walks and Quantum Resetting

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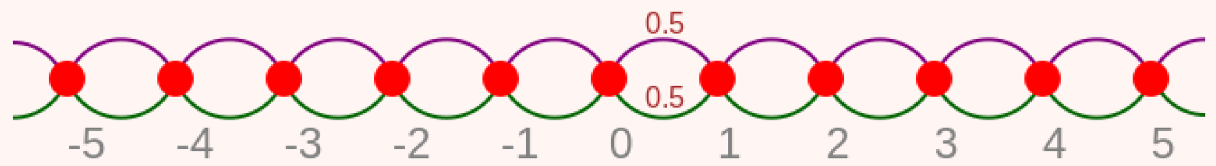
Introduction

Random walks are a commonly used tool in the arsenal of algorithms on Classical Computers to solve a variety of problems by sampling few solutions to come to a close solution. With the advent of quantum algorithms, quantum walks have arisen as an obvious extension of classical walks in the quantum domain.

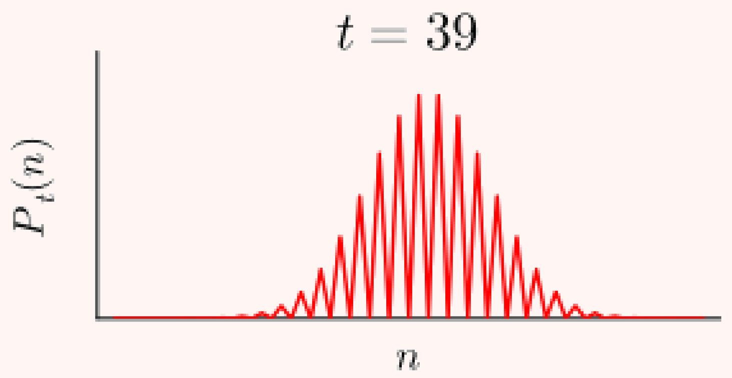
In search problems, we generally have a black box function $f(x) = \begin{cases} 0 & x \in G \\ 1 & x \in G^C \end{cases}$ where $G \cup G^C = S$ which is the search space. We are interested in developing an algorithm to output some $x \in G$ by querying f.

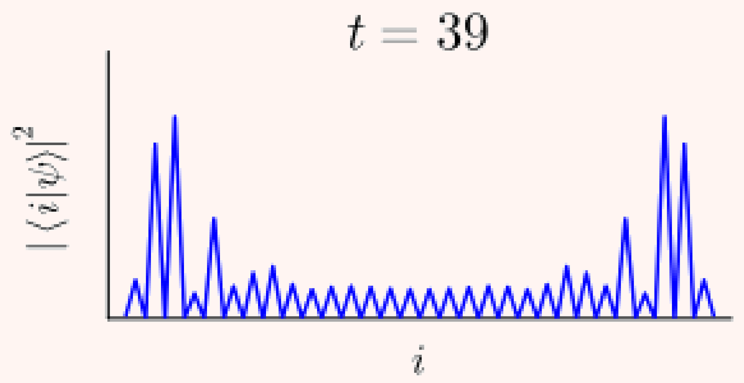
Naively, one can run a stochastic process over the domain $G \cup G^C$ and observe the system until we see an element in G. Naturally, we not only want high success rates, but we also want low mean hit times.

Walks on a 1D Chain



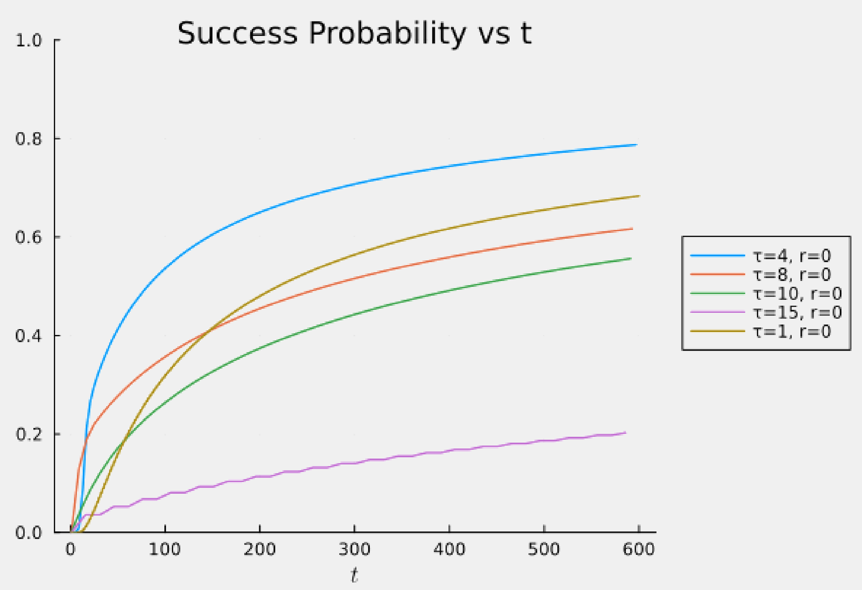
Classical	Quantum
$p_{ij} = \begin{cases} 1/2 & \lvert i-j\rvert = 1 \\ 0 & \text{otherwise} \end{cases}$ Irreducible Recurrent Evolution of probability $\implies$ No interference $\sigma \propto \sqrt{t}$	$S = \lvert 0\rangle\langle 0\rvert \otimes S_L + \lvert 1\rangle\langle 1\rvert \otimes S_R$ $S_L\lvert i\rangle = \lvert i-1\rangle,\ S_R\lvert i\rangle = \lvert i+1\rangle$ Irreducible Transient Evolution of probability amplitudes $\implies$ Possiblity of interference $\sigma \propto t$





Hitting a Particular Node

Mean time to hit a particular node δ. For the quantum case, we measure after τ steps, to prevent the **Quantum Zeno Effect** caused by continuous measurement. The $\tau = 1$ case reduces to the classical walk. Plotting the **Success Probability with Time**,



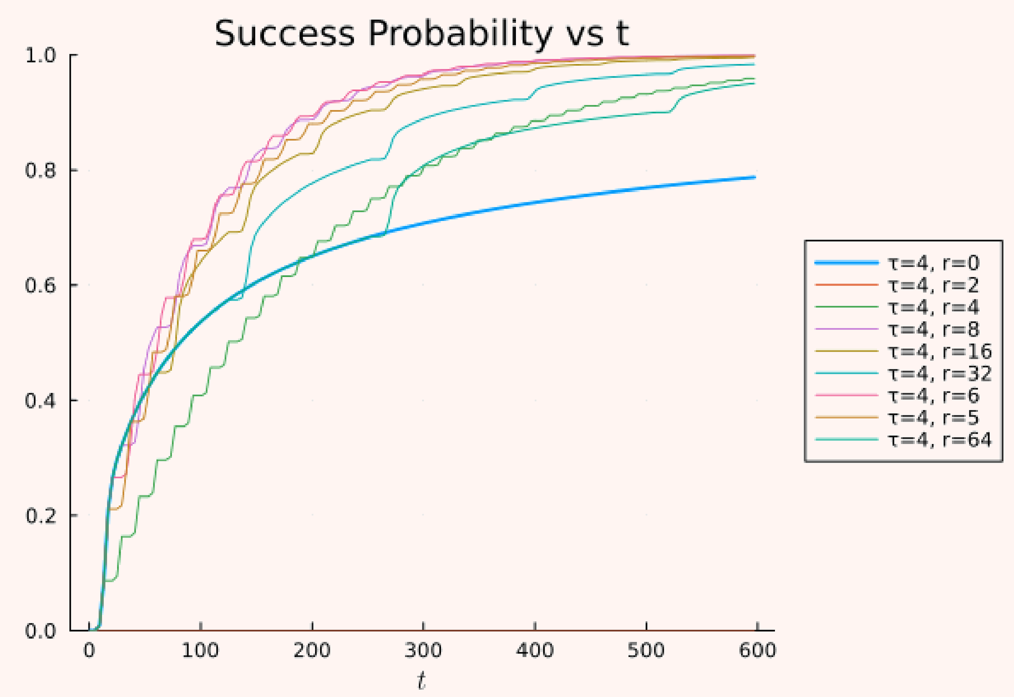
Classical	Quantum
Slow initial rise due to Gaussian spread	Fast initial rise due to ballistic spread
Asymptotic Success probability of 1 due to recurrence	Asymptotic Success probability < 1 due to transience

How can transient walks be made recurrent? How can we do that without losing the quantum advantage that we gained by superposition? Recent work [3], fixes this issue by restarting the walk.

Resetting of Walks

Can we **reset** the walk when the quantum walk starts saturating?

Classical	Quantum
$p_{ij} = \begin{cases} (1-\gamma)1/2 & \lvert i-j\rvert = 1 \\ \gamma & j = 0 \\ 0 & \text{otherwise} \end{cases}$	$O(\rho) = (1-\gamma)S(\rho) + \gamma\rho_0$
Ergodic Recurrent	Ergodic Recurrent



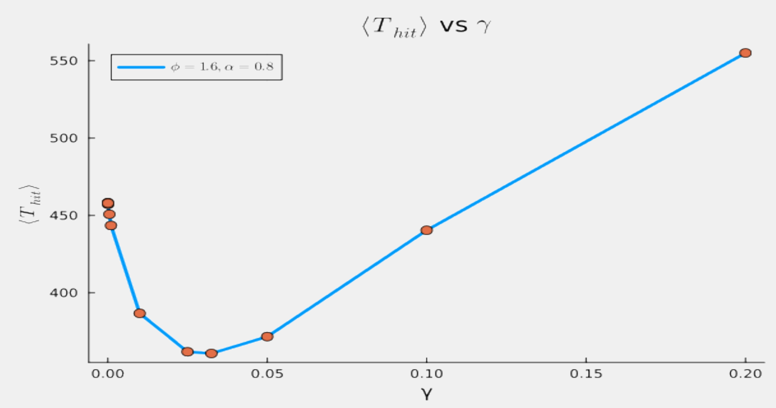
Thus by resetting, we see shorter hitting times, and an asymptotic success probability of 1.

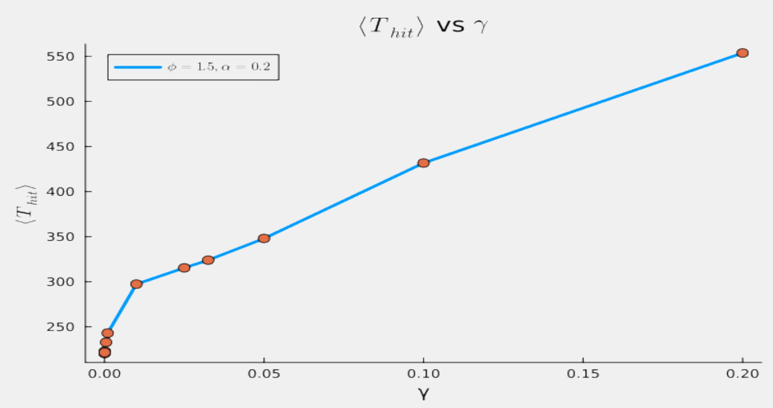
Quantum Resetting

Similar to how interference helped speed up the quantum walk, can we introduce interference between the evolve and the reset processes?

1. **Reset operation** - $\mathcal{R}$ by the Kraus operators $\{\mathcal{R}_{\rangle} = \lvert r_0\rangle\langle i\rvert\}_{i\in[-N,N]}$ on $\mathcal{H}_W$
2. **Evolve operation** - $\mathcal{U}$ by the unitary operation $S \circ H$ on $\mathcal{H}_C \otimes \mathcal{H}_W$
3. **Controlled reset** - $\mathcal{E}$ by the Kraus operators $\{\mathcal{E}_i = \lvert 0\rangle\langle 0\rvert \otimes I_2 \otimes \mathcal{R}_i + \frac{1}{\sqrt{N}}\lvert 1\rangle\langle 1\rvert \otimes \mathcal{U}\}_{i\in[-N,N]}$.
4. **Coin operator** - $\Gamma(\gamma) = \begin{bmatrix} \sqrt{1-\gamma} & \sqrt{\gamma} \\ \sqrt{\gamma} & -\sqrt{1-\gamma} \end{bmatrix}$

By plotting the mean hitting time for different initial state of the reset coin and γ, we see a non trivial dependence on both of these.





- Difficult to analyse
- Difficult to interpret
- Difficult to simulate

Main References

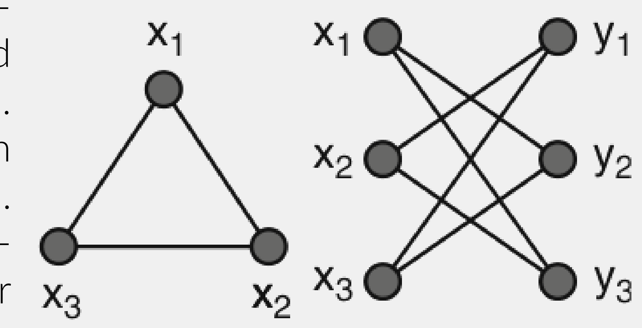
[1] Frédéric Magniez, Ashwin Nayak, Jérémie Roland, and Miklos Santha.
Search via Quantum Walk.
SIAM Journal on Computing, 40(1):142–164, January 2011.

[2] M. Szegedy.
Quantum Speed-Up of Markov Chain Based Algorithms.
In *45th Annual IEEE Symposium on Foundations of Computer Science*, pages 32–41, Rome, Italy, 2004. IEEE.

[3] Ruoyu Yin and Eli Barkai.
Restart expedites quantum walk hitting times, May 2022.
arXiv:2205.01974 [cond-mat, physics:quant-ph].

Unitary Reset, or Quantized Markov Chains

We want a unitary walk incorporating the resetting mechanism. Adirected chains don't naturally quantize via the coined walk formalism, thus we use the Szegedy walk formalism[2]. We start by duplicating nodes and drawing edges between nodes of different sets. For a state $\lvert\psi\rangle \in \mathcal{H}$, let $\Pi_\psi = \lvert\psi\rangle\langle\psi\rvert$. For a subspace $\mathcal{K}$ of $\mathcal{H}$ spanned by a set of mutually orthogonal states $\{\lvert\psi_i\rangle : i \in I\}$, let $\Pi_{\mathcal{K}} = \sum_{i\in I}\Pi_{\psi_i}$ be the projector onto $\mathcal{K}$ and $\mathcal{R}_{\mathcal{K}} = 2\Pi_{\mathcal{K}} - \mathrm{Id}$ be a reflection through $\mathcal{K}$.



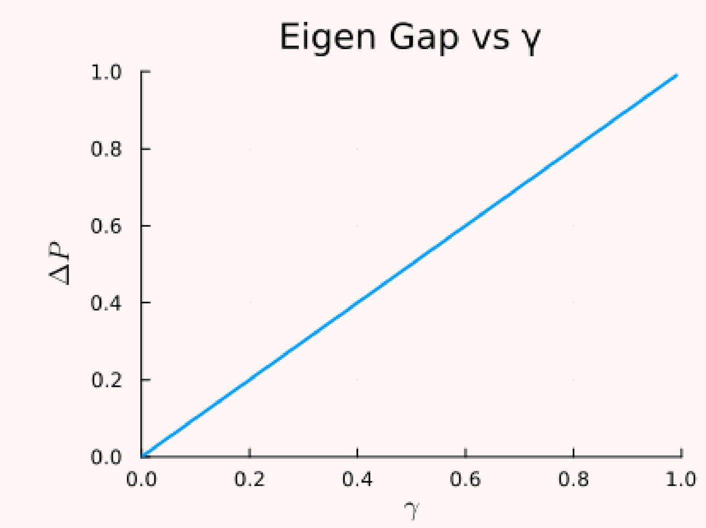
Let $\mathcal{A} = \mathrm{Span}(\lvert x\rangle\lvert p_x\rangle : x \in X)$ and $\mathcal{B} = \mathrm{Span}(\lvert p_y^*\rangle\lvert y\rangle : y \in Y)$ be subspaces of $\mathcal{H} = \mathbb{C}^{\lvert X\rvert\times\lvert X\rvert}$, where $\lvert p_x\rangle = \sum_{y\in X}\sqrt{p_{xy}}\lvert y\rangle$; $\lvert p_y^*\rangle = \sum_{x\in X}\sqrt{p_{yx}^*}\lvert x\rangle$ where p_{ij}, p_{ij}^* are elements of P, P^* respectively, and X is the set of nodes, Y is the set of nodes after duplication. Then, $W(P) = \mathcal{R}_{\mathcal{B}}\mathcal{R}_{\mathcal{A}}$ is the unitary of the quantum walk.

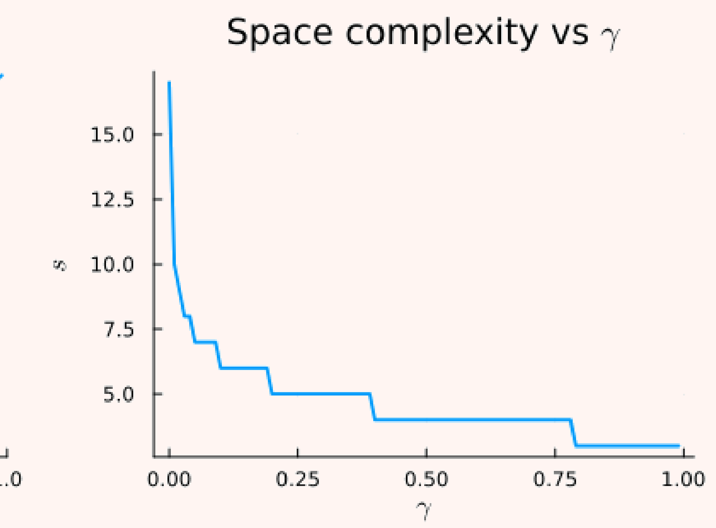
Szegedy Search

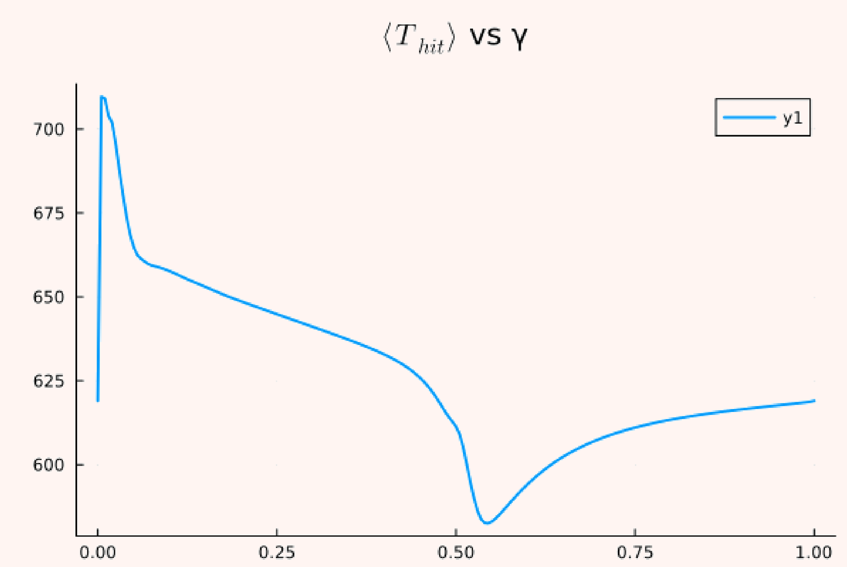
A search algorithm based on the Grover Search[1] showed that increasing the eigengap of the Markov chain corresponds to a speed up of the protocol and a reduction in the space required (s).

1. Prepare the initial state $\lvert\pi\rangle\lvert 0^T ks\rangle$, where $\lvert\pi\rangle_d = \sum_{x\in X}\sqrt{\pi_x}\lvert x\rangle\lvert p_x\rangle = \sum_{y\in X}\sqrt{\pi_x}\lvert x\rangle\lvert p_x\rangle$
2. Apply the Grover oracle $\mathcal{G}(\lvert x\rangle_d\lvert y\rangle_d\lvert z\rangle) = \begin{cases} -\lvert x\rangle_d\lvert y\rangle_d\lvert z\rangle, & \text{if } x \in G \\ +\lvert x\rangle_d\lvert y\rangle_d\lvert z\rangle, & \text{otherwise} \end{cases}$
3. Apply a phase estimation circuit to the quantum walk, repeated k times.
4. Repeat steps 2 and 3 T times.

We show that increasing γ leads to a corresponding increase in the eigengap ΔP.







Summary, Future Work and Acknowledgments

Thus, while our initial attempt at a Quantum Reset protocol does not show preliminary speed ups, our second attempt via Szegedy Walks shows a clear speedup for certain parameter ranges. The second advantage of our formalism is that it can be trivially extended to any other graph structure.

Thus the obvious next step would be to analyze the protocol for families of graphs. Another avenue of interest is applying the stochastic resetting protocol to the unitary reset walk, thus unifying both speedups.

This work would hardly be possible without the help and guidance of my teachers, friends and family. This space is hardly enough to contain my gratitude for Manab Da, Arnab Da, Gowri, Aabhas, Aarya, Anubhav, Neeraja, the Mods, friends in the QIQP and PaleoArcheo Labs, and my parents, who've supported me in more ways than one in this work.