Preferential Attachment Trees with Fitness

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Abstract

Preferential Attachment trees are a class of Random Graphs which were introduced by Barabàasi and Albert in 1999. Due to their scale free nature and small world property, they find wide applications in modelling real-life networks such as the citation networks of scientific papers, the World Wide web, collaboration networks. In this thesis, we study the degree distribution of preferential Attachment trees with additive and multiplicative fitness by using some well known results for Pòlya urns. We obtain analytical expressions for the first two moments of the degree of a fixed vertex. Further, we look at two special types of subtree structures and calculate their expected number in the tree at each time step using recurrence methods. Finally, we again appeal to Pòya urns to show that the proportion of these subtree structures in the entire tree converges to an almost sure limit.

Introduction to the Model

Preferential attachment trees are an example of growing graphs that are suitable to model real-life networks because of their small-world property and scale-free nature. Preferential attachment means that the vertices that have a higher degree (in other words, are more connected) are more attractive to the newcomers at each time step and because of this attachment law, sometimes this model is also referred to as 'rich grows richer' or 'success breeds success' model. This law gives rise to small-world property, which means that the distance between any two vertices in the tree is small and scale-free property which means that although most of the vertices have a small degree, few vertices with very high degrees can also be found (also referred to as hubs). Our model for Preferential Attachment trees with constant additive and multiplicative fitness is as follows.
At time $t = 0$, there is a root vertex labelled 0. At each subsequent time step, a vertex is added to the graph which attaches to an existing vertex with one edge. Given the tree at time t, $\mathcal{G}_t$, the probabilty that the incoming vertex $t+1$ will attach to an existing vertex v is given by

$$P(t+1 \sim v|\mathcal{G}_t) = \frac{ad_t(v)+b}{at+b(t+1)}$$

where the degree of vertex v at time t, $d_t(v)$, is the number of offsprings of v in $\mathcal{G}_t$.
For a fixed vertex v,

$$\mathbb{E}\left[d_t(v)\right] = \frac{b}{a}\left(\frac{\Gamma(t+1)\Gamma\left(v+\frac{b}{a+b}\right)}{\Gamma(v+1)\Gamma\left(t+\frac{b}{a+b}\right)} - 1\right)$$

and

$$Var\left(d_t(v)\right) = \left(\frac{b}{a}\frac{\Gamma\left(v+\frac{b}{a+b}\right)}{\Gamma\left(t+\frac{b}{a+b}\right)}\frac{\Gamma(t+1)}{\Gamma(v+1)}\right)^2 - \frac{b}{a}\frac{\Gamma(t+1)\Gamma\left(v+\frac{b}{a}+b\right)}{\Gamma(v+1)\Gamma\left(v+\frac{b}{a+b}\right)} + \frac{a(a+b)}{a^2}\frac{\Gamma\left(t+1+\frac{a}{a+b}\right)}{\Gamma\left(v+1+\frac{a}{a+b}\right)}\frac{\Gamma\left(v+\frac{b}{a+b}\right)}{\Gamma\left(t+\frac{b}{a+b}\right)}$$

Expected number of Subtree Structures

1. For $v \geq 0, n \geq 1$, let $\mathcal{L}_t^{(n)}(v)$ be the number of n-branches attached to a vertex v at time t. Then,

$$\mathbb{E}\left[\mathcal{L}_t^{(n)}(v)\right] = \begin{cases} \prod_{k=1}^{n}\left(\frac{b}{a(t-k)+b(t-k+1)}\right)\frac{\Gamma(t-n+1)\Gamma(t-v+1)\Gamma(v+\frac{b}{a+b})}{\Gamma(n+1)\Gamma(t-n+\frac{b}{a+b})\Gamma(t-v-n+1)\Gamma(v+1)} & \text{for } t \geq v+n \\ 0 & \text{otherwise.} \end{cases}$$

2. For $v \geq 0, n \geq 2$, let $\mathcal{C}_t^{(n)}(v)$ be the number of n-bunches attached to a vertex v at time t. Then,

$$\mathbb{E}\left[\mathcal{C}_t^{(n)}(v)\right] = \begin{cases} \prod_{k=1}^{n-1}\left(\frac{(k-1)a+b}{a(t-k)+b(t-k+1)}\right)\left(\frac{b}{a+b}\right)\frac{\Gamma(t-n+1)\Gamma(t-v+1)\Gamma(v+\frac{b}{a+b})}{\Gamma(n+1)\Gamma(t-n+\frac{b}{a+b}+1)\Gamma(t-v-n+1)\Gamma(v+1)} & \text{for } t \geq v+n \\ 0 & \text{otherwise.} \end{cases}$$

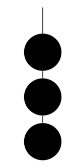
3. For $n \geq 1$, let $\mathcal{L}_t^{(n)}$ be the number of n-branches in the tree at time t. Then,

$$\mathbb{E}\left[\mathcal{L}_t^{(n)}\right] = \begin{cases} \frac{b^{n-1}}{\prod_{k=1}^{n} ak+b(k+1)}(at+b(t+1)) & \text{for } t \geq n \\ 0 & \text{otherwise.} \end{cases}$$

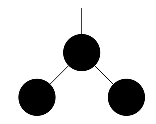
4. For $n \geq 1$, For $n \geq 1$, let $\mathcal{C}_t^{(n)}$ be the number of n-bunches in the tree at time t. Then,

$$\mathbb{E}\left[\mathcal{C}_t^{(n)}\right] = \begin{cases} \frac{\prod_{j=0}^{n-2}(ja+b)}{\prod_{k=1}^{n} ak+b(k+1)}(at+b(t+1)) & \text{for } t \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

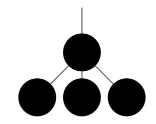
Asymptotic convergence results



$$\frac{\mathcal{L}_t^{(3)}}{t} \xrightarrow{a \cdot s} \frac{b(a+b)^2}{(a+2b)(2a+3b)(3a+4b)}$$



$$\frac{\mathbb{C}_t^{(3)}}{t} \xrightarrow{a.s.} \frac{b^2(a+b)}{(a+2b)(2a+3b)(3a+4b)}$$



$$\frac{\mathbb{C}_t^{(4)}}{t} \xrightarrow{a \cdot s_0} \frac{b(a+b)^2(2a+b)}{(a+2b)(2a+3b)(3a+4b)(4a+5b)}$$

Future Plans

1. To extend the model for deterministic and random fitnesses.
2. To look at more types of subtrees.

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