Modular Forms and its Applications

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Introduction

We provide a basic introduction to the reader on the theory of classical modular forms. Initially, we will give the required definitions and theorems; then, we will move on to the computation of elliptic points and cusps in a Modular curve, followed by computing dimensions of the space of modular forms as well as its subspaces. Then we will see what basis elements of the corresponding subspaces are, restricting only to Eigenspaces. Further, we will look into a few applications of Modular forms in Number theory.

Overview and Basic Ideas

We begin with the idea of a **Modular Group** which is the group of 2 by 2 matrices with integer entries and determinant 1. It is denoted by $SL_2(\mathbb{Z})$, and a Congruence subgroup of $SL_2(\mathbb{Z})$ is a finite index subgroup containing $\Gamma(N)$.For example $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$.

What is Modular Form?

A function f: $\mathcal{H} \longrightarrow \mathbb{C}$ is a modular form of weight k, with respect to Γ if it is holomorphic on $\mathcal{H}$, is weight-k invariant with respect to Γ and $f[\alpha]_k$ is holomorphic at ∞ for any $\alpha \in SL_2(\mathbb{Z})$. For example,the Zero function, Eisenstein Series $G_k(\tau)$, etc. A cusp form is a modular form vanishing at the cusps. For example the Discriminant function $\Delta(\tau)$

Complex Torus: A complex torus is a quotient of the complex plane by a lattice $(\Lambda = w_1\mathbb{Z} + w_2\mathbb{Z})$ which is of the form $\mathbb{C}/\Lambda = \{z + \Lambda : z \in \mathbb{C}\}$

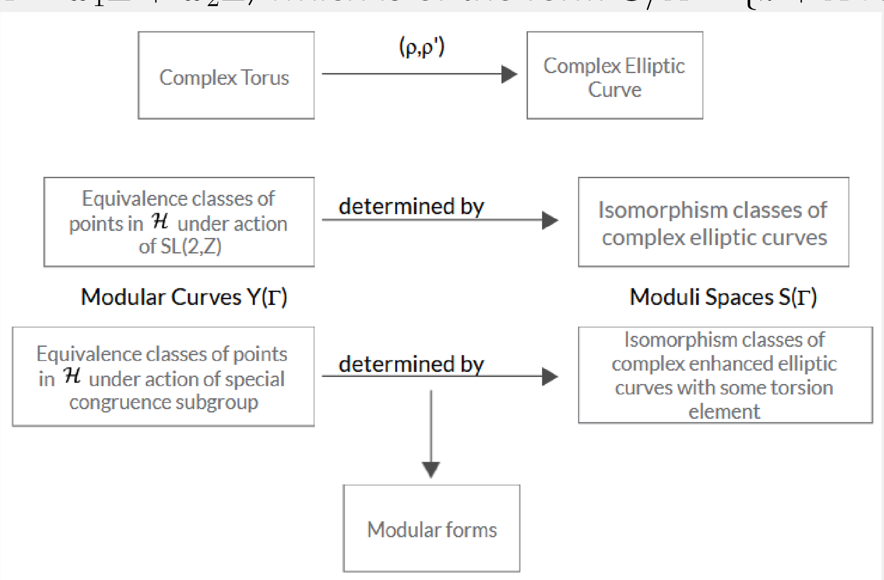

Figure 1. Relating Modular Forms with Complex Elliptic Curves via Complex Torus

Description of Modular Curve

The modular curve $Y(\Gamma) = \Gamma\backslash\mathcal{H}$ has a Riemann surface structure. In order to give a better model to visualize the modular curve, we define a Fundamental domain of $SL_2(Z)$.

Fundamental Domain: An open subset $R_G \in \mathcal{H}$ is called a fundamental domain of G if no two points of R_G are equivalent under G and If $\tau \in \mathcal{H}$, there is a point τ' in the closure of R_G, so that they are equivalent under G.

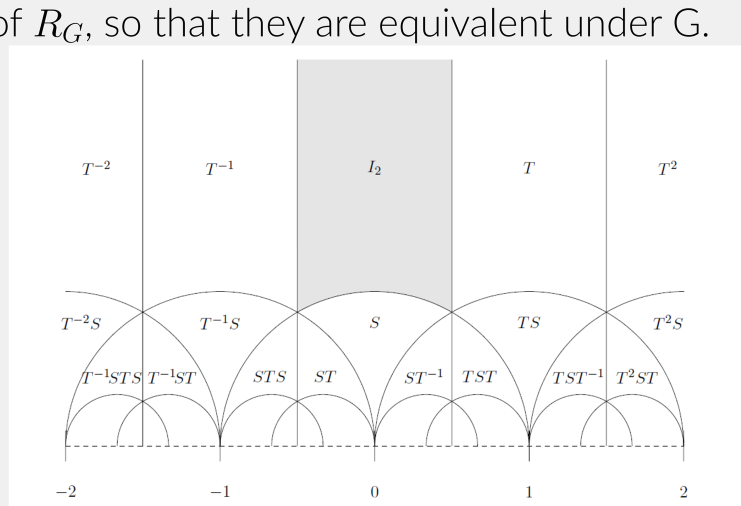

Figure 2. **The Fundamental Domain(grey) $\mathcal{H}$ of $SL_2(\mathbb{Z})$ along with its translates**

$Y(\Gamma)$ has some special points known as **elliptic points**, which are points having non-trivial stabilizer with respect to the action of Γ on $\mathcal{H}$. We associate a period $h_\tau = |\{\pm I\}\Gamma_\tau/\{\pm I\}|$ for each point $\tau \in \mathcal{H}$. The elliptic points for the modular group and its arbitrary congruence subgroup are given by the following table:-

Elliptic Points	Period 2	Period 3
Y(1)	i	μ_3
$Y(\Gamma)$	$\gamma_j(i)$	$\gamma_j(\mu_3)$

Here γ_j is one of the finite coset representatives of Γ with some conditions. The compactification of $Y(\Gamma)$ is done by adding the cusps, Γ-equivalence classes of $\mathbb{Q} \cup \{\infty\}$ making it a compact Riemann Surface $X(\Gamma)$.

Space of Modular Forms

The set of modular forms $\mathcal{M}_k(\Gamma)$ is a finite-dimensional vector space over $\mathbb{C}$. We consider the subspaces- the space of cusp forms $\mathcal{S}_k(\Gamma)$ and the Eisenstein space $\mathcal{E}_k(\Gamma) = \mathcal{M}_k(\Gamma)/\mathcal{S}_k(\Gamma)$.

We can define the evaluation map $\nu : \mathcal{M}_k(\Gamma) \longrightarrow \mathbb{C},\ \nu(f) = f(\infty)$. Then we have $ker(\nu) = \mathcal{S}_k(\Gamma)$, and all non-zero values are images of $\mathcal{E}_k(\Gamma)$ giving us the decomposition $\mathcal{M}_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus \mathcal{S}_k(\Gamma)$ We will now show the computation of dimension formulas for the space of modular forms.

Dimension Formulas

Valence formula for $SL_2(\mathbb{Z})$:

Let f be a non-zero automorphic form of weight k on $SL_2(\mathbb{Z})$. Then we have:-

$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_{\mu_3}(f) + \sum_{\tau \in X(1)} \nu_\tau(f) = \frac{k}{12}$ where the sum runs through the orbits in Y(1), other than i, μ_3.

The Valence formula gives us the following results:-

1. $dim(\mathcal{M}_k(SL_2(\mathbb{Z}))) = dim(\mathcal{S}_k(SL_2(\mathbb{Z}))) + 1$
2. $dim(\mathcal{M}_k(SL_2(\mathbb{Z}))) = \begin{cases} \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1, & \text{otherwise} \end{cases}$

The dimension formula for $\mathcal{M}_k(\Gamma)$ requires the following theorem in consideration:

Theorem(Riemann-Roch)

Let X be a compact Riemann surface of genus g. Let $div(\lambda)$ be a canonical divisor on X. Then, for any divisor D on X, we have:- $l(D) = deg(D) - g + 1 + l(div(\lambda) - D)$. where l(D) denotes the linear space of divisor D, and deg(D) is the degree of divisor.

Using Riemann-Roch theorem with certain assumption we get the following dimensional data :

Value of k	$dim(\mathcal{M}_k(\Gamma))$	$dim(\mathcal{S}_k(\Gamma))$
$k \geq 2$	$(k-1)(g-1) + \lfloor \frac{k}{4} \rfloor \epsilon_2 + \lfloor \frac{k}{3} \rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty$	$dim(\mathcal{M}_k(\Gamma)) - \epsilon_\infty$
$k = 0$	1	g
$k < 0$	0	0

Table 1. Dimension formulas for even k

Value of k	$dim(\mathcal{M}_k(\Gamma))$	$dim(\mathcal{S}_k(\Gamma))$
$k \geq 3$	$(k-1)(g-1) + \lfloor \frac{k}{3} \rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty^{\text{reg}} + \frac{k-1}{2}\epsilon_\infty^{\text{irr}}$	$dim(\mathcal{M}_k(\Gamma)) - \epsilon_\infty^{reg}$
$k = 1, \epsilon_\infty^{reg} > 2g-2$	$\epsilon_\infty^{reg}/2$	0
$k = 1, \epsilon_\infty^{reg} \leq 2g-2$	$\geq \epsilon_\infty^{reg}/2$	$dim(\mathcal{M}_k(\Gamma)) - \epsilon_\infty^{reg}/2$
$k < 0$	0	0

Table 2. Dimension formulas for odd k

Basis of space of Modular forms

The basis for $\mathcal{M}_k(SL_2(\mathbb{Z}))$ can be easily computed from corollaries of the Valence formula to be the set $\{E_4, E_6\}$ where E_k is the normalized Eisenstein series of weight k. The basis computation for $\mathcal{M}_k(\Gamma)$ is slightly difficult as that involves an explicit formula for modified Eisenstein series for $\mathcal{E}_k(\Gamma)$ and a bit of Hecke theory for $\mathcal{S}_k(\Gamma)$.We only list out the basis of $\mathcal{E}_k(\Gamma)$ for various congruence subgroups:-

Weight k	Eisenstein Space	Associated Eisenstein Series	Basis
$k \geq 3$	$\mathcal{E}_k(\Gamma(N))$	$E_k^{\bar{v}}(\tau)$ or $G_k^{\bar{v}}(\tau)$	$R_k = \{E_k^{\bar{v}}(\tau) \text{ or } G_k^{\bar{v}}(\tau) : \bar{v} \in y_\infty(\Gamma(N))\}$
	$\mathcal{E}_k(\Gamma_1(N))$	$G_k^{\psi,\varphi}(\tau)$	$S_k = \{E_k^{\psi,\varphi,t}(\tau) : (\psi,\varphi,t) \in A_{N,k}\}$
	$\mathcal{E}_k(N,\chi)$	$G_k^{\psi,\varphi}(\tau)$	$S_{k,\chi} = S_k \cap \{\psi\varphi = \chi\}$
$k = 2$	$\mathcal{E}_2(\Gamma(N))$	$g_2^{\bar{v}}$	$R_2 = \{g_2^{\bar{v}_k} - g_2^{\bar{v}_{k+1}} : k = 1, ..\epsilon_\infty - 1\}$
	$\mathcal{E}_2(\Gamma_1(N))$	$G_2^{\psi,\varphi}(\tau)$	$S_2 = \{E_2^{\psi,\varphi,t}(\tau) : (\psi,\varphi,t) \in A_{N,2}\}$
	$\mathcal{E}_2(N,\chi)$	$G_2^{\psi,\varphi}(\tau)$	$S_{2,\chi} = S_2 \cap \{\psi\varphi = \chi\}$
$k = 1$	$\mathcal{E}_1(\Gamma_1(N))$	$G_1^{\psi,\varphi}(\tau)$	$S_1 = \{E_1^{\psi,\varphi,t}(\tau) : (\psi,\varphi,t) \in A_{N,1}\}$
	$\mathcal{E}_1(N,\chi)$	$G_1^{\psi,\varphi}(\tau)$	$S_{1,\chi} = S_1 \cap \{\psi\varphi = \chi\}$

Table 3. Depicting the basis set for Eisenstein spaces

Application of Modular Forms

Modular forms can be used in proving certain number theoretic results- like the sum of Two squares, the sum of Four Squares, etc. We will go through one of the results, the Two Squares Theorem.

The Two Squares Theorem(Jordan)

Let n $\in \mathbb{Z}^+$, then we have -

$$r(n,2) = 4\left(\sum_{\substack{d|n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv 3 \pmod 4}} 1\right) = 4\left(\sum_{\substack{0<m|n \\ m \text{ is odd}}} (-1)^{(m-1)/2}\right).$$

Corollary(Fermat): Every prime $p \equiv 1 \pmod 4$ is a sum of two squares.

We consider the Jacobi theta function $\theta(\tau,k) = \sum_{n=0}^{\infty} r(n,k)q^n,\ \ q = e^{2\pi i\tau}$ where r(n,k) is representation number of n by k-squares. Following the transformation laws satisfied by $\theta(\tau,k)$, we find that $\theta(\tau,k) \in \mathcal{M}_{k/2}(\Gamma_1(4))$. Now for k=2, we find the dimension of this space to be 1, so constructing another modular form of weight 1 and level $\Gamma_1(4)$ would be just a multiple of $\theta(\tau,2)$. Now if we consider the modified Eisenstein series $E_1^{1,\chi,1}(\tau)$ where χ is the non-trivial character modulo 4, then since we know that it forms the basis of $\mathcal{E}_1(\Gamma_1(4))$, so we just have to equate series with the expansion of $\theta(\tau,2)$ to check the multiple and get the equation for r(n,2).

Glossary

y_∞ : Set of Cusps, $\epsilon_\infty^{\text{reg}}$ – # regular cusp, $\epsilon_\infty^{\text{irr}}$ – # irregular cusp

$D = \{\sum_{x\in X} c_x x; c_x \in \mathbb{Z}, c_x = 0 \text{ for almost all x}\}$, $\deg(D) = \sum_x c_x$

$L(D) = \{f \in C(X) : f = 0 \text{ or } div(f) + D \geq 0\},\ \lambda(\neq 0) \in \Omega^1(X)$

$G_k(\tau) = \sum{}'_{(c,d)}(c\tau + d),^{-k}\ \ E_k(\tau) = G_k(\tau)/(2\zeta(k)),\ g_2^{\bar{v}}(\tau) = G_2^{\bar{v}}(\tau) - (\pi/N^2 Im(\tau))$

$G_k^{\bar{v}}(\tau) = \sum{}'_{(c,d)\equiv v \text{ (mod N)}}(c\tau + d),^{-k},\ \ E_k^{\bar{v}}(\tau) = G_k^{\bar{v}}(\tau) \cap \{gcd(c,d) = 1\}$

$$G_k^{\psi,\varphi}(\tau) = \sum_{c=0}^{u-1}\sum_{d=0}^{v-1}\sum_{e=0}^{u-1} \psi(c)\overline{\varphi}(d) G_k^{\overline{(cv,d+ev)}}(\tau),\ \ E_k^{\psi,\varphi}(\tau) = (v^k/C_k g(\bar{\varphi}))G_k^{\psi,\varphi}(\tau),$$

$E_k^{\psi,\varphi,t}(\tau) = E_k^{\psi,\varphi}(t\tau), A_{N,k} = \{(\psi,\varphi,t) : (\psi\varphi)(-1) = (-1)^k, tuv|N\}$

References

[1] Tom M Apostol.
Modular functions and dirichlet series in number theory, 1990.

[2] Fred Diamond and Jerry Michael Shurman.
A first course in modular forms, volume 228.
Springer, 2005.

[3] Marc Masdeu.
Modular forms (ma4h9), 2015.

[4] James S Milne.
Modular functions and modular forms.
Course Notes of the University of Michigan, 1990.

[5] Rick Miranda.
Algebraic curves and Riemann surfaces, volume 5.
American Mathematical Soc., 1995.

[6] Toshitsune Miyake.
Modular forms.
Springer Science & Business Media, 2006.

[7] Steven Newberg.
Data of modular curves.
2017.

[8] Goro Shimura.
Introduction to the arithmetic theory of automorphic functions, volume 1.
Princeton university press, 1971.

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