Holographic Entanglement Entropy at Spatial Infinity

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Abstract

We study the calculation of Entanglement entropy in AdS/CFT using RT surfaces and introduce a natural analogue of a "spherical subregion" for the spatial infinity (spi) of asymptotically flat spacetime. The holographic data of such a spi-subregion can be encoded via the asymptotic behavior of extremal surfaces on a bulk spatial slice or a pair of points on $\mathfrak{I}^+$ and $\mathfrak{I}^-$. We define a holographic entropy associated with a spi-subregion as the area of an extremal Ryu-Takayanagi surface in the bulk, which is the waist of an Asymptotic Causal Diamond (ACD).

Holographic Principle

The holographic principle was inspired by black hole entropy equation, $S = {A_{bh}}/{4G_N}$ where, A_{BH} is the surface area of the black hole. The holographic principle suggests that a D-dimensional theory of quantum gravity can be equally represented as a (D-1) dimensional QFT written on the boundary of the space.

Statement of AdS/CFT Correspondence

Any conformal field theory on $\mathbb{R} \times \mathbb{S}^{d-1}$ is equivalent to a theory of quantum gravity in asymptotically $AdS_{d+1} \times M$ spacetime where M is some compact manifold.

Geometry of Anti-de Sitter space

AdS is a type of space-time geometry with constant negative curvature. It can be spanned by different coordinate systems. The coordinates systems are,

Global coordinates

In these coordinates the metric is given by eq (1)

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{\left(1+\frac{r^2}{L^2}\right)} + r^2 d\Omega_{d-1}^2 \tag{1}$$

Poincare Coordinates

In these coordinates the metric is given by eq (2)

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \left(d\vec{x}^2 - dx_0^2\right)\right) \tag{2}$$

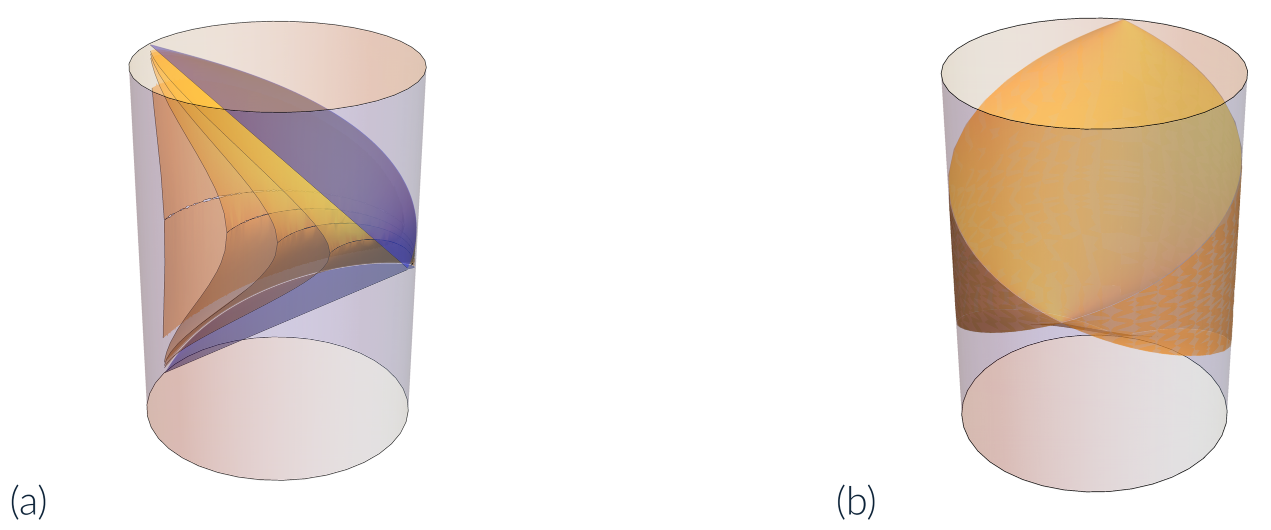

Figure 1. Poincare coordinates embedded in compactified global coordinate.In fig (a) the yellow surfaces are constant z surfaces and the yellow region in fig (b) is the space spanned by poincare coordinates

Ryu Takayanagi Surfaces

The RT prescription provides us a way to calculate the entanglement of sub regions on the boundary CFT. The prescription is as follows, we start with a boundary sub region (say A) on a constant time slice and then draw a co-dimension 2 surface (say γ_A) anchored to the boundary of the sub-region such that,

- γ_A has the same boundary as A.
- γ_A is homologous to A.
- γ_A extremizes the area. If there are multiple extremal surfaces, γ_A is the one with the least area.

Then the area of this extremal surface is proportional to the entropy of the sub region.

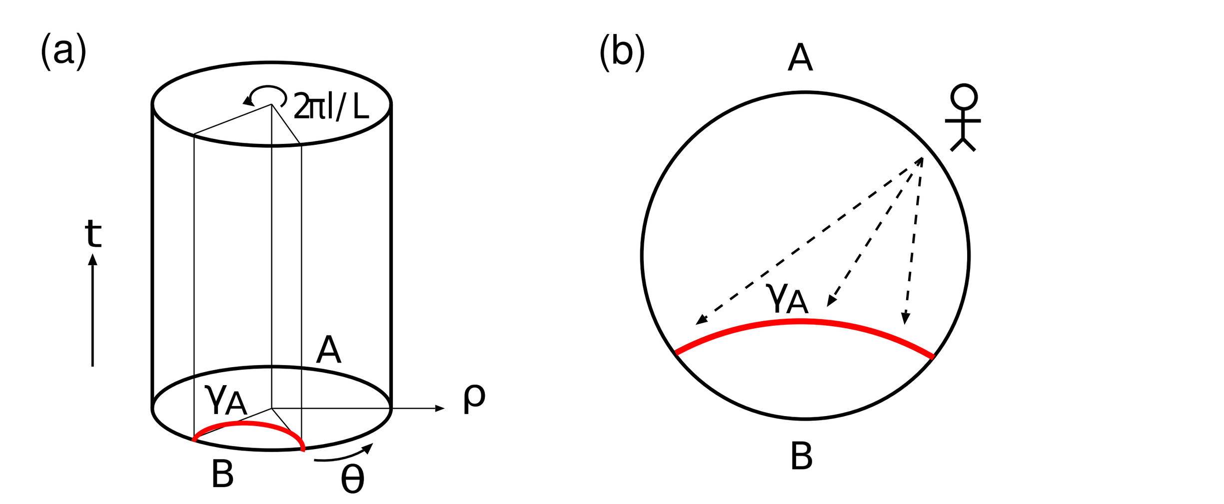

Figure 2. (a) AdS_3 space and CFT_2 living on its boundary and (b) a geodesics γ_A as a holographic screen

For empty AdS the entropy equation we get is,

$$\boxed{S_A = \frac{\text{Area of } \gamma_A}{4G} = \frac{1}{4G}\log\left(\frac{L}{\pi a}\sinh\left(\frac{\pi l}{L}\right)\right)} \tag{3}$$

where, L is the AdS-length scale, a is the UV-parameter and l is the size of the boundary sub-region.

Bulk with black holes

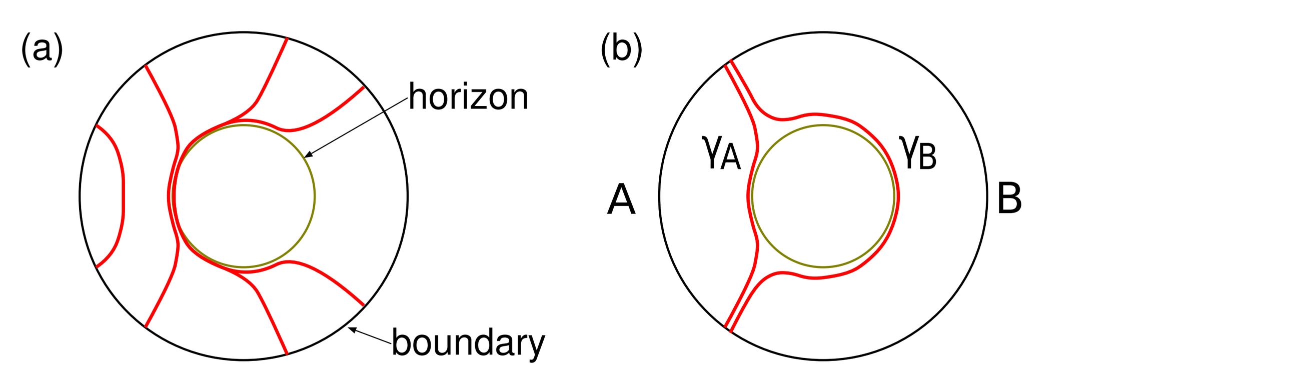

Figure 3. (a) Minimal surfaces γ_A for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon.

Flat Space Holography

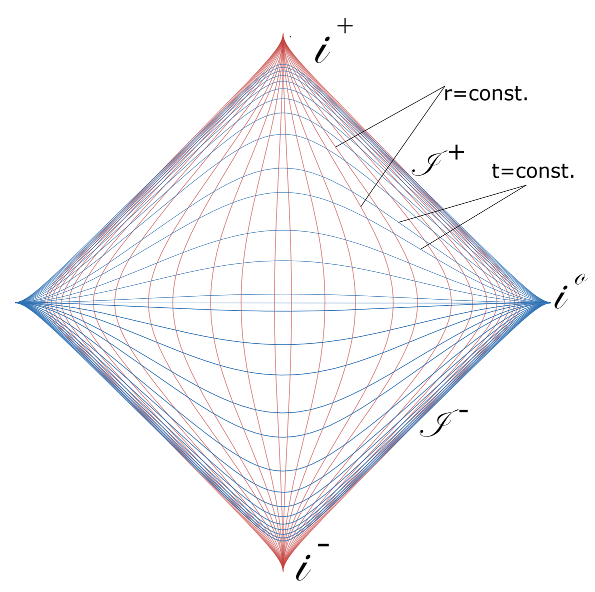

Figure 4. Penrose diagram

Flat space also has a compactified representation with the metric,

$$ds^2 = \Omega^2(-dt'^2 + dr'^2 + r'^2 d\phi^2) \tag{4}$$

with the transformations,

$$t + r = \tan\left(\frac{t'+r'}{2}\right) \qquad t - r = \tan\left(\frac{t'-r'}{2}\right) \tag{5}$$

Its difficult to construct a holographic theory for flat geometry as the boundary of flat space has a double null metric. Therefore, the notion of boundary sun-region isn't immediately obvious.

Asymptotic Causal Diamond

Def: An *Asymptotic Causal Diamond*, $\mathfrak{C}(p,q)$, is defined as the intersection of the past light cone of a point p at future null infinity $\mathcal{I}^+$ and the future light cone of a point q at past null infinity $\mathcal{I}^-$. That is

$$\mathfrak{C}(p,q) = \mathfrak{I}^-(p) \cap \mathfrak{I}^+(q), \text{ where } p \in \mathcal{I}^+,\ q \in \mathcal{I}^-. \tag{6}$$

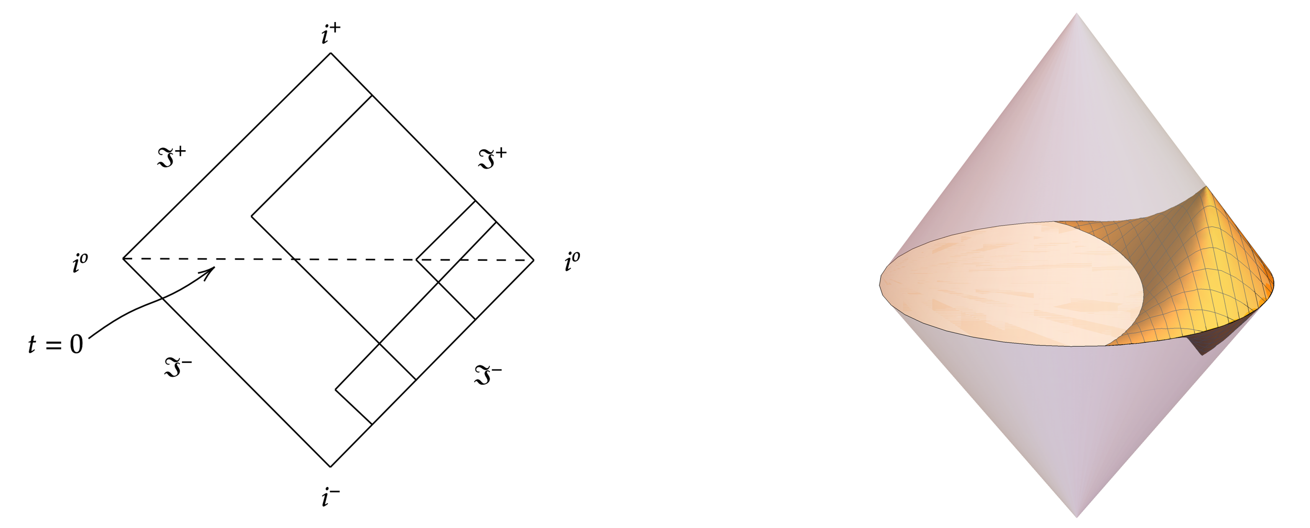

Figure 5. Illustration of a symmetric ACD inside a penrose diagram in $2+1$ dimensions.

The waist of the ACD is given by,

$$r\cos(\phi - \phi_0) = -\tan q_0 \tag{7}$$

In conformal coordinates the eq (7) becomes eq(8)

$$\tan(r'/2)\cos(\theta - \phi) = -\tan q_0 \tag{8}$$

The entropy is then given as the area of the waist,

$$A = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}\left(\Lambda^2 - \tan^2 q_0\right)^{(d-1)/2} \tag{9}$$

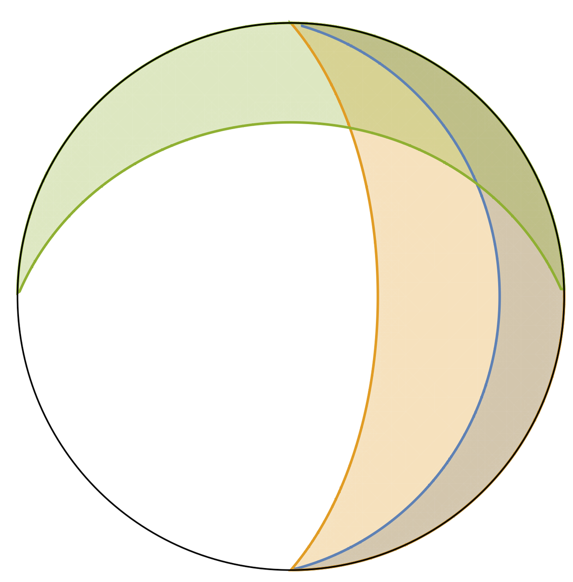

Figure 6. Plot of the waist ($t = 0$ slice) for different positions of the vertex

Bulk with black hole

The action functional for a co-dimension 2 surface in a schwarzchild metric,

$$A = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}\int r^{d-2}\sin^{d-2}(\theta)\sqrt{\frac{r'^2}{1-\left(\frac{r_s}{r}\right)^{d-2}} + r^2}\, d\theta \tag{10}$$

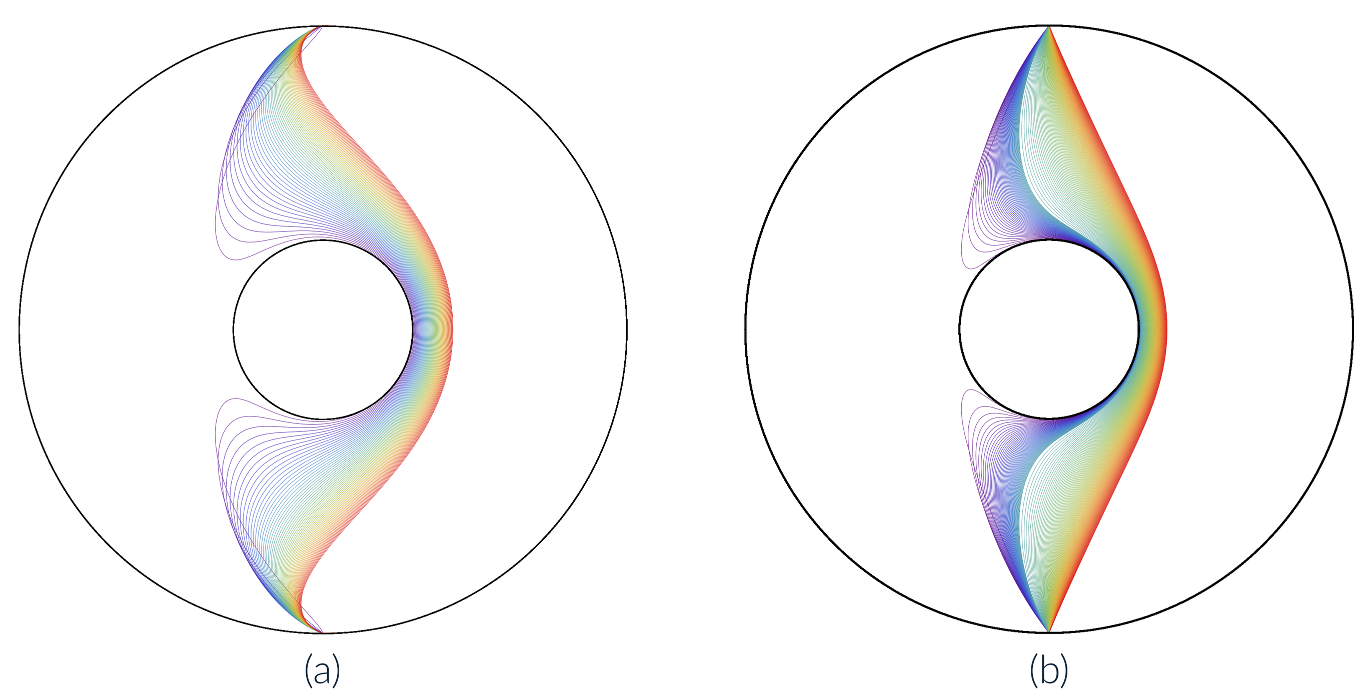

Figure 7. Extremal surfaces in (7a) 3+1 and (7b) 4+1 dimensions with a blackhole in the bulk

Conclusion and discussion

- In flat space the analogue of boundary data (sub-region) **seems** to be the angle at which the waist approaches the boundary
- The behaviour of extremal surface is similar in flat and AdS case. It suggests that the area could be associated to "boundary sub-region" defined by the asymptotic angle.
- Regulating the divergences in flat space case isn't as simple as in AdS case and is still work in progress.

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