Multi Agent Systems

Homework Assignment 1

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1 Game Theory: Concepts

1.1 Zero-sum game with infinite action space

In a zero sum game with infinite action space and two agents A and B, the respective utiliy of Agent A is given by:

$$u_A(x, y) = -x^2 - y^2 - 4xy + 4x + 2y - 2$$

The question now was to identify, if possible, a minimax/maximin solution. As this is a zero-sum-game they are, if possible, the same solution.

Answer / solution

Def.: minimax solution for agent A of a game:

- Given the strategy of his opponent, agent A will play hist **best response**:

$$\underset{x}{max}\, u_A(x, y)$$

- the opponent is aware and solely wants to punish A by minimizing his strategy:

$$\underset{y}{min}\ \underset{x}{max}\, u_A(x, y)$$

Bearing that definition in mind, at first the best response for agent A is sought by deriving the utility function according to x and setting it equal to zero:

$$\frac{du_A(x, y)}{dx} = -2x - 4y + 4 \overset{!}{=} 0$$
$$<=> \quad x = -2y + 2$$

With that best response, the second part of the above mentioned minimax solution is given by now minimizing for y after inserting the best response into the original utility function:

$$\begin{aligned} u_A(y) &= -(-2y+2)^2 - y^2 - 4(-2y+2)y + 4(-2y+2) + 2y - 2 \\ &= 3y^2 - 6y - 6 \end{aligned}$$

Deriving it for y:

$$\begin{aligned} \frac{du_A(y)}{dy} &= 6y - 6 \stackrel{!}{=} 0 \\ <=> \quad & y = 1 \end{aligned}$$

Plugging this into the best response of x yields

$$x = -2 \cdot 1 + 2 = 0 \ \rightarrow x = 0$$

So the minimax and in this case also maxmimin solution is given by (x,y)=(0,1).
The exercise also asks for the value of the game, which for zero sum games is the common value of minimax and maximin, so for agent A:

$$\begin{aligned} u_A(0,1) &= -(-1)^2 + 2 \cdot 1 - 2 \\ &= -1 \end{aligned}$$

1.2 Volunteers Dilemma

Pay-off matrix: In general, they both prefer the other one to clean up without cleaning

		Bob	
		clean	home
Alice	clean	1-c, 1-c	1-c, 1
	home	1, 1-c	0,0

up themselves. That is, why two pure strategy NE exist: (clean, home) and (home, clean).
So knowing the other one is cleaning, there is no incentive to then clean.
However, if the player cleaning in the game decides to stay home too, the playground remains dirty, and both earn 0.
As said in the slides, there is proof that all games have an odd number of equilibria. That is, why mixed strategies should be considered.
Starting with Alice's expected utility for cleaning:
Let q denote the probability for Bob to clean. That way:

$$\begin{aligned} Alice: \quad EU_{clean} &= 1 - c \\ EU_{home} &= q \cdot 1 + (1-q) \cdot 0 = q \end{aligned}$$

		Bob	
		clean (q)	home (1-q)
Alice	clean (p)	1-c, 1-c	1-c, 1
	home (1-p)	1, 1-c	0,0

Now, these expected utilities are set equal, allowing to solve for q:

$$EU_{clean} = EU_{home}$$
$$1 - c = q$$

As the game is symmetric, same holds for p. So in accordance with the mixed strategy NE, both players clean with probability 1-c and stay home with probability c. In that scenario, everthing depends on c.

Looking at the probability for the playground to remain dirty, one arrives at c^2. So, small c values increase chances for the playground to be cleaned, by whomever. The other way around, high c values increase the probability for the playground not to be cleaned at all.

1.3 Tragedy of the commons

Given

- n players; common source of size 1
- player i's share: x_i; $0 \leq x_i \leq 1$
- player i's pay-off/ utility: $u_i(x_i, x_{-\mathbf{i}}) = \begin{cases} x_i(1 - \sum_{j=1}^{n} x_j), & \text{if } \sum_j x_j < 1. \\ 0, & \text{otherwise.} \end{cases}$

Def.: Stable Strategy (= Nash equilibrium)

$x^\star = (x_1^\star, \ldots, x_n^\star)$ is a stable strategy, if $\forall$ players i, $\forall$ strategies $x_i' \neq x_i^\star : u(x_i^\star, x_{-i}^\star) > u(x_i', x_{-i}^\star)$

$\rightarrow$ A No regret / Self enforcing / strict NE is a stable strategy profile for which no agent has an incentive to unilaterally deviate:

Answer / solution

In order to find the stable strategy (NE), the best responses of each player need to be considered. Therefore, the utilty / pay-off of player i can be written as:

$$u_i(x_i) := max(0, x_i(1 - \sum_{j=1}^{n} x_j))$$
$$= x_i(1 - \sum_{j=1}^{n} x_j),$$

as the second part is always greater than zero. For the best response of agent i, $i \in \{1, \dots, n\}$ holds. Also, all other strategies according to which i finds a best response are denoted as s_{-i}. In terms of the sum that means $\sum_{i \neq j}^{n} x_j$. Thus, the utility is given by $u_i(x_i, x_{-i}) = x_i(1 - (\sum_{i \neq j}^{n} x_j + x_i)$. The best response for i is found by

$$\frac{d(\, x_i - x_i \cdot \sum_{i \neq j}^{n} x_j - x_i^2)}{dx_i} = 1 - \sum_{i \neq j}^{n} x_j - 2 \cdot x_i \stackrel{!}{=} 0$$

$$<=> x_i = \frac{1 - \sum_{i \neq j}^{n} x_j}{2}$$

So, pay-off is maximal for the shown x_i. The Nash equilibrium is then reached, when $\forall\ i \in \{1, \dots, n\}$ the above strategy x_i holds. As we now suppose all of the x_i to be equal, the following can be done

- There is no need to distinguish between j and i anymore, as all strategies are the same: $\sum_{i \neq j}^{n} x_j = \sum_{i=1}^{n-1} x_i$
- allowing to be inserted in here as such

$$\begin{aligned} x_i &= \frac{1 - \sum_{i \neq j}^{n} x_j}{2} \\ &= \frac{1 - \sum_{i=1}^{n-1} x_i}{2} \\ &= \frac{1 - (n-1) \cdot x_i}{2}, \end{aligned}$$

which can be solved for x_i as follows:

$$\begin{aligned} x_i &= \frac{1 - (n-1) \cdot x_i}{2} \\ &\to 2x_i = 1 - (n-1)x_i \\ &\to (n+1)x_i = 1 \\ &\to x_i = \frac{1}{n+1} \end{aligned}$$

So this solution holds for all agents $i \in \{1, \dots, n\}$. Each player's according pay-off then results as

$$\begin{aligned} u_i(x_i) &= x_i(1 - \sum_{j=1}^{n} x_j) \\ &= \frac{1}{n+1} \cdot (1 - n \cdot \frac{1}{n+1}) \\ &= \frac{1}{n+1} - \frac{n}{(n+1)^2} \\ &= \frac{n+1}{(n+1)^2} - \frac{n}{(n+1)^2} \\ &= \frac{1}{(n+1)^2} \end{aligned}$$

Taking this pay-off n times yields the social welfare for that stable strategy:

$$\sum_i u_i(x_i) = \frac{n}{(n+1)^2} \tag{1}$$

In order to now prove that this is **not optimizing** social welfare, it can be bound. So for

n=1:

$$\sum_{i=1}^{n} u_i(x_i) = \frac{1}{4}$$

The social welfare of (1) is a monotonically decreasing function in n. It can be shown by applying quotient rule, as the derivatives numerator is always negativ and its denominator always positiv making the whole derivative negative:

$$\begin{aligned} \frac{d\frac{n}{(n+1)^2}}{dn} &= \frac{1 \cdot (n+1)^2 - n \cdot 2(n+1)}{(n+1)^4} \\ &= \frac{-n^2}{(n+1)^4} \end{aligned}$$

So for $n > 1$, it can be said that:

$$\frac{n}{(n+1)^2} < \frac{1}{4}$$

As an example, there exist straegies, which lead to larger social welfare. E.g. $x_1 = \frac{1}{2}, x_2 = 0 \ldots, x_n = 0$. Then the social welfare is $\sum_{j=1}^{n} x_j = \frac{1}{2}$, which is larger than the stable's social welfare.

1.4 Cournot's Duopoly

- quantity to be produced by firm 1: q_1
- quantity to be produced by firm 2: q_2
- market price per unit: $P = 1000 - (q_1 + q_2)$
- cost per unit firm 1: $c_1 = 10$
- cost per unit firm 2: $c_2 = 20$

Following utility functions can be formulated:

Utility firm 1:

$$\begin{aligned} U_1(q_1, q_2) &= q_1 \cdot (1000 - (q_1 + q_2)) - q_1 \cdot c_1 \\ &= q_1 \cdot (1000 - (q_1 + q_2)) - q_1 \cdot 10 \\ &= -q_1^2 + 990q_1 - q_1 \cdot q_2 \end{aligned}$$

Utility firm 2:

$$\begin{aligned} U_2(q_1, q_2) &= q_2 \cdot (1000 - (q_1 + q_2)) - q_2 \cdot c_2 \\ &= q_2 \cdot (1000 - (q_1 + q_2)) - q_2 \cdot 20 \\ &= -q_2^2 + 980q_2 - q_1 \cdot q_2 \end{aligned}$$

Best response = maximum utility dependent on other firms action. (BR_i derive for q_i and set 0)

Best response for firm 1:

$$\begin{aligned} \frac{d\,U_1(q_1, q_2)}{d\,q_1} &= 990 - 2 \cdot q_1 - q_2 \stackrel{!}{=} 0 \\ \rightarrow q_1 &= -\frac{1}{2}(q_2 - 990) \\ &= 495 - \frac{1}{2}q_2 \end{aligned}$$

Best response for firm 2:

$$\begin{aligned} \frac{d\,U_2(q_1, q_2)}{d\,q_2} &= 980 - 2 \cdot q_2 - q_1 \stackrel{!}{=} 0 \\ \rightarrow q_2 &= -\frac{1}{2}(q_1 - 980) \\ &= 490 - \frac{1}{2}q_1 \end{aligned}$$

As it is a **finite strategic game**, there exists at least one NE according to Nash's Theorem (slides).
To find the NE, the two utility functions must be set equal. To obtain a solable system of equations, the respective best responses for both firms are filled in to get rid of the second variable. Thus, setting both functions, which are only dependent on one variable, equal leads to two values q_1 and q_2, which mark the NE.

$$U_1(q_1, q_2) = -q_1^2 + 990q_1 - q_1 \cdot q_2 \qquad with\, BR_1(q_2) = 495 - \frac{1}{2}q_2 \tag{2}$$

$$U_2(q_1, q_2) = -q_2^2 + 980q_2 - q_1 \cdot q_2 \qquad with\, BR_2(q_1) = 490 - \frac{1}{2}q_1 \tag{3}$$

In order to find the Nash equilibrium in this case, the pair (q_1, q_2) fulfilling the property

$$q_1 = BR_1(q_2) \; and \; q_2 = BR_2(q_1)$$

is sought.
That is

$$q_1 = 495 - \frac{1}{2}q_2 \; and \; q_2 = 490 - \frac{1}{2}q_1.$$

Inserting the one into the other yields:

$$q_1 = 495 - \frac{1}{2} \cdot (490 - \frac{1}{2}q_1)$$
$$= 250 \cdot \frac{4}{3} = 333.\overline{33}$$

Inserting into q_2 yields:

$$q_2 = 490 - \frac{1}{2} \cdot \frac{1000}{3} = \frac{970}{3} = 323.\overline{33}$$

Accounting that firms can only produce whole units, the Nash equilibrium is given by the pair $(333, 323)$, which results in the profit for firm 1 as

$$U_1(q_1, q_2) = 333 \cdot (1000 - (333 + 323)) - 333 \cdot 10 = 111222$$

and for firm 2 as

$$U_2(q_1, q_2) = 323 \cdot (1000 - (333 + 323)) - 323 \cdot 20 = 104652$$

1.5 Ice cream time!

- Ice cream vendors: Alice (A), Bob (B) and Charlize (C)
- Total length of beach: 1
- beach position parameter: $0 \leq x \leq 1$
- uniformly distributed amount of tourists

1.5.1 Question 1

- position of (A): $a = 0.1$
- position of (B): $b = 0.8$

Solution/Best response of (C):

The best spot for C would be anywhere between a and b. It does not matter at which exact place, C positions herself in between a and b. C should locate in any position c, so that $0.1^+ \leq c \leq 0.8^-$.

Explanations:

- 0.1^+: just right of $a = 0.1$
- 0.8^-: just left of $b = 0.8$

Proof

For $0 \leq c < a = 0 \leq c_s < 0.1$, the utility function of C is defined as:

$$u_c(a,b) = u_c(0.1, 0.8) = \frac{0.1 - c_s}{2} < 0.05$$

$\Rightarrow \forall c_s$, the utility of C is between 0 and less than 0.05.

For $a < c < b$, the utility function of C may be defined as:

$$u_c(a,b) = \frac{c-a}{2} + \frac{b-c}{2} \quad = \quad u_c(0.1, 0.8) = \frac{c-0.1}{2} + \frac{0.8-c}{2}$$

Choosing 3 arbitrary values for c, it can be shown that C's utility is equal $\forall c : a < c < b$.

Choose $c_1 = 0.1^+$, $c_2 = 0.45$ and $c_3 = 0.8^-$:

$\rightarrow u_{c1}(0.1, 0.8) = \frac{0.1^+ - 0.1}{2} + \frac{0.8 - 0.1^+}{2} = 0.35$

$\rightarrow u_{c2}(0.1, 0.8) = \frac{0.45 - 0.1}{2} + \frac{0.8 - 0.45}{2} = 0.35$

$\rightarrow u_{c3}(0.1, 0.8) = \frac{0.8^- - 0.1}{2} + \frac{0.8 - 0.8^-}{2} = 0.35$

$\Rightarrow$ Any position between a and b leaves C with a utility of 0.35. Therefore any position in between a and b is an equally well choice/positioning strategy.

For $c > b$ / $0.8 < c_u \leq 1$, the utility function of C is given as

$$u_{cu} = \frac{u_{cu} - 0.8}{2}$$

This leaves C with a ultility somewhere between 0 and 0.1.

$\Rightarrow$ As the utility $0.05 < 0.1 < 0.35$, it is proven, that C's best response is anywhere between a and b.

1.5.2 Question 2

Given:

- $a = 0.1$

- $a < b \leq 1$

Solution

Define h as the position, where for the utility u_h it holds that $u_{h^-} = u_{h^+}$.

Explanations:

- u_{h^-} = C's utility left of h
- u_{h^+} = C's utility right of h

At this position h, it holds that:

$$\begin{aligned} & \frac{h-a}{2} = 1-h \\ \Leftrightarrow \quad & h - a = 2 - 2h \\ \Leftrightarrow \quad & h - 0.1 = 2 - 2h \\ \Leftrightarrow \quad & 3h = 2.1 \\ \Leftrightarrow \quad & h = \frac{2.1}{3} = 0.7 \end{aligned}$$

Answer

The best position for C is $h = 0.7$. If $h < b$, the minimal utility is 0.3 and also if $h > b$, the minimal utility is 0.3. h maximizes the worst ces outcome for C's utility. As such, it is her best response.

1.5.3 Question 3

Solution

The best strategy for B is to place his stall at $b = h = 0.7$. If B places his stall there, then C would place herself just right of b or anywhere in between a and b.

- For $c = b^+ : u_c = 1 - 0.7 = 0.3$
- For $a < c < b : u_c = \frac{0.7-0.1}{2} = 0.3$

Choosing $b = 0.7$ is B's Maximin strategy. Wherever C places her stall to maximize her own revenue, B's utility can never be less than 0.3. For any other b, Bob would have a lower worst-case-revenue.

1.5.4 Question 4

Solution

If A arrives at the beach before B and C, it is not the wisest decision to place her stall at $a = 0.1$.

Depending on whether B (who i assumed to arrive second) knows about C and her will to maximize her own revenue, there are two alternative solutions.

Alternative 1: *The second vendor B does not know that C is still to come.*

In this case it is wisest for A to place herself left or right from the middle point 0.5. Assume she would choose $a_1 = 0.49$. Then B would come to the beach and place his stall right in the middle at $b = 0.5$. They would both have approximately the same utility of $u_A = 0.49$ and $u_B = 0.51$. Still A's choice was wisest, because when C comes, she would definitely take the place just right of B at $c_1 = 0.51$. Now B is surrounded with $u_B = 0$ whereas $u_A = 0.49$ and $u_C = 0.49$. The same would hold true if A chose $a_2 = 0.51$, B again $b = 0.5$ and C would take $c_2 = 0.49$. One could say, Alice played a trick on the second vendor B. On the other hand, it might be good for all three vendors to be maximally close to each other, so that all the people come to their spot and finally choose the ice cream they like best of all 3 vending stalls.

Alternative 2: *The second vendor B knows about the third vendor C to come.*

Also here, it is not wisest for A to choose $a = 0.1$. It would be wisest to play a Maximin strategy, both for A and for B. The highest revenue, A can guarantee for herself would be 0.25. A should position herself at $a_1 = 0.25$ (or mirrored at $a_2 = 0.75$). These are her Maximin strategies. Thereby, she guarantees herself a revenue of at least $u_A = 0.25$. As B (who arrives second) knows that C is going to come and maximize his revenue, B would position himself at $b = 0.75$. As such, B is also playing his Maximin strategy, guaranteeing himself a revenue of at least 0.25. Now that C arrives at the beach, he can choose any position he wants , but cannot decrease A's and B's revenues under a value of 0.25. In the best case, where C positions his stall just right or left of B, A could reach the maximum revenue of $u_A = 0.5$. Of course, she could also choose $a_2 = 0.75$ and B choose $b_2 = 0.25$. This would lead to the same revenues u_A and u_B.