Modern Methods of Amplitude Calculation

Dishant Sisodia [1] Dr. Ambresh Shivaji [1]

[1]Indian Institute of Science Education and Research Mohali

Abstract

Quantum field theory enables us to extract feynman rules from a Lagrangian, using which we write the expressions for $i\mathcal{M}$. These amplitudes form the building blocks of any field theory. Most of the observables of interest like cross sections, decay rates, branching ratios are related to $|\mathcal{M}|^2$. A common observation for many such calculations using the standard approach is that the final expression of $|\mathcal{M}|^2$ is compact, but the intermediate stages of calculations explode. In this thesis work we mainly focus on novel methods of calculating the tree level and 1-loop amplitudes which are far more insightful and efficient.

Spinor Helicity Techniques

The essence is to compute individual helicity contributions to the amplitude instead of squaring the full amplitude first and then summing over spins/polarizations. We write the two independent solutions to Dirac equation as:

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \quad ; \quad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \tag{1}$$

$$\bar{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix} \quad ; \quad \bar{u}_+(p) = \begin{pmatrix} [p|^a & 0 \end{pmatrix} \tag{2}$$

The angle and square brackets are nothing to be scared of, they are simply two component commuting spinors that solve the massless dirac equation in the weyl representation. The idea is to to use these structures in external line feynman rules for massless fermions.

Formalism for Spin-1 particles:

For outgoing spin-1 massless vectors we write:

$$\epsilon^\mu_-(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[qp]} \qquad \epsilon^\mu_+(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} \tag{3}$$

where q is not equal to p, and q denotes an arbitrary reference spinor. The arbitrariness in the choice of reference spinor reflects gauge invariance in QED.

Useful properties for Simplification:

For manipulation of angle and square brackets, there are some identities for simplification.

Antisymmetry: $\langle pq\rangle = -\langle qp\rangle \quad ; \quad [pq] = -[qp]$ (4)

Orthonormality: $\langle pq] = 0 \quad ; \quad \langle pp\rangle = [qq] = 0$ (5)

Connection with external momenta: $\langle pq\rangle[pq] = 2p.q = (p+q)^2$ (6)

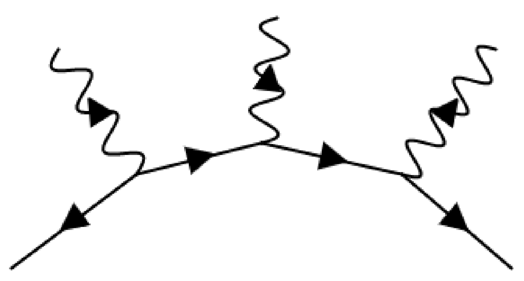
Fierz Identity: $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$ (7)

Momentum Conservation: $\sum_{i=1}^{n}\langle qi\rangle[ik] = 0$ (8)

Schouten Identity: $\langle ri\rangle\langle jk\rangle + \langle rj\rangle\langle ki\rangle + \langle rk\rangle\langle ij\rangle = 0$ (9)

Three photon Scattering:

The result for general 3-photon scattering $e^+e^- \longrightarrow \gamma+\gamma+\gamma$ is already known in the literature. We used the above technique for its calculation.



Since there are 3! ways to arrange the three photon lines, so we have six different diagrams. We are in an outgoing convention so we label outgoing e^-'s momentum as p_1^-, e^+'s momentum as p_2^+ and three outgoing photons as p_3, p_4, p_5. To label the diagrams, the label $3^-, 4^+, 5^+$ for example, would imply that the leftmost photon in figure is 3^-, middle one is 4^+ and rightmost one is 5^+. Solving for all helicity cases we get:

Configuration	Amplitude (proportional)
$3^-, 4^+, 5^+$	$\frac{\langle 13\rangle[45]}{[13][25]\langle 14\rangle}$
$4^+, 3^-, 5^+$	vanishes
$3^-, 5^+, 4^+$	$\frac{\langle 23\rangle\langle 13\rangle[35]}{[13]\langle 25\rangle\langle 14\rangle\langle 24\rangle}$
$4^+, 5^+, 3^-$	vanishes
$5^+, 3^-, 4^+$	$\frac{\langle 12\rangle\langle 13\rangle(\langle 12\rangle[12]+\langle 14\rangle[14])}{\langle 25\rangle\langle 15\rangle[13]\langle 24\rangle\langle 14\rangle}$
$5^+, 4^+, 3^-$	vanishes

Table 1. Three photon scattering

So to obtain the net result, we just have to add all of them which gives:

$$\frac{\langle 12\rangle\langle 13\rangle^2}{\langle 14\rangle\langle 24\rangle\langle 15\rangle\langle 25\rangle} \longrightarrow \frac{s_{12}s_{13}^2}{s_{14}s_{24}s_{15}s_{25}} \tag{10}$$

which matches with the known result.

Loop Techniques

The computation of loop level diagrams is much more complicated than tree level because we have to evaluate tensor integrals.The techniques described here are so powerful that they completely bypass the need for tensor integration. We use tree level amplitudes as building blocks to construct the loop amplitudes. Our starting point will be the general decomposition of a one-loop amplitude into a basis of scalar integral functions:

$$A_n^{1-\text{ loop}} = \mathcal{R}_n + r_\Gamma \frac{(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}}\left(\sum_i b_i B_0\left(K_i^2\right) + \sum_{ij} c_{ij} C_0\left(K_i^2, K_j^2\right) + \sum_{ijk} d_{ijk} D_0\left(K_i^2, K_j^2, K_k^2\right)\right) \tag{11}$$

Generalized Unitarity Method

The fundamental property of S-matrix is unitarity, which leads to the result:

$$-i\left(T_{oi}^{(1-\text{loop})} - T_{io}^{(1-\text{loop})*}\right) = \int d\mu T_{o\mu}^{(\text{tree })} T_{\mu i}^{(\text{tree })}$$

The left hand side of this equation can be viewed as the Discontinuity across the branch cut. To compute the discontinuity, Cutkowsky showed that we need to cut the propagators making them on-shell. The idea is to reduce the loop diagram to product of trees by applying $Disc$ on both sides of equation 11 and find coefficients by comparison to reconstruct the full one-loop amplitude.

Box Coefficients:

Even though several scalar boxes can share the same branch cut, but it turns out that their leading singularity is unique. So a quadruple cut, which imposes four on-shell conditions on loop momenta isolates a unique box configuration, hence the corresponding box coefficient. We select a convenient parametrisation of loop momentum and evaluate the product of resulting four tree amplitudes to get the box coefficient as:

$$d_k = \frac{i}{2}\sum_{i=1}^{2} A_1^{(k)}(l_i)A_2^{(k)}(l_i)A_3^{(k)}(l_i)A_4^{(k)}(l_i) \tag{12}$$

Triangle Coefficients:

Making a triple cut indeed isolates a particular triangle coefficient, but the contribution from higher topology i.e. the scalar boxes is also present. In the momentum parametrisation, there is one free parameter as we only have three on-shell conditions. Examining the behaviour of the integrand as this unconstrained parameter approaches infinity then allows for a straightforward separation of the desired coefficient from any extra contributions. The triangle coefficient is given as:

$$c_j = -\left[\text{Inf}_t\, A_1^{(j)}A_2^{(j)}A_3^{(j)}\right](t)\Big|_{t=0} \tag{13}$$

Bubble Coefficients:

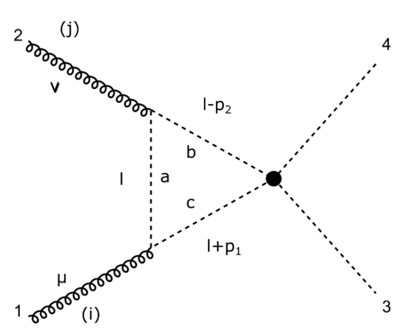
Here we'll have only two on-shell conditions which will give two unconstrained parameters in out cut-loop momentum. The bubble coefficient consists of two terms:

$$\begin{aligned} b_i = &- i\left[\text{Inf}_y\left[\text{Inf}_t\, A_1^{(i)}A_2^{(i)}\right]\right](t,y)\Big|_{t\to 0, y^n\to\frac{1}{n+1}} \\ &- \frac{1}{2}\sum_{\text{third cut}}\sum_{y=y_\pm}\left[\text{Inf}_t\, \tilde{A}_1^{(i)}\tilde{A}_2^{(i)}\tilde{A}_3^{(i)}\right](t)\Big|_{t^n\to T(n)} \end{aligned} \tag{14}$$

Application to $gg \longrightarrow \chi\chi^*$

This process has three contributing diagrams, let's look at $1^+, 2^+$ helicity combination:

Diagram 1:

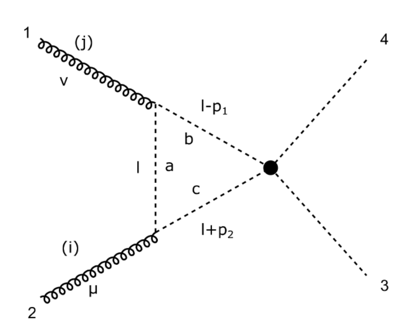


The triangle coefficient comes out to be zero, out of three possible bubbles only one of the bubble $B_0((p_1+p_2)^2)$ contributes and the coefficient is:

$$\frac{ig_s^2\lambda_s\delta^{ij}[21]}{2\langle 12\rangle}\left(1 - \frac{\langle q_1 2\rangle\langle q_2 1\rangle}{\langle q_1 1\rangle\langle q_2 2\rangle}\right) \tag{15}$$

Diagram 2

:



Here also the triangle coefficient comes out to be zero, out of three possible bubbles the same bubble $B_0((p_1+p_2)^2)$ contributes and the coefficient is:

$$\frac{ig_s^2\lambda_s\delta^{ij}[21]}{2\langle 12\rangle}\left(1 - \frac{\langle q_1 2\rangle\langle q_2 1\rangle}{\langle q_1 1\rangle\langle q_2 2\rangle}\right) \tag{16}$$

Diagram 3

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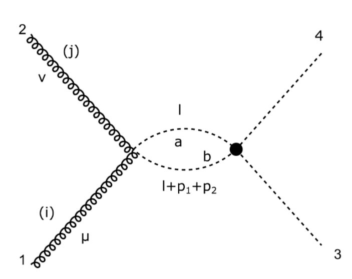

Here only one bubble $B_0((p_1+p_2)^2)$ can contribute and the result is:

$$-ig_s^2\lambda_s\delta^{ij}\frac{\langle q_1q_2\rangle[21]}{\langle q_1 1\rangle\langle q_2 2\rangle} \tag{17}$$

Adding all the three coefficients together gives zero, which implies that only $\mathcal{R}_n$ contributes to the amplitude. Physically this makes sense because the amplitude must be U.V finite. This amplitude can also be computed via the traditional method called tensor reduction and the explicit calculation using that method also gives the same result.

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