Multi Agent Systems

Homework Assignment 2

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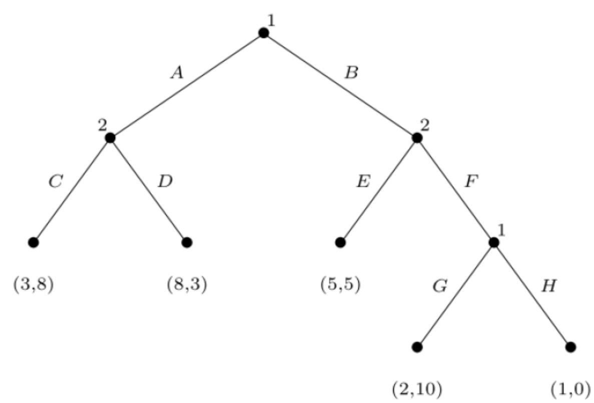
3 Sequential Games

3.1 Sequential Game with perfect information

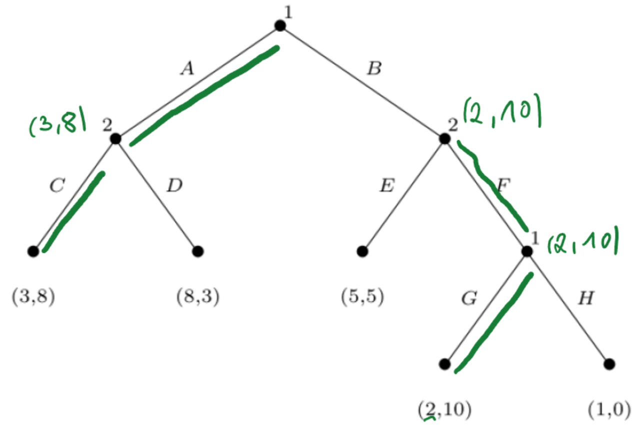
Consider the following game tree:



Solve by backward induction

Following the strategy discussed in the lecture:
Consider each SG, Find NE for SG, replace the SG by a new terminal node that has the same equilibrium pay offs. One arrives at:

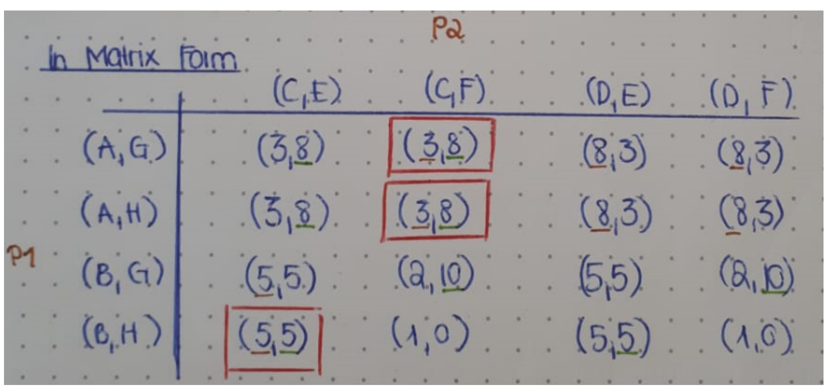
Consider the following game tree:



According to that, the solution provided by Backward Induction is given by $\{(A, G), (C, F)\}$ and leads to payoffs (3,8).

Write in matrix form and find all Nash equilibria

It can be transformed in to Matrix form the following way:



In Matrix Form

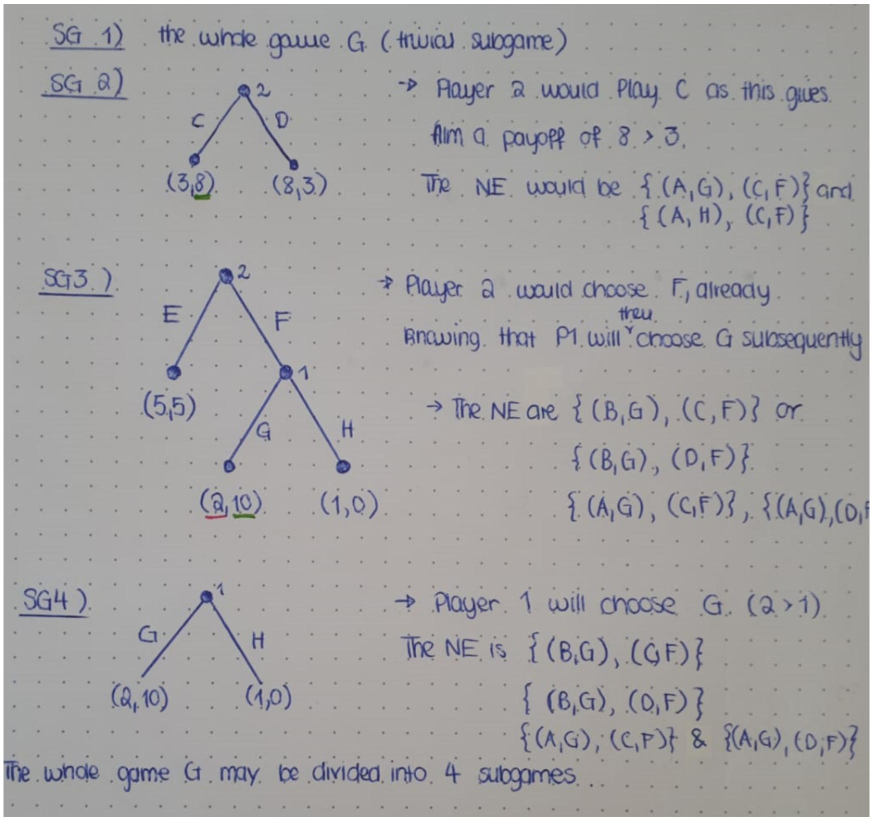
P2

P1	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	(3,8)	(3,8)	(8,3)	(8,3)
(A,H)	(3,8)	(3,8)	(8,3)	(8,3)
(B,G)	(5,5)	(2,10)	(5,5)	(2,10)
(B,H)	(5,5)	(1,0)	(5,5)	(1,0)

$\Rightarrow$ Every finite, complete information game hat at least one PNE.
The whole game has 3 PNE:

1. $\{(A, G), (C, F)\}$
2. $\{(A, H), (C, F)\}$
3. $\{(B, H), (C, E)\}$

Find all possible subgames and find the subgame-perfect equilibrium



Thus, equilibrium (2) is not subgame perfect, as H is not the NE in SG4, but G is. Equilibrium (3) is not subgame perfect either, as E is not part of the NE in SG3. This leaves only equilibrium (1). As $\{(A, G), (C, F)\}$ is a NE in all 4 Subgames, it is a subgame perfect equilibrium.

3.2 Sequential Game with imperfect information

Can we solve this game with BI, if affirmative what is the solution?

This game is a simultaneous game and according to the lecture slides, Backward induction only works for finite horizon and sequential games with perfect information. However, this game can still be solved by BI, as Player 2 will always go for action r. First, each subgame of the game is checked for NE, then the according payoffs serve as the next terminal node. This is done until the initial node is reached:

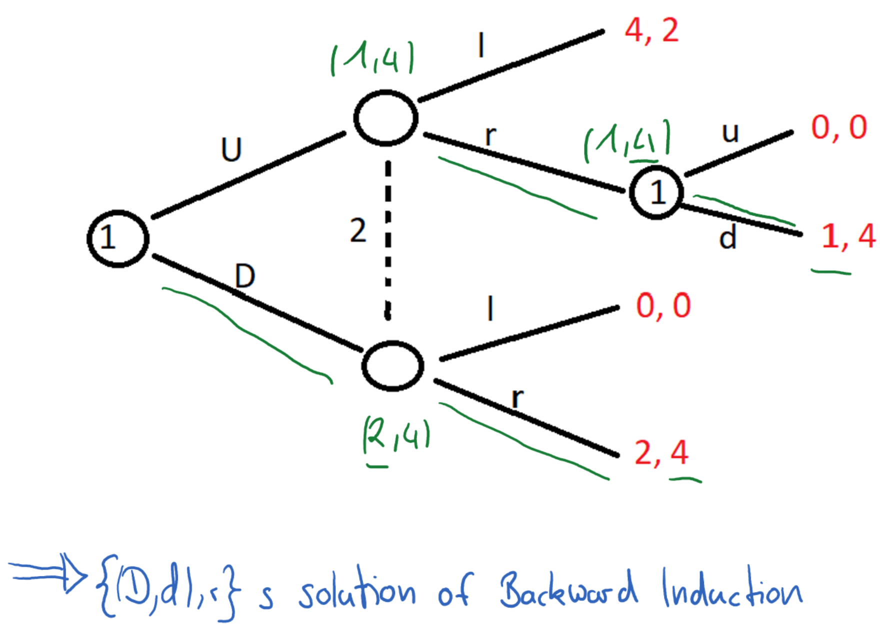


Figure 1:

Following the strategy from above, the final winner strategy is $\{(D,d),(r)\}$ with respective payoff(2,4).

Identify all possible subgames

Similar to the lecture, subgames are circled in the according game plan:

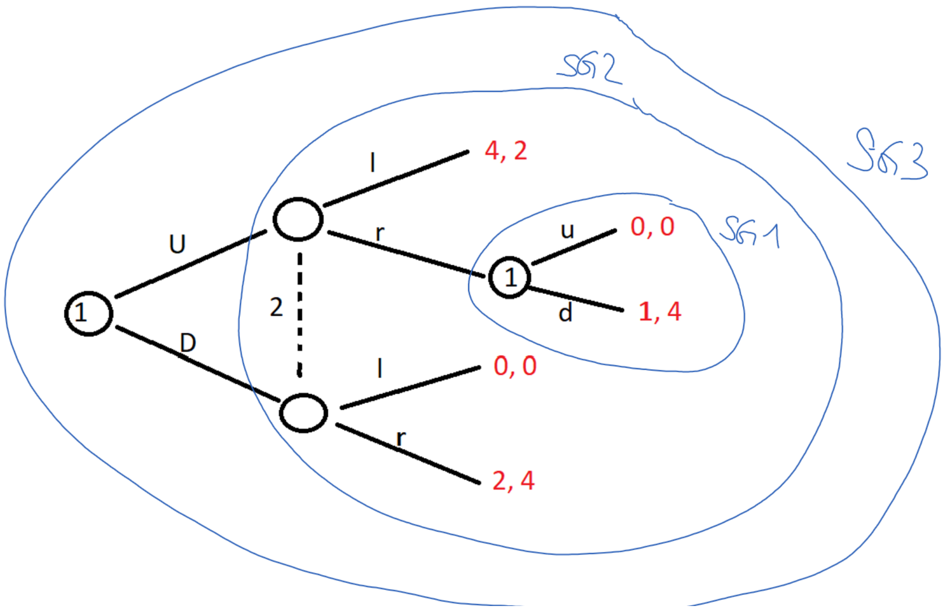


Figure 2:

Rewrite the game in normal form and identify all pure Nash equilibria

Therefore, the Game tree is transformed into a Matrix Game as follows: There are three

P2

P1		l	r
	Uu	(4,2)	(0,0)
	Ud	(4,2)	(1,4)
	Du	(0,0)	(2,4)
	Dd	(0,0)	(2,4)

Table 1:

pure Nash equilibria found, which are marked with circles:

		P2	
		l	r
P1	Uu	(4,2)	(0,0)
	Ud	(4,2)	(1,4)
	Du	(0,0)	(2,4)
	Dd	(0,0)	(2,4)

Figure 3:

So, $\{Uu, l\}$, $\{Du, r\}$ and $\{Dd, r\}$ are the respective NE's

Which NE is subgame perfect?

Looking at Figure 1 reveals that the NE (4,2) caused by $\{Uu, l\}$ is not an NE in the Subgame 2 declared in Figure 2. Thus, this NE is not subgame perfect. On the other hand (2,4) is achieved by $\{Du, r\}$ and $\{Dd, r\}$. As the NE of Subgame 1 is $\{d\}$, $\{Du, r\}$ can not be subgame perfect either because of the d. This leaves only one subgame perfect strategy: $\{Dd, r\}$.

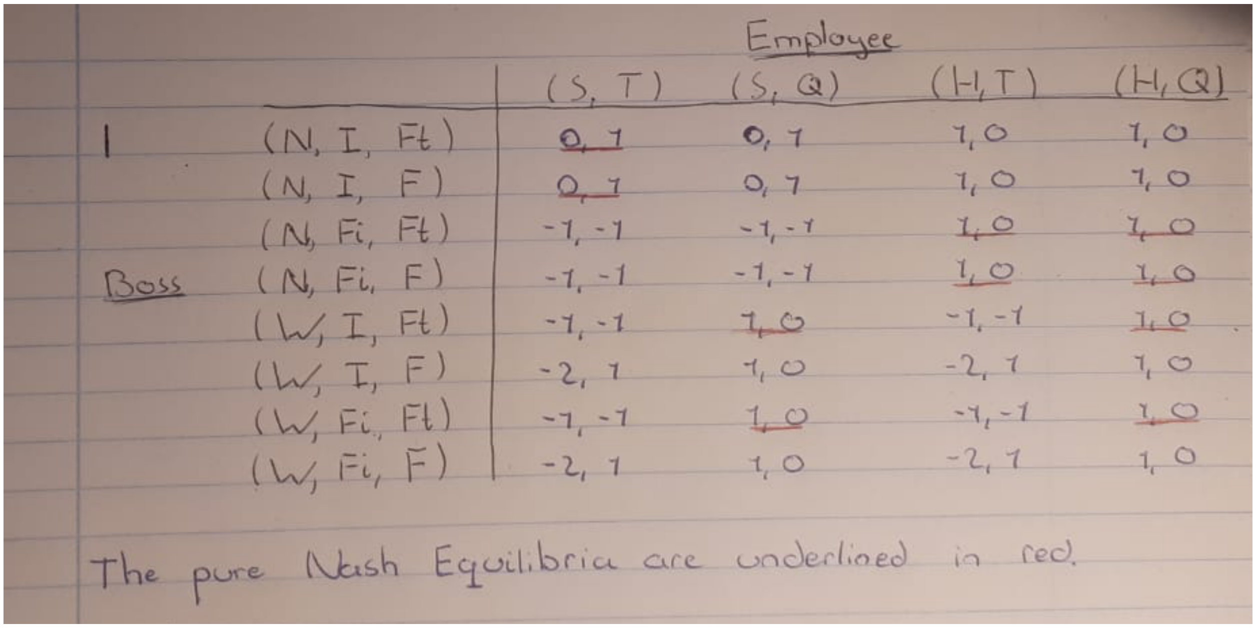
3.3 Boss and stealing employer

What are the pure actions for the two players (boss and employee)? Construct the normal form matrix. Use this matrix to identify all the pure Nash equilibria of the normal form game.

The pure actions for the Boss are: N, W, I, Fi, F, Ft

The pure actions for the Employee are: S, T, H, Q

See Figure 4 for the normal form matrix and pure Nash equilibria.



		Employee			
		(S, T)	(S, Q)	(H, T)	(H, Q)
Boss	(N, I, Ft)	0, 1	0, 1	1, 0	1, 0
	(N, I, F)	0, 1	0, 1	1, 0	1, 0
	(N, Fi, Ft)	-1, -1	-1, -1	1, 0	1, 0
	(N, Fi, F)	-1, -1	-1, -1	1, 0	1, 0
	(W, I, Ft)	-1, -1	1, 0	-1, -1	1, 0
	(W, I, F)	-2, 1	1, 0	-2, 1	1, 0
	(W, Fi, Ft)	-1, -1	1, 0	-1, -1	1, 0
	(W, Fi, F)	-2, 1	1, 0	-2, 1	1, 0

The pure Nash Equilibria are underlined in red.

Figure 4: The normal form matrix

Determine the subgame-perfect equilibrium (equilibria?) by eliminating all the Nash equilibria that fail to induce a NE in subgames

In Figure 5 we see the sub games and the sub game perfect equilibriums for each sub-game.

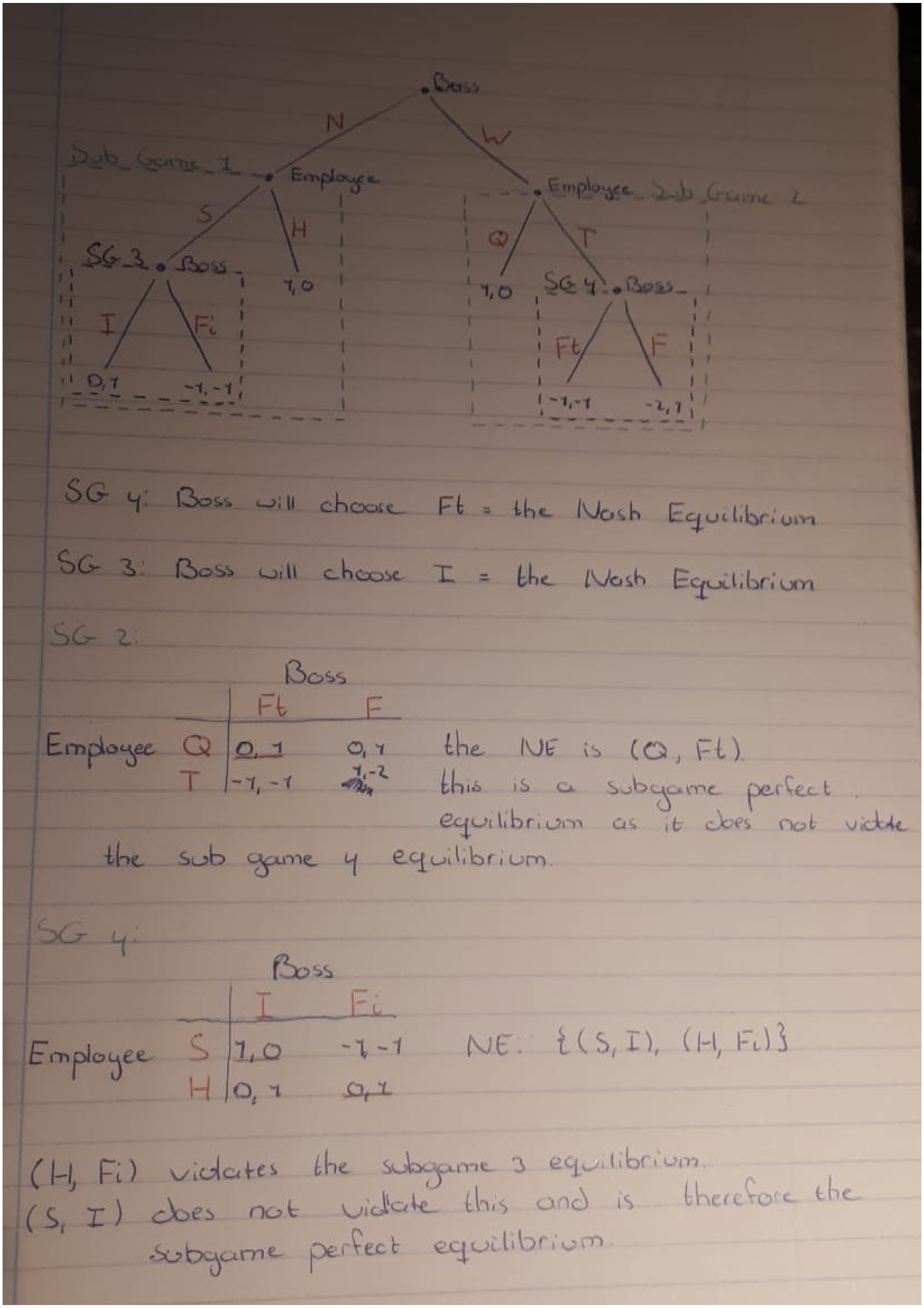


Figure 5: Sub game equilibriums

Next, for each Nash equilibrium we will check if it is sub game perfect:

($NIFt$,ST) violates sub game 2 and is therefore not perfect.
(NIF,ST) violates sub game 2 and is therefore not perfect.
($NFiFt$,HT) violates sub game 3 and is therefore not perfect.
($NFiFt$,HQ) violates sub game 3 and is therefore not perfect.
($NFiF$,HT) violates sub game 4 and is therefore not perfect.
($NFiF$,HQ) violates sub game 4 and is therefore not perfect.
($WIFt$**,**SQ) **does not violate a sub game and is therefore the subgame perfect equilibrium!**
($WIFt$,HQ) violates sub game 1 and is therefore not perfect.
($WFiFt$,SQ) violates sub game 3 and is therefore not perfect.
($WFiFt$,HQ) violates sub game 3 and is therefore not perfect.

Solve the same problem using backward induction. Do you arrive at the same conclusion?

See Figure 6 for the backwards induction. As you can see it gives the same conclusion as in 3.3.3

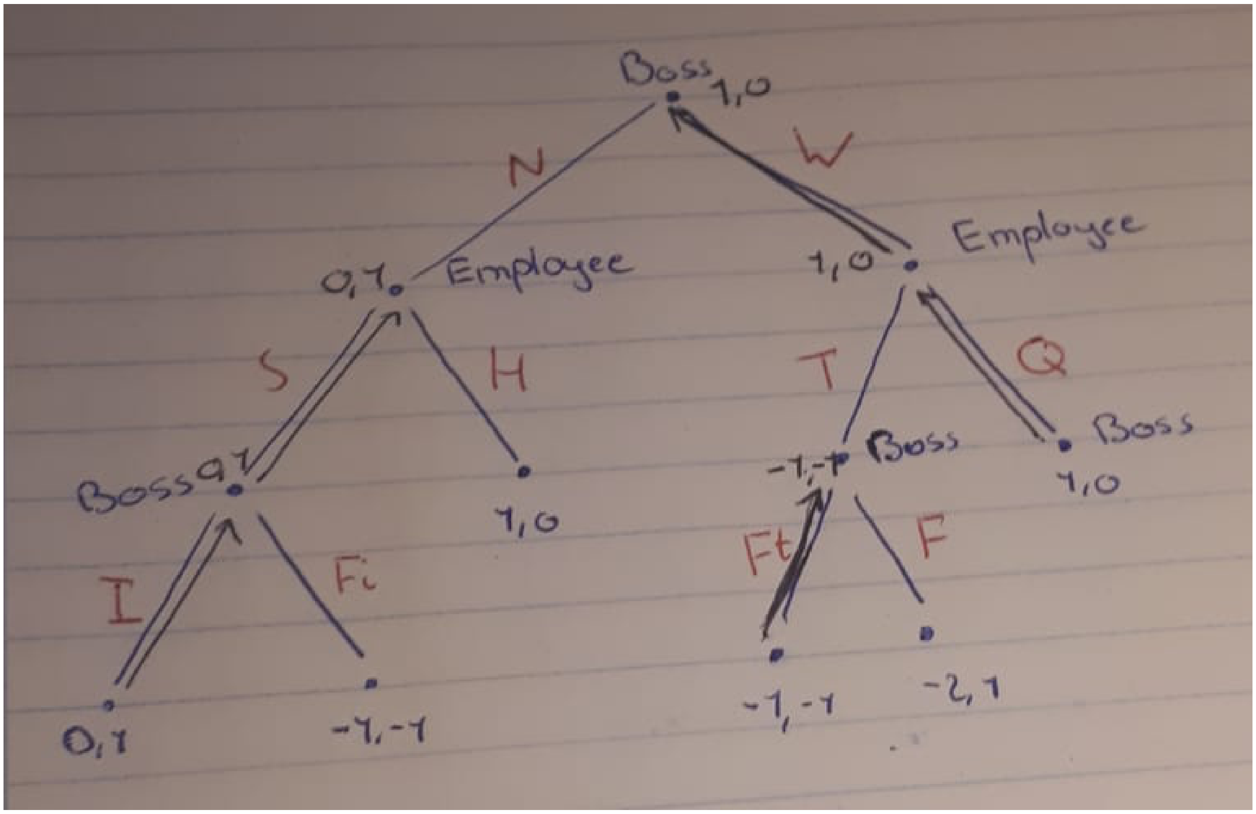


Figure 6: Backward Induction

3.4 Stackelbergs Duopoly Model

Use backward induction to determine the optimal quantities for both firms

In order do derive the backward induction solution, the Best response of firm 2 is calculated in dependence of all the constants:

$$u_2(q_1, q_2) = q_2(a - b(q_1 + q_2)) - c \cdot q_2$$

$$\Rightarrow \frac{du_2}{dq_2} = a - b \cdot q_1 - 2q_2 \cdot b - c \stackrel{!}{=} 0$$

$$<=> q_2^\star = \frac{a - b \cdot q_1 - c}{2b}$$

So, $q_2^\star$ is the best response of q_2. Now, this is plugged into the utility of firm 1 in order to find the respective q_1 maximizing their utility.

$$u_2(q_1, q_2^\star) = q_1 \cdot (a - b(q_1 + q_2^\star)) - c \cdot q_1$$

$$= q_1 \cdot \left(a - b\left(q_1 + \left(\frac{a - b \cdot q_1 - c}{2b}\right)\right)\right) - c \cdot q_1$$

$$<=> \frac{du_2(q_1, q_2^\star)}{dq_1} = \frac{a}{2} - \frac{c}{2} - b \cdot q_1 \overset{!}{=} 0$$

$$<=> q_1 = \frac{a - c}{2b}$$

Then taking this q_1 value and plugged into the Best response of firm 2 yields:

$$q_2 = \frac{a}{2b} - \frac{a - c}{4b} - \frac{c}{2b}$$

So we solved the game with backwards induction obviously following the conditions for a,b and c.

Compare your results to the once obtained for the Cournot (simultaneous) model. Is there a first mover advantage?

In Assignment 1, a was 1000 and b=1. If plugged into our results of the Stackelbergs model, one obtains:

$$q_1 = \frac{1000 - c}{2}$$

$$q_2 = 500 - \frac{1000 - c}{2} - \frac{c}{2}$$

$$= 250 - \frac{c}{4}$$

If we then plug those into both utility functions and u_1 is always larger than u_2 independent of c, then there is a first mover advantage. Thus,

$$u_1(q_1, q_2) = \left(\left(a - b\left(\frac{1000-c}{2} + \left(250 - \frac{c}{4}\right)\right)\right) \cdot \left(\frac{1000-c}{2}\right)\right) - c \cdot \left(\frac{1000-c}{2}\right)$$

$$= \ldots$$

$$= \frac{1}{8}(c-1000)^2$$

and

$$u_2(q_1, q_2) = \left(\left(a - b\left(\frac{1000-c}{2} + \left(250 - \frac{c}{4}\right)\right)\right) \cdot \left(250 - \frac{c}{4}\right)\right) - c \cdot \left(250 - \frac{c}{4}\right)$$

$$= \ldots$$

$$= \frac{1}{16}(c-1000)^2.$$

As $\frac{1}{8}(c-1000)^2 > \frac{1}{16}(c-1000)^2$ and thus $u_1(q_1, q_2) > u_2(q_1, q_2)$, there is a first mover advantage within the Stackelbergs Duopoly.