Spatial Complexity and Neighborhood Structure in Cities

v0.0

Phil Chodrow

May 29th, 2016

The problem of identifying natural, socio-economically defined neighborhoods arises in applied contexts including Census reporting, measurement of segregation, and dimension reduction in urban computing. This problem is also of interest for urban theory, since the difficulty of identifying neighborhoods may be viewed as a measure of socio-spatial complexity.

We develop a rigorous, information-theoretic approach to this topic, using open data on race in American cities as a case study. First, we formulate the *mean local information* $J(X, Y)$ as a localization of the mutual information between spatial and socio-economic variables. The measure $J(X, Y)$ is closely related to the Fisher information of the underlying joint distribution, and is therefore a measure of the intrinsic spatial complexity of an urban phenomenon. Unlike standard global information measures, the mean local information clearly distinguishes between cities like Detroit–which is dominated by a few huge, monoracial superclusters–and cities like Philadelphia–which is an intricate patchwork of small, racially-distinct neighborhoods. Second, we provide a practical algorithm for identifying natural neighborhoods through greedy information maximization, and relate this algorithm's behavior to the mean local information. Questions raised by this work include the social, economic, and policy determinants of socio-spatial complexity in cities, and the potential use of spatial information measures in quantifying temporal changes in socio-economic structure, on time scales ranging from days to decades.

1 Introduction

1.1 Neighborhoods, Segregation, and Information

The neighborhood in which an individual lives has profound impact on that individual's mental and physical well-being [15, 16]. Urban plan-

ners may aim to revitalize neighborhoods, and an entire cottage industry in sociology and epidemiology has grown around the idea of "neighborhood effects" (see, for example, [12, 25] for overviews of older work). Despite the persistent import of neighborhoods in social sciences, little consensus exists on how to define or demarcate neighborhoods in a non-arbitrary way. Many studies define "neighborhoods" by the boundaries supplied in their data sets, such as those determined by the U.S. Census [11]. Such an approach follows the limitations of the data provided, rather than a theoretically-principled concept of neighborhood. However, recent progress in non-arbitrary demarcations of urban form serve as proofs of concept for the theoretically-principled measurement of heterogeneity in urban form [23, 24].

We take up the question of neighborhoods in the context of segregation studies in sociology, and will argue that the measurement of segregation is intricately linked with the identification of spatial demographic structure. Many indices have been proposed in the last half-century to measure various aspects of segregation; see [18, 19] for useful synthesizing overviews. Many such approaches have also taken Census-defined boundaries as constitutive of neighborhoods, but more recent work has focused "spatialized" measures that are independent of neighborhood demarcations[20, 21, 22, 27, 28, 13]. Other work has quantified the tendency of different social groups to cluster together [14].

The approach we develop is grounded in information measures of spatial demographic structure. Information theory maps the physical concept of structure to the epistemic concept of predictability. A complex system has enough structure to be at least partially predictable, but enough variability to make prediction challenging. Information-theoretic concepts have already been introduced in the study of cities, finding application in urban planning problems, in predicting population distributions, and in quantifying difference between neighborhoods [4, 1, 2, 3, 5, 26]. Attractive features of information measures include their deep relationship to statistical inference [9, 10], their generalizability to a wide variety of phenomena, and their conceptual linkage of cities with the methods of statistical physics. In the context of the measurement of urban demographic structure, information measures provide answers to questions such as:

1. Supposing you choose an individual randomly from a city, how accurately could I guess their race?
2. Supposing you then told me where that person lived within the city, how much would this new information increase my accuracy?

The following considerations connect these questions to the measurement of diversity and structure. If I can guess a random individual's race with 100% accuracy given no information, then it must be the case that the city is monoracial; increased diversity reduces my ability to guess. If knowing

where a person lives dramatically increased my ability to guess, then there must be substantial structural dependence of demographics on space, a prerequisite of segregation.

We make two primary contributions. The first is to the field of quantitative segregation studies in sociology. We formulate the *mean local information* $J(X, Y)$ as a measure of spatiosocial structure. We then show that this novel measure, in concert with the mutual information $I(X, Y)$, comprise a unified, spatially-aware methodology for the measurement of diversity and segregation. These metrics possess many of the traditionally desirable properties of sociological indices, and can be easily computed from open data such as that provided by the U.S. Census. Our second contribution is to provide a simple, theoretically-motivated, and computationally-tractable method for agglomerating spatio-social data into "natural neighborhoods" across which demographic trends are approximately constant. This method has multiple applications. Sociologists can use this agglomeration procedure to study racially coherent neighborhoods, rather than the partially-arbitrary administrative boundaries such as those produced by the U.S. Census. Computational urban planners can use the method for *dimension-reduction*, in which a large data set is made more computationally tractable while preserving spatio-social relationships of interest. Finally, urban physicists may be interested in the scaling behavior of the mutual information across different levels of aggregation in analogy with coarse-graining in statistical mechanics [5]. The hierarchical character of agglomerative clustering defines a non-arbitrary way to perform this aggregation.

1.2 Structure of the Essay

In Section 2, we motivate and develop the two core components of our mathematical framework. The first of these is the mean local information $J(X, Y)$, a measure of the intrinsic spatial complexity of a compositional phenomenon. In the context of racial trends in cities, the mean local information measures the granularity of racial neighborhood structure. It is low in a city like Detroit, which consists of a few large, monolithic tracts of constant racial composition. It is high in a city like Philadelphia, in which many (spatially) smaller neighborhoods are knit together in an intricate patchwork. The second component of our mathematical framework is a simple algorithm for identifying clusters that are both spatially and compositionally coherent through agglomerative hierarchical clustering and greedy information maximization. In the context of race in cities, this algorithm may be viewed as an automated means to identify "natural" neighborhoods.

In Section 3, we apply these techniques to the analysis of spatial trends in race across U.S. cities. We show that the mean local information $J(X, Y)$

and the mutual information $I(X, Y)$ jointly supply simple and intuitive measures of spatial complexity for various cities. We also note an intriguing scaling relationship between spatial complexity and population density. We next evaluate our information-theoretic clustering method, and show that the neighborhoods it identifies can usefully shed light on traditional concerns of quantitative sociology, such as racial affinities and spatial concentration. Next, by defining a global evaluation measure on each clustering, we show that the mean local information $J(X, Y)$ does indeed measure the "clusterability" of a spatial compositional data set.

Section 4 is a discussion of our findings and prospects for future work. Finally, we include an Appendix supplying some mathematical details not present in the text, including a proof of our assertion that the mean local information is related to a fundamental statistical property of spatial compositional phenomena, and a thorough specification of algorithms and computations.

2 Mathematical Framework

2.1 Information and the Checkerboard

We begin by developing two fundamental measures of information theory, the entropy of one random variable and the mutual information of two. Let X be a spatial variable, such as a set of coordinates in space or a neighborhood name. Let Y be a compositional variable that we aim to study, defined on an alphabet $\mathcal{Y}$. In the simple case in which $\mathcal{Y} = \{\text{Black, White}\}$, Figure 1 illustrates a range of possible joint distributions $p(X, Y)$. Comparing the completely undiverse city (a) and the spatially uniformly diverse (b) motivates the first fundamental information measure, the entropy:

$$H(Y) \triangleq -\mathbb{E}_Y[\log p(Y)] = -\sum_{y \in \mathcal{Y}} p(y) \log p(y) \, . \tag{1}$$

The entropy measures how evenly the global marginal distribution $p(Y)$ is distributed over the alphabet $\mathcal{Y}$. Epistemically, $H(Y)$ measures the difficulty in guessing the random variable Y, given no further information. $H(Y) = 0$ when Y always takes a single value, such as in city (a) – perfect guessing is possible. On the other hand, $H(Y)$ achieves its maximum of $H(Y) = \log |\mathcal{Y}|$ when Y is uniformly distributed on $\mathcal{Y}$, as in city (b). The entropy $H(Y)$ is this sufficient to distinguish the presence of global diversity from its absence. On the other hand, the entropy H is unable to distinguish between the spatial uniformity of (b) and the spatially variability of (c). To distinguish these two cities we may use the mutual information,

which is defined in terms of the Kullback-Leibler divergence D:

$$D[p(Z)\|q(Z)] \triangleq \sum_z p(z) \log \frac{p(z)}{q(z)} \tag{2}$$

$$I(X,Y) \triangleq D[p(X,Y)\|p(X)p(Y)] = \mathbb{E}_X[D[p(Y|X)\|p(Y)]] \tag{3}$$

Though not a true metric, the divergence D is interpretable a measure of distance in the space of probability distributions, and so $I(X,Y)$ may be interpreted as the distance between the true joint distribution $p(X,Y)$ and the product of marginals $p(X)p(Y)$. Since the latter expresses statistical independence between X and Y, $I(X,Y)$ measures the extent to which X and Y are dependent. Epistemically, $I(X,Y)$ measures the extent to which knowledge of Xincreases possible accuracy in guessing Y. In city (b), X and Y are independent: knowing X (where an individual lives) conveys no information about that Y (that individual's race). In city (c), on the other hand X and Y are completely dependent: if you know where someone lives, you know their race with 100% confidence, and $I(X,Y)$ achieves its maximum.

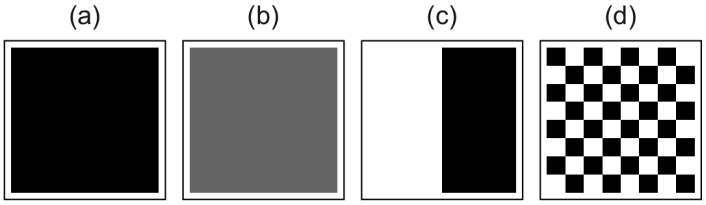
City (d) shares with city (c) the fact that race is completely determined by residence. However, city (d) embodies the "checkerboard problem": measures that use only the joint distribution $p(X,Y)$ without additionally considering the spatial information contained within X will evaluate city (d) to be the same as city (c), despite their considerably different patterns of racial separation and potentially very different implications for planning and policy. A recent working paper [22] provides one highly operational approach to this problem, using road network topology and a weighting function that decays with distance to define a localized measure based on the Kullback-Leibler divergence. Here we pursue an alternative strategy, showing that a localization of the mutual information both measures spatial variation in race and corresponds to the estimation of a fundamental statistical property of the distribution $p(X,Y)$.

2.2 Measuring Socio-Spatial Complexity

We will now derive a measure that solves the "checkerboard problem" by distinguishing the toy cities (c) and (d). To motivate our methods, we consider an idealized scenario in which we have a differentiable field $p(y|x)$ of observed probability distributions for each $x \in M \subset \mathbb{R}^n$, where the metric space M is interpretable as the "map" on which we work.

Fix a point $x_0 \in M$ and a radius $r > 0$. Let $B_r(x_0)$ denote the ball of radius r about x_0. Then, define the *local mutual information in radius r* as the mutual information between X and Y, restricted to the small ball $B_r(x_0)$ about x_0:

$$I_r(x_0) \triangleq \mathbb{E}_X[D[p(\cdot|X)\|p(\cdot|X \in B_r(x_0))]|X \in B_r(x_0)] \tag{4}$$



City	$H(Y)$	$I(X,Y)$	$J(X,Y)$
(a)	0.0	0.0	0.0
(b)	0.7	0.0	0.0
(c)	0.7	0.7	0.6
(d)	0.7	0.7	2.7

Figure 1: Information theory and the checkboard problem: model cities with various kinds of spatial diversity can be distinguished through progressively more subtle spatial information measures.

Intuitively, $I_r(x_0)$ measures how much knowing the "location" x adds to our information about the category Y, or, equivalently, how much the probability field $p(y|x)$ varies with x in a small neighborhood of x_0. It is therefore a natural measure of local complexity, aligned in approach with the global mutual information but designed to detect local variation. As expected, $I_r(x_0) = 0$ if and only if $p(y|x)$ is constant for each y in the ball $B_r(x_0)$; that is, if within $B_r(x_0)$ the field $p(y|x)$ resembles toy city (b) in Figure 1. In the definition (4), r may be thought of as the spatial resolution at which we conduct analysis, and $I_r(x_0)$ is highly dependent on r. However, it is possible to show that $I_r(x_0)$ is related to a fundamental statistical property of the probability field $p(y|x)$ that is resolution-independent. Under the stated conditions, the following approximation holds:

$$\frac{I_r(x)}{r^2} \cong \frac{1}{4}\text{trace } J_Y(x) \ , \tag{5}$$

where $J_Y(x)$ is the Fisher information matrix in Y about x, defined as

$$J_Y(x) \triangleq \mathbb{E}_Y \left[(\nabla_x \log p(Y|x))(\nabla_x \log p(Y|x))^T \right] \ . \tag{6}$$

A formal statement and proof of (5) are provided in Appendix 2. The Fisher information J_Y is a fundamental quantity in statistics and information theory. From a geometric perspective, J_Y provides the natural intrinsic metric in the geometric space of probability distributions parameterized by the spatial variable x. Equation (5) therefore expresses a relationship between the local mutual information $I_r(x)$ and the information geometry of the underlying probability field. Corresponding to the fact that

$I_r(x_0)$ vanishes if and only if $p(y|x)$ is constant in $B_r(x_0)$, $J_Y(x_0) = 0$ if and only if $\nabla_x p(y|x_0) = 0$ for all y. This implies that x_0 is a stationary point, about which the probability field $p(y|x_0)$ exhibits only small (2nd order or smaller) changes with respect to changes in x.

Since the Fisher information is a strictly local measure of statistical variability around x, we can aggregate the Fisher information to derive a measure of average local variability. The *mean local information* is

$$J(X, Y) \triangleq \mathbb{E}_X[\text{trace } J_Y(X)] \tag{7}$$

As demonstrated in Figure 1, $J(X, Y)$ distinguishes cities (c) and (d), thereby addressing the "checkerboard problem" head on. We propose the aggregate quantity $J(X, Y)$ as a third measure–alongside the entropy $H(Y)$ and mutual information $I(X, Y)$– as a tool for the information-theoretic structure of spatial compositional complexity.

We note that, since

$$\text{trace } J_Y(x) = \sum_i \mathbb{E}_Y \left[\left(\frac{1}{p(Y|x)} \frac{\partial p(Y|x)}{\partial x_i} \right) \right]^2 \tag{8}$$

may be viewed as a weighted norm of the gradient $\nabla_x p(Y|x)$, the quantity

$$J(X, Y) = \mathbb{E}_X[\text{trace } J_Y(X)] \tag{9}$$

$$= \int_M \text{trace } J_Y(X) d\mathbb{P}_X \tag{10}$$

may be viewed as a cousin to total variation measures often encountered in analysis.

The application of this methodology to a discrete data set is conceptually simple. For a given set of tracts, overlay an evenly spaced grid of radius r, and measure the mutual information $I_r(x)$ in each grid cell. Then, when data and grid resolutions are sufficiently high, $4I_r(x)/r^2$ approximates the quantity trace $J_Y(x)$, which can then be aggregated across the data set. We provide a more formal statement of this computation in the appendix.

2.3 Information Measures and Segregation Studies

Considerable attention has been paid to relating segregation indices to intuitive concepts of diversity and segregation. We note here that the mutual information $I(X, Y)$ and mean local information $J(X, Y)$ correspond closely to the two dimensions of segregation formulated in [19]. The authors of [19] describe *evenness* as "the extent to which groups are similarly distributed in residential space" (page 126), and *exposure* as "the extent that members of one group encounter members of another group...in their local spatial environments." The mutual information $I(X, Y)$ may be viewed as a measure of (lack of) evenness, since to say that groups are similarly distributed

in residential space is to say that knowing a spatial location conveys little about the race of the people who live there. Large $I(X,Y)$ reflects highly uneven distributions of demographic groups. Complementarily, the mean local information $J(X,Y)$may be viewed as a measure of spatial exposure *for a fixed level of* $I(X,Y)$, as is illustrated by cities (c) and (d) in 1. Though distinct, these dimensions are not independent. To see this, note that the most thorough kind of exposure is achieved when $I(X,Y) = J(X,Y) = 0$, as in city (b). Since the concept of exposure only applies in a city with spatial differences, $J(X,Y)$ must be considered jointly with $I(X,Y)$ as a measure of spatial exposure.

There has also been much work showing the desirable properties of various segregation indicies. To give a brief sampling, indices should be invariant to changes in overall population size; they should decrease when populations "even themselves out," and they should behave predictably under aggregation. Since we are presenting $I(X,Y)$ and $J(X,Y)$ as a suite of complementary information measures, we consider each in turn. The author of [21] shows that the mutual information $I(X,Y)$ (which she calls the "Divergence Index") satisfies these and other desirable properties, generally as well or better than existing alternatives. We highlight one property of $I(X,Y)$ for special note, as this property will be central to our development of natural neighborhood identification. The principle of "additive decomposability" [19] stipulates that, when the data is grouped along either racial or spatial axes, a good segregation index should split into "within group" and "between group" components. In the context of the mutual information $I(X,Y)$, additive decomposability is simply the familiar chain rule of mutual information. For concreteness, let C be a random variable giving the cluster label of location X; importantly, C is completely determined by X. Then, the chain rule expresses additive decomposability as

$$I(X,Y) = I(C,Y) + I(X,Y|C)\ ; \tag{11}$$

i.e. the information I have about Y given that I know X is equal to the information I have if I first learn the cluster C, plus the amount of additional information I gain if I subsequently learn the exact location X as well. The first term is interpretable as the between-group information, while the second is interpretable as the within-group information. It is therefore comparable to various "sum of squares" decompositions that frequently appear in classical statistics. Indeed, it is possible to show that, when the differences between locations are small, the mutual information can be interpreted as a variance, in which case (11) expresses the sum of squares composition directly. A similar version of the chain rule expresses additive decomposability for aggregation of racial groups rather than spatial locations.

What of $J(X,Y)$? Below, we provide a brief overview of the properties of $J(X,Y)$. $J(X,Y)$ possesses a variety of intuitive properties, and those

that it does not possess either underscore the importance of reading it in conjunction with $I(X,Y)$ or do not apply to explicitly spatial measures. See [19, 20] for further discussion of these criteria. The mean local information $J(X,Y)$ satisfies:

Organizational Equivalence: Both the theoretical definition (7) or the procedure to compute it from tract data remain unchanged when a tract is subdivided into smaller tracts, each of which with identical demographic structure.

Size and Density Invariance: Since $J(X,Y)$ is completely determined by the marginal distribution $p(X,Y)$, it is invariant under changes in population density.

Additive Group Decomposability: When demographic groups are aggregated into super-groups, the chain rule of mutual information applied to the demographic variable Y provides an an additive decomposition of the form $J(X,Y) = J(X,G) + J(X,Y|G)$, which can be interpreted as the sum of a between-groups term and a within-groups term as needed.

Scale Interpretability: We have $J(X,Y) = 0$ if and only if $p(Y|X)$ is constant on each connect component of the metric space M. When only one connected component exists, this implies that every locale has the same demographic structure as the global environment. When multiple connected components exist (e.g. the city is divided by a river), demographics must be constant in space on each side of the division. $J(X,Y)$ does not satisfy the additional condition of achieving its maximum value when all locales are monoracial, but this point only emphasizes that $J(X,Y)$ and $I(X,Y)$ should be read jointly. When all locales are monoracial, $I(X,Y)$ achieves its maximum value and $J(X,Y) = 0$.

Boundary Independent: $J(X,Y)$ is defined in terms of a continuous underlying probability distribution $p(X,Y)$, rather than arbitrarily-defined tracts. In computational practice, $J(X,Y)$ is indeed sensitive to the boundaries supplied with the data; however, (5) guarantees that this sensitivity vanishes as the resolution grows sufficiently small.

Exchanges: Exchanges that tend to "smooth out" demographic distributions in space reduce both $I(X,Y)$ and $J(X,Y)$.

The mean local information $J(X,Y)$ fails to one of the criteria enumerated in [19, 20]; that of **additive spatial decomposability**. This criterion states that "If X spatial subareas are aggregated into Y larger spatial areas, then a segregation measure should be decomposable into a sum of within- and between-area components" [20], page 136. However, considering that we have imposed additional spatial structure on our model, this criterion does not appear to be motivated. Consider, for example, Figure 2. Aggregating

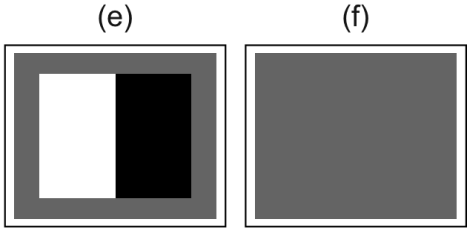


Figure 2: A problem with additive spatial decomposability: aggregating the inner two regions transforms (e) into (f), smoothing out all spatial variability. A "between groups" term must therefore vanish, while a "within-groups" term considers only the inner boundary; not the outer ones.

the two central regions transforms (e) into (f), erasing all spatial variability. Any measure should therefore have a between-groups component equal to 0. On the other hand, the within-groups component can only consider the middle white/black boundary between the central two regions. The sum of the between-groups and within-groups components must therefore consist only in information included in the middle boundary. However, the spatial variability in city (e) is not exhausted by this middle boundary; there are a total of six other frontiers of racial difference that should be considered in any spatial segregation measure. We therefore content that additive decomposability is a desideratum of nonspatial measures (it is satisfied by $I(X, Y)$), but not of explicitly spatial ones. We have not exhausted the full complement of desiderata for segregation measures. However, many of the remaining ones – such as composition invariance – are either vague or controversial, and we will not consider further.

2.4 Identifying Natural Neighborhoods

Equation (11) motivates a simple scheme for identifying natural neighborhoods based on maximizing the between-group term $I(C, Y)$. If all I had access to were the cluster labels C and not the locations X, then the information I had about Ywould be $I(C, Y)$. A "good" clustering therefore maximizes the between-cluster information $I(Y, C)$, which entails minimizing the within-cluster information $I(X, Y|C)$. Solving this problem exactly is a challenging discrete optimization problem, and may not be computationally tractable. We can, however, construct a greedy algorithm which leads to satisfactory results. Suppose that we face the problem of choosing a pair of tracts $\{i*, j*\}$ to cluster together. The reduction in information

associated with aggregating the locations I into a single cluster is

$$d(i,j) \triangleq \sum_{k \in \{i,j\}} p(X=k) D[p(Y|X=k) \| p(Y|X \in \{i,j\})] \tag{12}$$

$$- p(X \in \{i,j\}) D[p(Y|X \in \{i,j\}) \| p(Y)] \tag{13}$$

where $p(Y) = \sum_x p(x,Y)$ is the global marginal distribution. the first term is the information associated with the two separate tracts, while the second is the information associated with a merged tract. Equation (13) defines a natural information distance between locations i and j. Like the KL divergence, this distance is strictly nonnegative; unlike the KL divergence, it is symmetric, and defines an axiomatic metric on the space of tracts. Importantly, we can therefore use the distance $d(i,j)$ as a dissimilarity measure for the purposes of clustering. Our greedy procedure is simple: at each iteration, determine

$$(i^*, j^*) = \underset{i \text{neighbors} j}{\operatorname{argmin}} \; d(i,j), \tag{14}$$

and then combine i^* amd j^* into a single tract, repeating until only one cluster remains. This procedure defines a form of agglomerative hierarchical clustering distinguished by two characteristics: its spatial constraints and its pursuit of minimal information loss at each step. As a greedy algorith, it possesses no guarantees for optimal solutions, but in practice its performance leads to intuitive, racially-coherent regions. It thereby enables a study of spatial difference using non-arbitrarily-defined regions.

3 Findings

3.1 Data Used

We assembled block-group level data from the 2010-2014 American Community Survey (ACS), conducted by the U.S. Census Bureau, on race and ethnicity for counties housing 51 large US cities. We then aggregated the detailed racial and ethnic groups into five meta-categories: 'Asian', 'Black', 'Hispanic', 'Other', and 'White'.

3.2 Information Measures

For each city, we computed the entropy $H(Y)$ and the mutual information $I(X,Y)$. To compute the estimated aggregate Fisher information $J(X,Y)$, we tiled the map with a hexagonal grid of cell radius 0.5km. We then computed the estimated mutual information within each grid cell, and averaged the results weighted by population population. A technical specification of this approach is provided in Appendix 1, and an illustration of it in Figure 3.

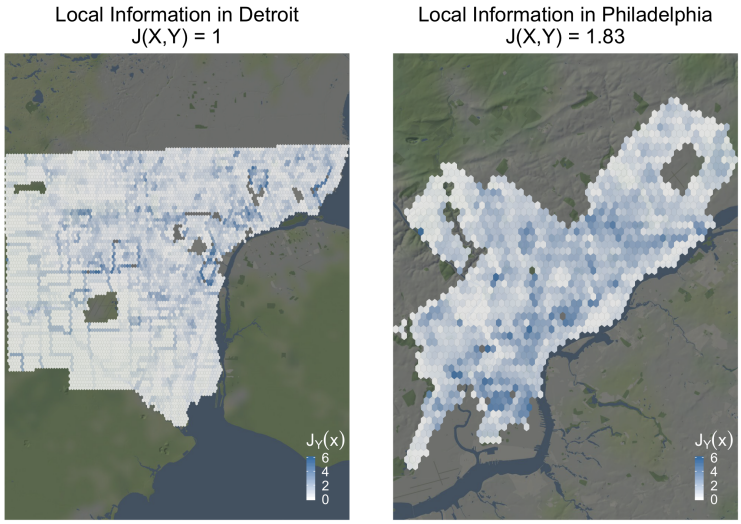


Figure 3: Illustration of methodology for estimating local information in Suffolk County, MA (Boston). We first cover the map in a hexgrid, and compute the mutual information within each hex from the Census tracts that overlap the hex, weighted by density.

Figure 4 shows the relationship of the mutual information (or evenness) $I(X,Y)$ and mean local information (or exposure) $J(X,Y)$. The global positive trend reflects the fact that global spatial variability is a prerequisite for local variability: if the city is uniform (like Figure 1(a)), then no local differences either exist. On the other hand, there are substantial variations in $J(X,Y)$ even in cities with comparable global variability $I(X,Y)$. For example, the cities of Detroit and Philadelphia provide a striking contrast. While they have mutual information $I(X,Y)$, Philadelphia's mean local information $J(X,Y)$ is substantially higher. This reflects the fact that Detroit is composed of large, highly-segregated, monoracial neighborhoods, whereas Philadelphia has a much more fine-grained, intricate neighborhood structure. These patterns illustrate how combinations of the information measures $H(Y)$, $I(X,Y)$, and $J(X,Y)$ can be used to construct taxonomies diversity for American cities. It may be useful summarise the two measures by calling cities close to the bottom-right "most segregated", but we recommend reporting both $I(X,Y)$ and $J(X,Y)$. It is best to compare $J(X,Y)$ only between cities with similar $I(X,Y)$; thus, while it may be right to say that Detroit has lower levels of exposure than Phildelphia, when comparing Detroit to Baltimore it should be kept in mind that Baltimore is more even in its overall sociospatial structure.

Intriguingly, the mean local variability $J(X,Y)$ appears obey a scaling relation with respect to urban density. In Figure 5, we plot $J(X,Y)$ against

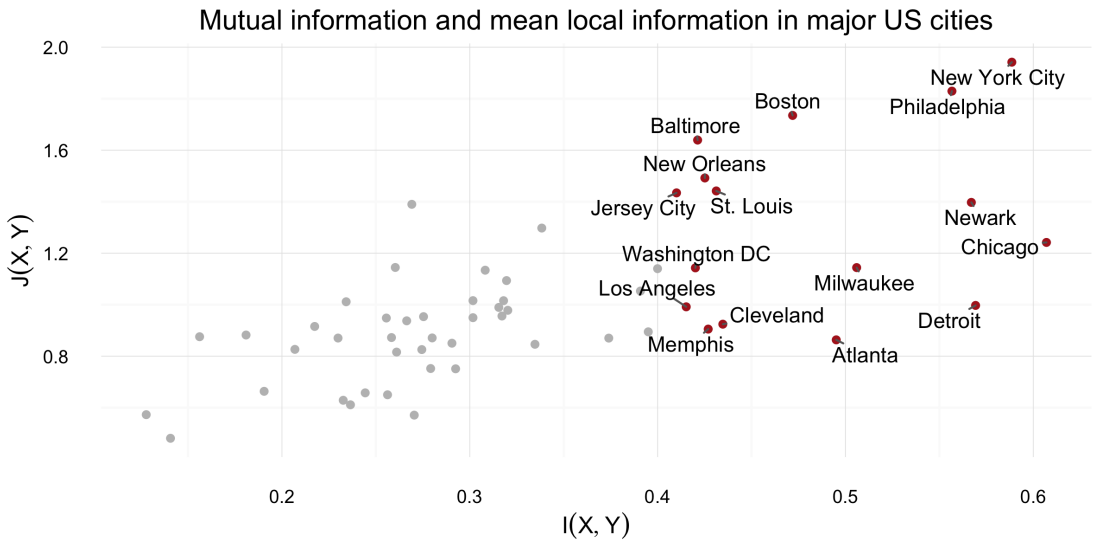


Figure 4: Relationship of global mutual information $I(X, Y)$ and mean local information $J(X, Y)$.

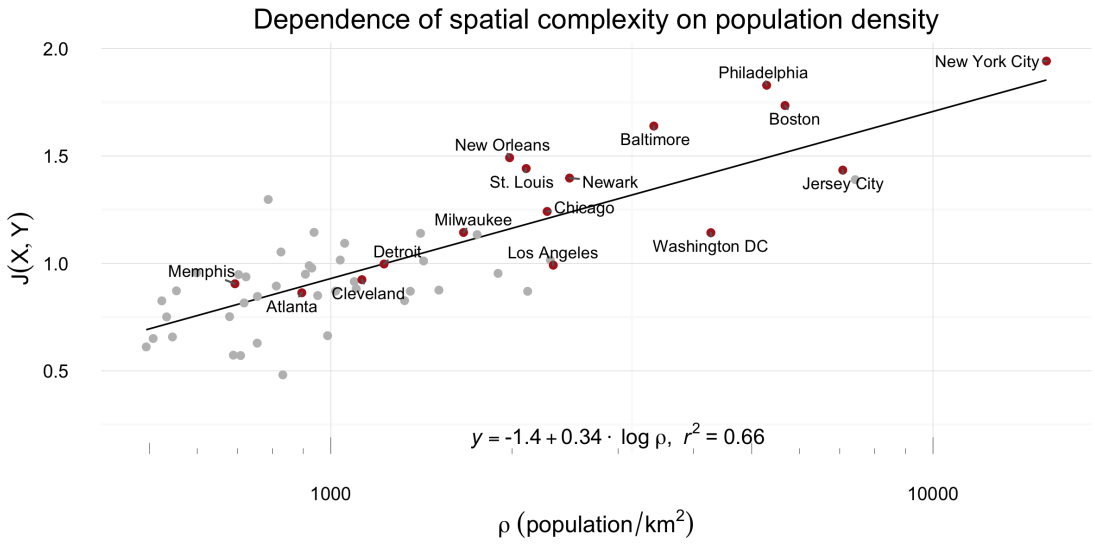


Figure 5: The mean local information $J(X, Y)$ scales with the logarithm of population density.

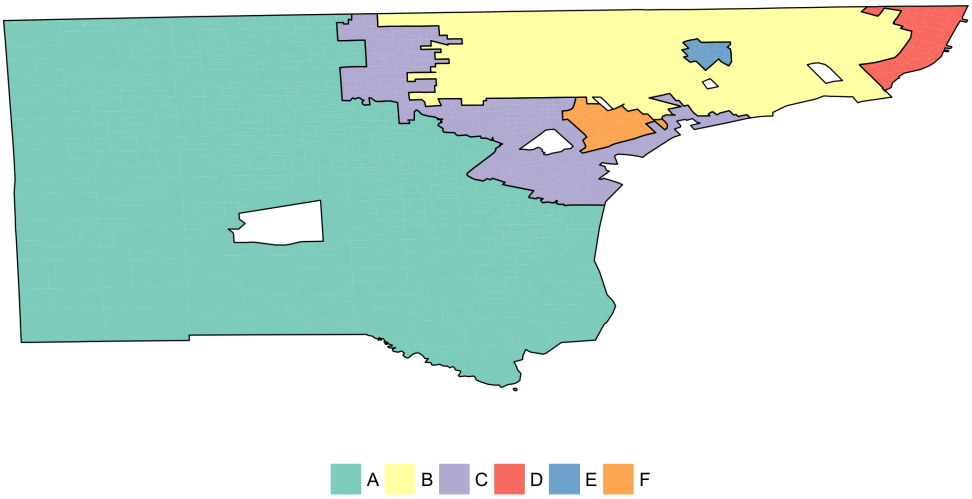


Figure 6: Clustering of Detroit into 6 neighborhoods based on race.

the population density ρ of our studied cities. It appears that $J(X, Y)$ grows linearly with the logarithm of density. We interpret this trend as reflecting a compression of social space in dense urban areas: the same structure of racial variability fits into much less geographical area in New York than in Phoenix. One aspect of diversity in large, dense cities is that one need walk much less distance in order to reach a neighborhood with substantially different racial trends than one's own. It may be of interest for later studies to consider why this trend might arise through dynamical processes of urban formation.

3.3 Learning From Natural Neighborhoods

Figure 6 shows an illustrative clustering of Wayne County (housing the city of Detroit) into six regions using the greedy information maximization procedure defined by equation (13). The procedure cleanly divides the city into racially coherent zones. Cluster A is predominantly white and B predominantly black, demarcating the major racial divide in the city. Cluster C is a small transitional zone in which the two races overlap. Clusters D and E are the demographically distinct independent cities of Hamtramck and Grosse Pointe, respectively, while Cluster F is a Hispanic community known as Mexicantown. The existence of clusters like C reflects that not every racially coherent tract is interpretable as a "neighborhood"; some may be more naturally thought as transitions between neighborhoods.

Cities vary sharply in the level of cluster detail needed to convey equal amounts of demographic information. Figure 7 shows example cluster di-

visions for four major cities. Each clustering conveys approximately equal demographic information, as measured by the mutual information $I(C, Y)$ between cluster labels and racial distribution. In sharply segregated Atlanta, just two clusters–one predominantly white, the other predominantly black–suffice to convey as much information as five in Boston, where the clusters are substantially more nuanced. Boston's Cluster A is predominantly white, cluster D majority Hispanic, and cluster E majority black. Cluster B is again majority white, but is distinct in that its minority citizens are almost all Hispanic. Cluster C is approximately equally black, Hispanic, and white.

Clusterings such as those shown in Figure 7 shed useful light on traditional concerns in the study of segregation. We have already argued that $I(X, Y)$ and $J(X, Y)$ naturally encode the high-level ideas of evenness and exposure endorsed by [20]. We now consider the further concepts of clustering and concentration. According to **[?]**, "*clustering* measures the degree to which minority areas are located adjacent to one another" and "*concentration* refers to the degree of a group's agglomeration in urban space" (pages 309-310). When areas with similar racial composition are located close to one another, fewer spatial clusters are necessary to convey equivalent information about racial trends. Thus, 7 suggests that Atlanta is more clustered than Boston, a suggestion that we can quantify using the AUC methodology developed below. The concept of concentration invites us to consider the composition and density of each cluster. In Detroit, for example, whites make up 70% of "their" Cluster A, while black residents make up a full 86% of "their" Cluster B. Cluster B is also more concentrated in that it is more densely populated than Cluster A: it contains 34% of the Detroit's population, but accounts for just 19% of land in the analyzed area. This makes Cluster B more than twice as dense as Cluster A. Thus, in Detroit, "black" neighborhoods tend to be more racially homogeneous and more densely packed in urban space than "white" ones. Finally, we note that these clusterings allow us to examine the somewhat abstract concept of "exposure" in considerable detail. Figure 7 indicates that across all cities, Asians are much more likely to be found in predominantly white clusters than they are in predominantly black or Hispanic ones, indicating significant spatial integration between these two groups. In Chicago, some groups of Hispanics are exposed primarily to Hispanics and other whites (Cluster B), while other groups are exposed primarily to black residents (Cluster D).

3.4 Complexity and Neighborhood Structure

We expect that spatially complex cities with high $J(X, Y)$ would require more complicated models in order to capture similar levels of spatiosocial structure. Testing this expectation requires a quality measure defined on

Figure 7: Example clusters for different cities, and the associated mutual information contained in each. The number in each box reflects the percentage of the population in the labeled racial group and cluster.

Figure 8: Evaluation of clustering using an Area Under the Curve (AUC) metric. The AUC is the fraction of the bounding box lying under the information gain curve. Boston's AUC is smaller than Detroit's, showing that Boston's spatial complexity (measured by $J(X,Y)$) requires more complex models than Detroit's.

cluster models for each city. The natural measure of loss for a clustering is the mutual information $I(C,Y)$ between cluster and racial labels.

To develop a global evaluation of the hierarchical clustering of a city according to the method defined by (13), we borrow the methodology of the "area under the curve" (AUC) used extensively in the field of statistical learning. By plotting the information $I(C,Y)$ against the number of clusters n, we obtain a curve reflecting how the information evolves at varying scales of aggregation. Figure 8 illustrates these curves for Boston and Detroit, with N plotted on a logarithmic scale.

Figure 8 also shows the bounding rectangle, whose top right corner is defined by the number N of tracts in the census data set and the full mutual information $I(X,Y)$. The AUC is defined as the ratio of the shaded area in Figure 8 to the bounding rectangle. An AUC of 1 indicates that just one region carries full information; this can only occur in a city with no spatial variation, such as city (b) in 1. A larger AUC indicates that sample models with few regions capture more information about spatial variation in the city. In Detroit, there exists a clear dividing line between a predominantly white region on the west and a predominantly black region in the east. A two-cluster model therefore captures much of the information in the city, giving a high AUC of 0.74. In Boston, on the other hand, no such clear divide exists, and more complex models are necessary to capture similar

	AUC
$I(X, Y)$	-0.184^{***}
	$(-0.292, -0.075)$
$J(X, Y)$	0.450^{***}
	(0.374, 0.525)
Intercept	-0.147^{***}
	$(-0.239, -0.055)$
Constant	0.811^{***}
	(0.733, 0.889)
Observations	56
R^2	0.764
Adjusted R^2	0.750
Residual Std. Error	0.040 (df = 52)
F Statistic	55.959^{***} (df = 3; 52) (p = 0.000)

Figure 9: Relationship between information measures $I(X, Y)$, $J(X, Y)$, and the clustering AUC. The AUC has been geographically adjusted to account for cities like New York, whose Census blockgroups form eight disconnected components, which would mildly distort results without adjustment. Table produced using [17]

.

amounts of information. Thus, Boston has a lower AUC of 0.67.

Based on our discussion so far, we would expect that the AUC is related to the information measure $J(X, Y)$. Figure 9 confirms this expectation. Overall, two factors explain much about the "clusterability" of a city as measured by the AUC. The most important factor is the overall mutual information $I(X, Y)$. This reflects a simple intuition: when there is little spatial variation, there is not much to cluster. The second factor is the mean local information $J(X, Y)$, reflecting that spatial complexity makes clustering harder. A simple linear regression makes these insights precise. As predictors of the AUC, both $I(X, Y)$ and $J(X, Y)$ are highly significant, and the coefficient of $J(X, Y)$ is negative. The two measures jointly account for an adjusted 70% of the variation in AUC.

4 Discussion

It is worthwhile to emphasize the characteristics of the data that make useful clustering possible. The unclustered data as provided by the U.S. Census presents a drawback: its boundaries are arbitrary. However, it also has an opportunity: its resolution is much higher than the resolution of the phenomena we aim to investigate. While Detroit contains over 1,800 block-

groups, it is clear that there are not 1,800 distinctive neighborhoods; rather, cluster analysis suggests that there are perhaps ten. We can think of our clustering method as a means of exchanging excess resolution for meaningful boundaries, with the mutual information quantifying how much meaning is lost in the exchange.

We have formulated a novel measure of urban spatial complexity, and applied it to an analysis of spatial distributions of race in American cities. We emphasize that the mean local spatial variability $J(X, Y)$ is novel piece of a more complete information-theoretic characterization that also includes the entropy $H(Y)$ and the global variability $I(X, Y)$. Our results suggest two major directions of further exploration:

One major physical question raised by our results is the origin of the scaling relation seen in 5. This scaling relation may suggest a dynamical process of neighborhood formation and growth common to many American cities. If so, a physical understanding of this process in terms of individual-level behavior is called for. It is tempting to view the scaling relationship as an interplay of spatial preferential attachment and intrinsic limits to neighborhood density due to housing availability, but this view is speculative at this point. A theory to explain this behavior would be most welcome.

Another question raised is more operational. In many urban planning contexts, the amount of available data may make direct computations prohibitive. In such contexts, it may be useful to construct a simplifying model of the data the data (such as a clustering) in order to reduce dimensionality and make "the big picture" more visible. When a particular demographic phenomenon (such as race) is under consideration, it may be important to ensure that the model preserves the major patterns of variation in the granular data. We conjecture that the local variability $J(X, Y)$ measures the "model-ability" of such data sets. Intuitively, a city like Detroit with relatively low $J(X, Y)$ and large, monoracial neighborhoods should be easily representable with relatively few modeled clusters, one for each of the relatively few major neighborhoods. On the other hand, relatively speaking, a city like Philadelphia with very intricate neighborhood structure may require substantially greater model complexity to represent without large loss of information. A promising course of further study is to design an information-theoretic clustering algorithm designed to model urban patterns, and then compare this algorithm's performance to $J(X, Y)$.

A third question and final question relates to the time-dependence of spatial structure. On one time scale, daily movement around a city for work or leisure activities has the impact of "mixing" separated residents in public spaces, potentially leading to very different spatial patterns of racial difference. On another time scale, these measures may track how the spatial structure of cities evolves over decades, potentially shedding further light on the dynamics of city formation.

A final question is....

References

[1] Michael Batty. Spatial Entropy. *Geographical Analysis*, 6(1):1–31, 1974.

[2] Michael Batty. Entropy in spatial aggregation. *Geographical Analysis*, 8(1):1–21, 1976.

[3] Michael Batty. Space, Scale, and Scaling in Entropy-Maximizing. 44(0):0–28, 2010.

[4] Michael Batty. Entropy and Spatial Geometry. *Royal Geographical Society*, 25(5):269–283, 2014.

[5] Luis M A Bettencourt, Joe Hand, and José Lobo. Spatial Selection and the Statistics of Neighborhoods Spatial Selection and the Statistics of Neighborhoods. 2015.

[6] Roger Bivand, Tim Keitt, and Barry Rowlingson. rgdal: bindings for the Geospatial Data Abstraction Library, 2014.

[7] Roger Bivand and Nicholas Lewin-Koh. maptools: Tools for reading and handling spatial objects, 2014.

[8] Roger Bivand and Colin Rundel. rgeos: Interface to Geometry Engine - Open Source (GEOS), 2014.

[9] Thomas M Cover and Joy A Thomas. *Elements of Information Theory*. John Wiley and Sons, New York, 1991.

[10] Imre Csiszar and Paul C Shields. Information Theory and Statistics: A Tutorial. *Foundations and Trends^{TM} in Communications and Information Theory*, 1(4):417–528, 2004.

[11] Robert D. Dietz. The estimation of neighborhood effects in the social sciences: An interdisciplinary approach. *Social Science Research*, 31(4):539–575, 2002.

[12] A V Diez Roux. Investigating neighbourhood and area effects on health. *Am J Public Health*, 91(11):1783–1789, 2001.

[13] Barret A Lee, Sean F Reardon, Glenn Firebaugh, Chard R Farrell, Stephen A Matthews, and David O'Sullivan. Beyond the Census Tract: Patterns and Determinants of Racial Segregation at Multiple Geographic Scales. *American Sociological Review*, 73(5):766–791, 2008.

[14] Rémi Louf and Marc Barthélemy. Patterns of residential segregation. 2015.

[15] Jens Ludwig, Greg J Duncan, Lisa A Gennetian, Lawrence F Katz, Ronald C Kessler, Jeffrey R Kling, and Lisa Sanbonmatsu. Neighborhood Effects on the Long-Term Well-Being of Low-Income Adults. *Science*, 337(September):1505–1510, 2012.

[16] Jens Ludwig, Greg J. Duncan, Lisa A. Gennetian, Lawrence F. Katz, Ronald C. Kessler, Jeffrey R. Kling, and Lisa Sanbonmatsu. Long-term

neighborhood effects on low-income families: Evidence from moving to opportunity. *American Economic Review*, 103(3):226–231, 2013.

[17] Hlavac Marek. stargazer: Well-Formatted Regression and Summary Statistics Tables., 2015.

[18] Douglas S Massey and Nancy A Denton. The Dimensions of Residential Segregation. *Social Forces*, 67(2):281–315, 1988.

[19] Sean F Reardon and G Firebaugh. Measures of multigroup segregation. *Sociological Methodology*, 32:33–67, 2002.

[20] Sean F. Reardon and David O'Sullivan. Measures of Spatial Segregation. *Sociological Methodology*, 34(1):121–162, 2004.

[21] Elizabeth Roberto. Measuring Inequality and Segregation. 2015.

[22] Elizabeth Roberto. The Spatial Context of Residential Segregation. *arXiv.org*, pages 1–27, 2015.

[23] Hernán D Rozenfeld, Diego Rybski, José S Andrade, Michael Batty, H Eugene Stanley, and Hernán a Makse. Laws of population growth. *Proceedings of the National Academy of Sciences of the United States of America*, 105(48):18702–18707, 2008.

[24] Hernán D Rozenfeld, Diego Rybski, Xavier Gabaix, and Hernán a Makse. The area and population of cities: New insights from a different perspective on cities. *American Economic Review*, 101(5):2205–2225, 2011.

[25] Robert J. Sampson, Jeffrey D. Morenoff, and Thomas Gannon-Rowley. Assessing "Neighborhoods Effects": Social Processes and New Directions in Research. *Annual Review of Sociology*, 28(2002):443–478, 2002.

[26] Michael J. Webber. *Information Theory and Urban Spatial Structure*. Croon Heml Ltd, London, 1979.

[27] David Wong. Comparing Traditional and Spatial Segregation Measures: A Spatial Scale Perspective. *Urban Geography*, 25(1):66–82, 2004.

[28] David W. S. Wong. Geostatistics As Measures of Spatial Segregation. *Urban Geography*, 20(7):635–647, 1999.

5 Appendix

Relation of mutual and Fisher informations

Let X be a continuous random variable taking values in $\mathbb{R}^n$, and let Y be a discrete random variable define on finite alphabet $\mathcal{Y}$. Suppose further that $p(y|x) > 0$ and that $p(y|x)$ is differentiable as a function of x for all $x, y \in \mathbb{R}^n \times \mathcal{Y}$. Fix $x_0 \in \mathbb{R}^n$, and define $B_r \triangleq B_r(x_0) = \{x \in R^n \mid \|x - x_0\| \leq$

$r\}$. Additionally, define the *local mutual information* in B_r as the mutual information between X and Y where X is restricted to B_r:

$$I_r(x_0) \triangleq \mathbb{E}_X[D[p(\cdot|X)\|p(\cdot|X \in B_r)]|X \in B_r] \tag{15}$$

$$= \int_{B_r} p(x|X \in B_r) D[p(\cdot|x)\|p(\cdot|X \in B_r)] d^n x \; . \tag{16}$$

where $D[p\|q] \triangleq \sum_y p(y) \log \frac{p(y)}{q(y)}$ is the Kullback-Leibler divergence of q from p.

Theorem 1. *Under the stated conditions,*

$$\lim_{r \to 0} \frac{I_r(x_0)}{r^2} = \frac{n}{2(n+2)} \text{trace } J_Y(x_0) \; . \tag{17}$$

where the Fisher information matrix J_Y is given by

$$J_Y(x) \triangleq \mathbb{E}_Y \left[\nabla_x S_Y(x) \nabla_x S_Y(x)^T \right] \tag{18}$$

$$S_y(x) \triangleq \log p(y|x) \; . \tag{19}$$

The proof of Theorem 1 proceeds by the application of a number of Taylor approximations, in tandem with a fundamental relationship of information geometry. We first expand out $I_r(x_0)$ explicitly as

$$I_r(x_0) = \int_{B_r} p(x|X \in B_r) D[p(\cdot|x)\|p(\cdot|X \in B_r)] d^n x \; . \tag{20}$$

Lemma 1. *The following approximation relationships hold for the components of* (20)*:*

(a) $p(X \in B_r, Y) = p(x_0, Y) v(B_r) + O(r^{n+2})$

(b) $p(Y|X \in B_r) = p(Y|x_0) + e_y$ *where the error terms* e_y *satisfy* $e_y \in O(r^2)$ *and* $\sum_{y \in \mathcal{Y}} e_y = 0$.

(c) $p(x|X \in B_r) = \frac{1 + O(r)}{v(B_r)}$

Proof. For each approximation, we directly apply Taylor expansions about $X = x_0$.

(a) We have

$$p(X \in B_r, Y) = \int_{B_r} p(x, Y) \; d^n x \tag{21}$$

$$= \int_{B_r} p(x_0, Y) + \frac{\partial p(x_0, Y)}{\partial x}(x - x_0) + O(\|x - x_0\|^2) \; d^n x \tag{22}$$

$$= p(x_0, Y) v(B_r) + \frac{\partial p(x_0, Y)}{\partial x} \int_{B_r} (x - x_0) \; d^n x \tag{23}$$

$$+ O\left(\int_{B_r} \|x - x_0\|^2 \; d^n x \right) \tag{24}$$

$$= p(x_0, Y) v(B_r) + O(r^{n+2}) \; , \tag{25}$$

where the middle term vanishes due to spherical symmetry.

(b) The fact that the error terms e_y must satisfy $\sum_{y\in\mathcal{Y}} e_y = 0$ follows from the fact that $p(Y|X \in B_r)$ must be a valid probability distribution over $\mathcal{Y}$. We'll now show that $e_y \in O(r^2)$. First,

$$p(X \in B_r) = \sum_{y\in\mathcal{Y}} p(X \in B_r, y) \tag{26}$$

$$= \sum_{y\in\mathcal{Y}} \left[p(x_0, y)v(B_r) + O(r^{n+2})\right] \tag{27}$$

$$= p(x_0)v(B_r) + O(r^2); \tag{28}$$

from part (a). Next,

$$p(Y|X \in B_r) = \frac{p(X \in B_r, Y)}{p(X \in B_r)} \tag{29}$$

$$= \frac{p(x_0, Y)v(B_r) + O(r^{n+2})}{p(x_0)v(B_r) + O(r^{n+2})} \tag{30}$$

$$= p(Y|x_0) + O(r^2)\ , \tag{31}$$

which completes this part of the argument.

(c) First,

$$p(X \in B_r) = \int_{B_r} p(x)\ d^n x \tag{32}$$

$$= \int_{B_r} \left[p(x_0) + \nabla p(x_0)(x - x_0) + O(r^2)\right]\ d^n x \tag{33}$$

$$= p(x_0)v(B_r) + O(r^{n+2})\ , \tag{34}$$

where the middle term again vanishes through spherical symmetry. Thus, for $x \in B_r$, we have

$$p(x|X \in B_r) = \frac{p(x)}{p(X \in B_r)} \tag{35}$$

$$= \frac{p(x_0) + \nabla p(x_0)(x - x_0) + O(r^2)}{p(x_0)v(B_r) + O(r^{n+2})} \tag{36}$$

$$= \frac{1 + O(r)}{v(B_r)}\ . \tag{37}$$

□

Lemma 2. *The following approximation holds for the divergence factor in the integral* (20)

$$D[p(\cdot|x)\|p(\cdot|X \in B_r)] = D[p(\cdot|x)\|p(\cdot|x_0)] + O(r^3) \tag{38}$$

Proof. We compute directly:

$$
\begin{aligned}
D[p(\cdot|x)\|p(\cdot|X\in B_r)] &= \sum_{y\in\mathcal{Y}} p(y|x)\log\frac{p(y|x)}{p(y|X\in B_r)} && (39)\\
&= -H[Y|X=x] - \sum_{y\in\mathcal{Y}} p(y|x)\log p(y|X\in B_r) && (40)\\
&= -H[Y|X=x] - \sum_{y\in\mathcal{Y}} p(y|x)\log\big(p(y|x_0)+e_y\big) \\
& && \text{(from Lemma 1)}\\
&= -H[Y|X=x] - \sum_{y\in\mathcal{Y}} p(y|x)\left[\log p(y|x_0) + \frac{e_y}{p(y|x_0)} + O(e_y^2)\right] && (41)\\
&= D[p(\cdot|x)\|p(\cdot|x_0)] + \sum_{y\in\mathcal{Y}} \frac{p(y|x)}{p(y|x_0)} e_y \\
& && \text{(quadratic terms negligible)}\\
&= D[p(\cdot|x)\|p(\cdot|x_0)] && (42)\\
&\quad + \sum_{y\in\mathcal{Y}} \left(1 + \frac{1}{p(y|x_0)}\nabla p(y|x_0)(x-x_0) + O(r^2)\right) e_y && (43)\\
&= D[p(\cdot|x)\|p(\cdot|x_0)] + \sum_{y\in\mathcal{Y}} \left[e_y + O(r^3)\right] \\
& && (e_y \in O(r^2))\\
&= D[p(\cdot|x)\|p(\cdot|x_0)] + O(r^3) && (\textstyle\sum_{y\in\mathcal{Y}} e_y = 0)
\end{aligned}
$$

□

Lemma 3. *For any positive-semidefinite matrix $A \in \mathbb{R}^{n\times n}$,*

$$\int_{B_r} \langle x - x_0, A(x-x_0)\rangle \, d^n x = \frac{n}{n+2} r^2 v(B_r)\mathrm{trace}(A)$$

Proof. Since A is positive-semidefinite, there exist an orthonormal matrix P and a diagonal matrix D such that $A = P^T D P$. Furthermore, the entries of D are the eigenvalues $\{\lambda_i\}$ of A. Then,

$$
\begin{aligned}
\int_{B_r} \langle x - x_0, A(x-x_0)\rangle \, d^n x &= \int_{B_r} \Big\langle x - x_0, P^T D P(x-x_0)\Big\rangle \, d^n x && (44)\\
&= \int_{B_r} \langle P(x-x_0), DP(x-x_0)\rangle \, d^n x \,. && (45)
\end{aligned}
$$

We can regard P as a reparameterization of B_r; since $\det P = 1$, we have

$$\int_{B_r} \langle P(x - x_0), DP(x - x_0) \rangle \, d^n x = \int_{B_r} \langle x - x_0, D(x - x_0) \rangle \, d^n x \tag{46}$$

$$= r^n \int_{B_n} \langle rx, rDx \rangle \, d^n x \tag{47}$$

$$= r^{n+2} \int_{B_n} \langle x, Dx \rangle \, d^n x \ , \tag{48}$$

where B_n is the unit n-ball. We also let $S_n(r)$ be the n-sphere of radius r. Continuing,

$$r^{n+2} \int_{B_n} \langle x, Dx \rangle \, d^n x = r^{n+2} \int_{B_n} \sum_{i=1}^n x_i^2 \lambda_i \, d^n x \tag{49}$$

$$= r^{n+2} \sum_{i=1}^n \lambda_i \int_{B_n} x_i^2 \, d^n x \tag{50}$$

$$= \frac{r^{n+2}}{n} \sum_{i=1}^n \lambda_i \int_{B_n} \|x\|^2 \, d^n x \tag{51}$$

(spherical symmetry)

$$= \frac{r^{n+2}}{n} \text{trace}(A) \int_{B_n} \|x\|^2 \, d^n x \tag{52}$$

(spherical symmetry)

$$= \frac{r^{n+2}}{n} \text{trace}(A) \int_{\rho \in [0,1]} \rho^2 S_{n-1}(\rho) d\rho \tag{53}$$

$$= \frac{r^{n+2}}{n} \text{trace}(A) \int_{\rho \in [0,1]} \rho^{n+1} S_{n-1}(1) d\rho \tag{54}$$

$$= \frac{r^{n+2}}{n} \text{trace}(A) \frac{1}{n+2} S_{n-1}(1) \tag{55}$$

$$= \frac{r^2}{n+2} \text{trace}(A) n r^n v(B_n(1)) \tag{56}$$

$$= \frac{n}{n+2} r^2 v(B_r) \text{trace}(A) \ , \tag{57}$$

as was to be shown. □

Fact. *The Kullback-Leibler divergence and the Fisher information J_Y are related according to the approximation*

$$D[p(\cdot|x) \| p(\cdot|x_0)] = \frac{1}{2} \langle x - x_0, J_Y(x_0)(x - x_0) \rangle + O(\|x - x_0\|^3) \tag{58}$$

We are finally ready to prove Theorem 1. Computing directly, we have

$$I_r(x_0) \triangleq \mathbb{E}_X[D[p(\cdot|X)\|p(\cdot|X \in B_r)]|X \in B_r] \,. \tag{59}$$

$$= \int_{B_r} p(x|X \in B_r) D[p(\cdot|x)\|p(\cdot|X \in B_r)] d^n x \tag{60}$$

$$= \int_{B_r} \left[\frac{1+O(r)}{v(B_r)}\right] D[p(\cdot|x)\|p(\cdot|X \in B_r)] d^n x \qquad \text{(Lemma 1(c))}$$

$$= \left[\frac{1+O(r)}{v(B_r)}\right] \int_{B_r} \left(D[p(\cdot|x)\|p(\cdot|x_0)] + O(r^3)\right) d^n x \qquad \text{(Lemma 2)}$$

$$= \left[\frac{1+O(r)}{v(B_r)}\right] \int_{B_r} \left(\frac{1}{2}\left\langle x - x_0, J_Y(x_0)(x-x_0)\right\rangle + O(\|x-x_0\|^3) + O(r^3)\right) d^n x \tag{61}$$

$$= \frac{1}{2}\left[\frac{1+O(r)}{v(B_r)}\right] \int_{B_r} \left(\left\langle x - x_0, J_Y(x_0)(x-x_0)\right\rangle + O(r^3)\right) d^n x \tag{62}$$

$$= \frac{1}{2}\left[\frac{1+O(r)}{v(B_r)}\right] \left(\frac{n}{n+2} r^2 v(B_r) \text{trace}(J_Y(x_0)) + v(B_r) O(r^3)\right) \tag{63}$$

$$= r^2 \frac{n}{2(n+2)} \left[1 + O(r)\right] \left(\text{trace}(J_Y(x_0)) + O(r^3)\right) \,. \tag{64}$$

Dividing through by r^2 and computing the limit as $r \to 0$ proves the result.

Computational Methods and Assumptions

In this section, we provide a specification of the computational procedure used to estimate $J(X, Y) = \mathbb{E}_x[J_Y(X)]$ using blockgroup level data from the U.S. Census.

For fixed Census blockgroup i, let P_i be the population, let A_i be the area, let $\rho_i = P_i / A_i$ be the population density, and let $p_Y^i(y)$ be the observed proportion of racial group y. For hex k in our hexagonal grid, let N_k be the set of overlapping Census blockgroups. We also define $p_I^k(i) = \rho_i / \sum_{i \in N_k} \rho_i$ as the estimated proportion of population within hex k residing in blockgroup i. This definition embodies a computationally-simplifying assumption that each blockgroup in N_k overlaps hex k with equal area. Finally, $p_Y^k(y) = \sum_{i \in N_k} p_I^k(i) p_Y^i(y)$ is the estimated overall racial composition of hex k. Then, we estimate the mutual information in hex k as

$$I(k) = \sum_{i \in N_k} p_I^k(i) D[p_Y^i(\cdot)\|p_Y^k(\cdot)] \,. \tag{65}$$

Using (**??**), the estimated Fisher information is

$$J(k) \approx \frac{4I(k)}{r^2} \tag{66}$$

where r is the grid radius. The estimated population in hex k is $P_k = A_k \sum_{i \in N_k} \rho_i$, where A_k is the cell area. We finally estimate $\mathbb{E}_X[J(X)]$ as

$$J(X, Y) = \mathbb{E}_X[J_Y(X)] \approx \frac{1}{\sum_k P_k} \sum_k P_k J(k) \tag{67}$$

5.1 Specification of Spatially-Constrained Information-Theoretic Clustering