Solutions: Introducing Graphs and Graph Theory

Problem 1

Consider the sequence of integers:

$$(4, 4, 3, 2, 2, 2, 1, 1)$$

Could this sequence be the degree sequence of a graph? If so, find the number of nodes and edges in this graph.

> This can't be the degree sequence of a graph because the sum of the degrees is odd.

Counting

- How many *possible* edges are there in a graph with n vertices? What is a big-O expression describing how this number grows as n becomes large?
- How many possible graphs on n labeled nodes are there?

> - There are $\binom{n}{2}$ ways to choose a pair of vertices between which to draw an edge, so that is the number of possible edges.
> - For each of the possible $\binom{n}{2}$ edges, I have 2 choices – draw a node, or don't! So, applying the principle of multiplication, there are $2^{\binom{n}{2}}$ possible graphs on n vertices.

Interlude

A **path** from vertex u to vertex v is a sequence of edges $(u, w_1), (w_1, w_2), \dots, (w_n, v)$ that begins with u and ends with v, such that no intermediate vertices w_i are revisited and such that the second vertex in edge each is the first vertex of the next edge.

A **cycle** is a path from vertex u to itself.

A **tree** is a graph in which between every pair of nodes there is exactly one path.

Problem 2

Prove the following theorem:

Theorem: A tree contains no cycles.

We'll use proof by contradiction. Suppose that my graph contained a cycle. Then, it must have a sequence of edges of the form $(u, w_1), (w_1, w_2), \dots, (w_n, u)$. Let's pick a single vertex on this cycle, such as w_1. Let's show that we can find two paths from u to w_1. First of all, there's a path with just a single edge: (u, w_1). But there's also another path: $(u, w_n), (w_n, w_{n-1}), \dots, (w_2, w_1)$. This is a contradiction, since a tree contains exactly one path between each pair of vertices. We conclude that our graph could not have contained any cycles.