Optimizing Bell CHSH Violations in Continuous Systems

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Abstract

Bell violations in a continuous variable (CV) system were achieved in an experiment as late as 2018. In this thesis we explore the optimal strategy for state preparation and measurement to maximize the Bell CHSH violation achieved, for Gaussian states. The two constraints are that we use homodyne measurements, and the two parties can only perform linear optical operations on their ends. We give a scheme to come up with the optimal Bell operator, and to calculate the expectation value for any state. An important parameter in continuous variable systems is the squeezing strength and we notice that as squeezing strength decreases, systems tend to show stronger Bell violations.

Introduction

Given a bipartite quantum system with subsystems A and B, and given pairs of operators A_1, A_2 and B_1, B_2 with eigenvalues ± 1 acting on subsystems A and B respectively, the Bell CHSH inequality is:

$$\left\langle \hat{A}_1 \otimes \hat{B}_1 + \hat{A}_2 \otimes \hat{B}_1 + \hat{A}_1 \otimes \hat{B}_2 - \hat{A}_2 \otimes \hat{B}_2 \right\rangle \leq 2.$$

An n-mode CV state is the composition of n Fock spaces. A **squeezed state** is created by "squeezing" the vacuum:

$$|\psi_r\rangle = \exp\left[\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]|0\rangle .$$

We would like to use such states to create entangled states which show Bell violations, using homodyne measurements, which are measurements of "position" and "momentum". A **Gaussian Unitary** $\hat{U}_G$: exponentiation of a Quadratic Hamiltonian $\hat{U}_G = \exp\left\{-i\hat{H}\right\}$.

A Gaussian state is specified completely by its $2n \times 2n$ **covariance matrix** σ, with elements

$$\sigma_{ij} = \langle \{\hat{r}_i, \hat{r}_j\} \rangle .$$

Here $\hat{r}_i \in (\hat{q}_1, \hat{q}_1, ..., \hat{p}_1, \hat{p}_2, ...)$. When a Gaussian state with covariance matrix σ is subjected to a Gaussian Unitary $\hat{U}_G$, it undergoes a linear transformation:

$$\sigma \longrightarrow S\sigma S^T,$$

Where S is a symplectic matrix specified by $\hat{U}_G$.

Equivalence Between Qubits and CV systems

A single qubit density matrix can be written in terms of Pauli matrices as:

$$\hat{\rho} = \sum_{\mu=0}^{3} S_\mu \sigma_\mu,$$

with $S\mu = \langle \sigma_\mu \rangle$. A two qubit density matrix may be written as:

$$\hat{\rho} = \frac{1}{4}\sum_{\mu,\nu=0}^{3} S_{\mu\nu}\sigma_\mu \otimes \sigma_\nu,$$

where $S_{\mu\nu} = \langle \sigma_\mu \otimes \sigma_\nu \rangle$.

Encoding a single qubit in a two mode state: Define a set of two mode operators, called the Schwinger representation of the Pauli matrices [1]:

$$\boldsymbol{S} = \{\hat{h}^\dagger \hat{h} + \hat{v}^\dagger \hat{v},\ \hat{h}^\dagger \hat{v} + \hat{v}^\dagger \hat{h},\ i(\hat{v}^\dagger \hat{h} - \hat{h}^\dagger \hat{v}),\ \hat{h}^\dagger \hat{h} - \hat{v}^\dagger \hat{v}\}.$$

To encode a qubit in a two mode CV state, simply adjust the CV state so as to match the condition

$$\left\langle \hat{S}_\mu \right\rangle = S_\mu.$$

Similarly, a four mode state is equivalent to a two qubit state when

$$\left\langle \hat{S}_\mu \otimes \hat{S}_\nu \right\rangle = S_{\mu\nu}.$$

To Optimize the Bell Violation

For a CV state $\hat{\sigma}$, find the two qubit density matrix $\hat{\rho}$, then find the optimal Bell operator for $\hat{\rho}$:

$$\hat{B}_{opt} = \hat{n}_1 \cdot \vec{\sigma} \otimes (\hat{m}_1 + \hat{m}_2) \cdot \vec{\sigma} + \hat{n}_2 \cdot \vec{\sigma} \otimes (\hat{m}_1 - \hat{m}_2) \cdot \vec{\sigma}.$$

Each of the terms in this operator needs to be measured individually. **The measurement outcomes will be the same for the CV state and its corresponding two qubit state**, since the Stokes' parameters are the same by construction.

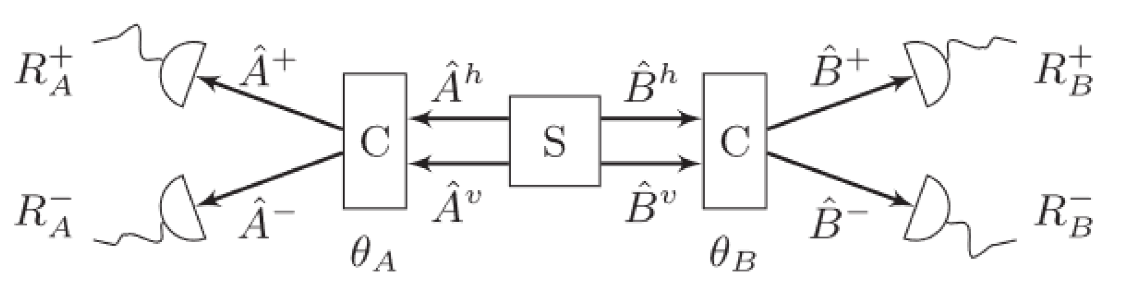

Figure 1. Schematic of a generic Bell experiment.

C are operations which depend on the operator $\hat{n}_i \cdot \vec{\sigma} \otimes \hat{m}_j \cdot \vec{\sigma}$ which is being measured [2].

To measure $\hat{n} \cdot \vec{\sigma} \otimes \hat{m} \cdot \vec{\sigma}$: Over the course of several measurements, let $R_{ij}(\hat{n}, \hat{m})$ be the number of counts recorded by detector i on Alice's end and detector j on Bob's end. Then, the expectation value of the operator is:

$$E(\hat{n}, \hat{m}) = \frac{R_{++}(\hat{n}, \hat{m}) - R_{+-}(\hat{n}, \hat{m}) - R_{-+}(\hat{n}, \hat{m}) + R_{--}(\hat{n}, \hat{m})}{R_{++}(\hat{n}, \hat{m}) + R_{+-}(\hat{n}, \hat{m}) + R_{-+}(\hat{n}, \hat{m}) + R_{--}(\hat{n}, \hat{m})} \quad (1)$$

Example: A Bell State

If Fig. 2, let $\hat{\sigma}_r$ be the state of the CV system with initial squeezing r, and $|\psi_r\rangle$ be the corresponding qubit state. We observe that

$$\lim_{r \to 0} |\psi_r\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

This is because the full wave function of the CV state,

$$|\tilde{\psi}_r\rangle = \sum_{n=0}^{\infty} c_n \hat{U}_{BS}|n\rangle|0\rangle|0\rangle|n\rangle$$

under the limit $r \longrightarrow 0$, becomes

$$|\tilde{\psi}_r\rangle = \alpha|0\rangle + \beta|\phi\rangle + \gamma|\xi\rangle ,$$

where $|0\rangle$ does not contribute to the Bell value, $|\phi\rangle$ is a superposition of maximally entangled states, and $\gamma \longrightarrow 0$ in the limit of zero squeezing.

How to calculate the Stokes' parameters?

- **For qubits:** To measure $\hat{n} \cdot \vec{\sigma} \otimes \hat{m} \cdot \vec{\sigma}$, we need to project the state onto the eigenvectors of the operator. This can be done by transforming the state with a local unitary $\hat{U}(\hat{n}) \otimes \hat{U}(\hat{m})$ so that the canonical basis is transformed into the set of eigenvectors of the operator, then projecting onto the canonical basis.
- For every $n \times n$ unitary

$$U = X - iY,$$

there is a unique $2n \times 2n$ orthogonal symplectic matrix

$$S = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

- The local transformation $\hat{U}_1 \otimes \hat{U}_2$ would correspond to $S_1 \oplus S_2$, which is applied to the covariance matrix σ.
- Finally, we use Eqn. 1, except

$$R_{ij}(\hat{n}, \hat{m}) = \left\langle \hat{a}_i^\dagger \hat{a}_i \otimes \hat{b}_j^\dagger \hat{b}_j \right\rangle,$$

where $\hat{a}_{1,2} = \hat{h}, \hat{v}$ and $\hat{b}_{1,2} = \hat{h}', \hat{v}'$, which are the field operators for Alice and Bob. This can be evaluated with the covariance matrix.

$$R_{ij}(\hat{n}, \hat{m}) = \left\langle \hat{a}_i^\dagger \hat{a}_i \right\rangle \left\langle \hat{b}_j^\dagger \hat{b}_j \right\rangle + \left\langle \hat{a}_i^\dagger \hat{b}_j^\dagger \right\rangle \left\langle \hat{a}_i \hat{b}_j \right\rangle + \left\langle \hat{a}_i^\dagger \hat{b}_j \right\rangle \left\langle \hat{a}_i \hat{b}_j^\dagger \right\rangle$$

An example setup

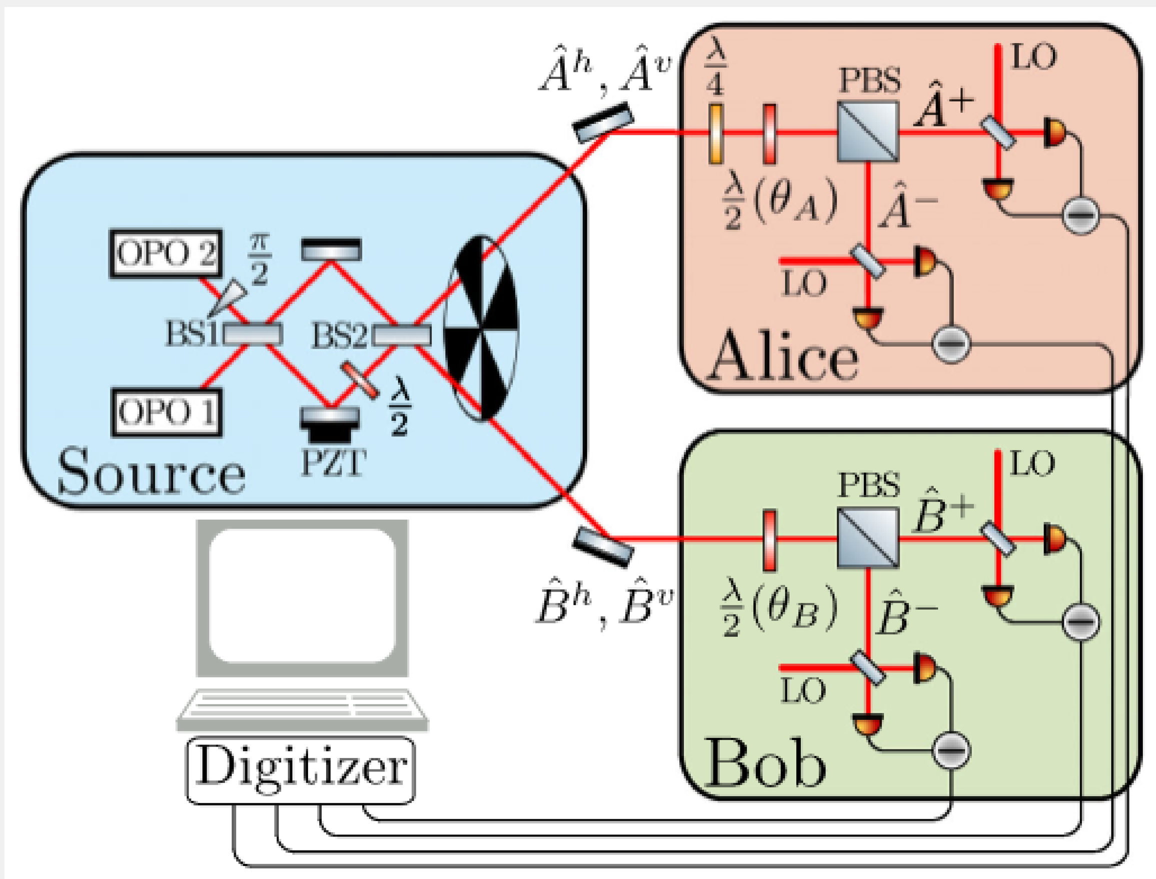

Figure 2. The setup used in the experiment by Thearle et al. [3] to achieve Bell violations. Two orthogonal squeezed states in horizontal polarization are mixed on a beam splitter. One arm of the state is rotated into the vertical polarization, and the state is mixed on a beam splitter again and sent to two parties Alice and Bob. Alice and Bob perform unitaries on their states depending on the observable they are measuring. The resultant modes are homodyned.

The Laboratory

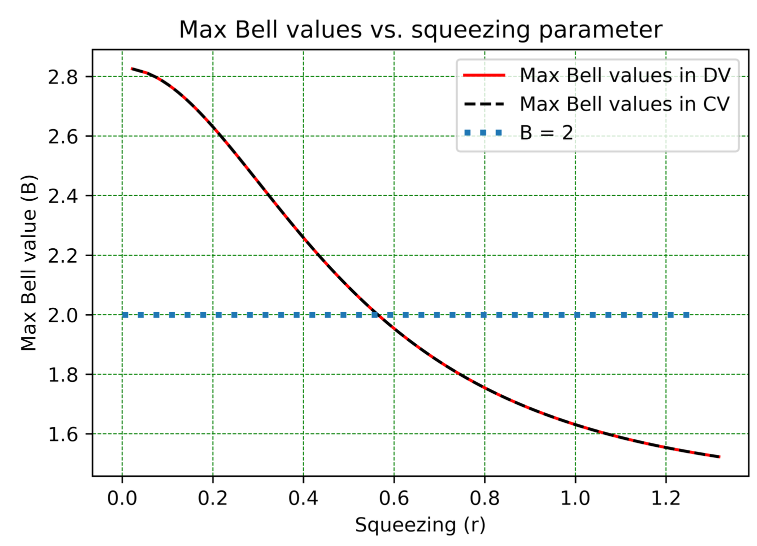

Figure 3. The optimized Bell values as a function of the squeezing strength for the setup in Fig. 2. As the squeezing strength decreases, the Bell violations become stronger.

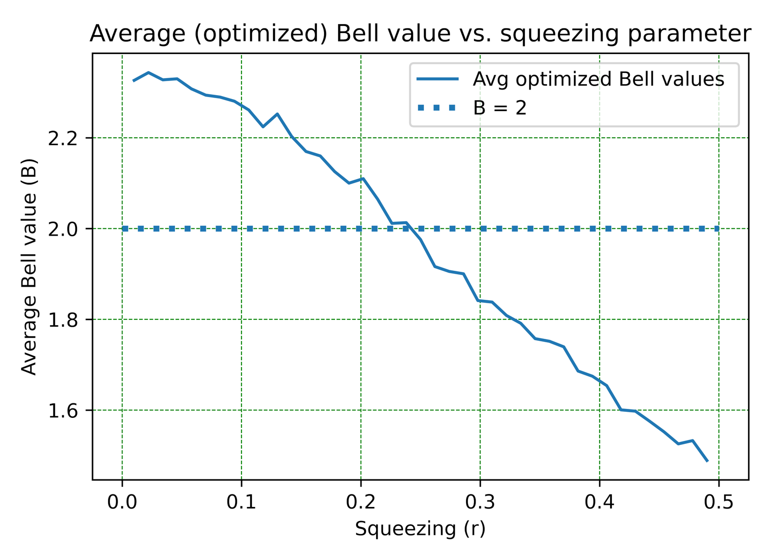

Figure 4. Here, for each value of initial squeezing, we apply 500 random orthogonal symplectic transformations (which represent linear optical devices like beam splitters, phase shifters, half wave plates and quarter wave plates), find the Bell values and take the average of all of them. The result is that the trend displayed by the system in Fig. 2 is not unique to that system, but it is a general trend of such systems, i.e. Bell violations become stronger with low squeezing.

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