A Quasianalytical Constitution of Physical Space

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Introduction . Carnap’s constitution theory of the *Aufbau* has been subjected to many a criticism. Maybe the most incisive one is Quine’s who maintained that the *Aufbau* account of the constitution of the physical world is fatally flawed: "Statements of the form "Quality q is at point−instant x;y;z;t" were, according to [Carnap’s] canons, to be apportioned truth values in such a way as to maximize and minimize certain over−all features, and with growth of experience the truth values were to be progressively revised in the same spirit. I think this is a good schematization ... of what science really does; but it provides no indication, not even the sketchiest, of how a statement of the form "Quality q is at x;y,z;t" could ever be translated into Carnap’s initial langu−age of sense data and logic. The connective "is at" remains an added undefined connective … " (Quine 1953 (1961), p. 40). In this paper I sketch how the "is at"−connective may be defined in terms of Carnap´s initial language of elementary experiences and their relations thereby providing a coordinatisation of the class of quality points.

In the *Aufbau* Carnap took physical space as the real vector space $\mathbf{R}^4$ considered as a coordinating space for 4−dimensional spacetime. In order not to overburden the paper with mathematical technicalities I will replace $\mathbf{R}^4$ by $\mathbf{R}^2$ and show the following: given a class S of “quality points” endowed with a binary similarity relation ~, a faithful representation r: S −−−−> $\mathbf{R}^2$ can be constructed. Then, if the quality point a is mapped by r onto r(a) $\mathbf{R}^2$ this is to be interpreted as “a is at r(a)”. As is well known the same type of coordinatisation works for $\mathbf{R}^4$. Thus, contrary to Quine’s verdict, physical space can be constituted from quality points and a similarity relation alone. Hence, Carnap could have constituted spacetime by quasianalytical methods, he wasn’t forced to fall back on other, conventionalist methods of constitution (cf. Richardson 1998).

Of course, not any class S of quality points can be endowed with such a coordinatisation, rather, it has to satisfy certain structural requirements. The point is that all these requirements can be expressed in terms of Carnap´s initial language. Hence, Quine’s criticism against the *Aufbau*, up to now unchallenged even by the most radical revisionist interpretations of this work, can be defused. This should not be considered as a minor detail: Quine took Carnap’s alleged failure of constructing the "is at"−connective as the decisive evidence that any reductionist program must fail. The present construction shows that Carnap’s omission cannot be taken as such an evidence.

The outline of this paper is as follows: in section 2, we explicate the “essence” of Carnap’s quasianalytical approach as a theory of structural representations of similarity structures (cf. Mormann 1994). Section 3 contains the main novelty of this paper, namely the insight that quasianalytical representations and geometric incidence structures are intimately related. This will enable us to constitute structures like Euclidean space(time) from similarity structures. Section 4 is dedicated to the construction of the coordinatisation map. This allows us to construct the affine structure of the plane $\mathbf{R}^2$ from the apparently very weak similarity structure that can be defined in terms of Carnap’s initial language of quality points and their similarity relations. In section 5 we close with some general remarks on the role the concepts of similarity and quasianalysis may play in philosophy and science.

2. Quasianalysis as a Representational Theory of Similarity Structures. For the first time, the term quasianalysis (“Quasizerlegung”) appears 1923 in the unpublished manuscript “Quasianalysis − A Method to Order Non−homogenous Sets by Means of the Theory of Relations” (RC−081−04−01, *Quasizerlegung* in the following). In this paper, quasianalysis is conceived as the basic methodological tool of a general theory of constitutional systems. A constitution system aims to reconstruct all scientific concepts in an orderly way solely from the basis of a class of elementary experiences endowed with a binary similarity relation. This relation is assumed to be reflexive and symmetric, it need not be transitive. Thus, the basic level of any constitutional system may be described as a relational system (S, ~), S being the class of basic elements and ~ $\prod$ S x S a similarity relation.

Let us start by considering the relation between similarity and properties (or qualities). In *Quasizerlegung* Carnap proposes some axioms which govern the relation between properties and the similarity relation. For our purposes they may be formulated in the following way (cf. *Quasizerlegung* p. 4/5):

(2.1) Basic Assumptions concerning the Relation between Similarity and Quasi−properties.

(C1) Two elements are similar if and only if they share at least one quasipro−perty.

(C2) No quasiproperty can be discarded unless (C1) is violated.

The requirement (C1) also appears in the *Aufbau*. It may be considered as an almost analytic condition for any reasonable relation between similarity and the sharing of properties. The condition (C2) is a kind of Occam's razor for eliminating superfluous properties. In the *Aufbau*, (C2) is not mentioned, probably because Carnap took it for granted. In our construction of a coordinating map it will play a crucial role.

In the constitutional system treated in most detail in the *Aufbau*, the basic elements are elementary experiences and the only basic relation is a similarity relation. However, as Carnap emphasizes (cf. Friedman 1987), this gestaltist interpretation of the basic elements is not essential for the quasianalytical account. Quasianalysis is a general method (cf. Proust 1986). Hence, the class S of basic elements may be inter−preted as a domain of elementary experiences, quality points, situations, spacetime regions or whatsoever. In any case, the first main task of quasianalysis is the constitution of (quasi)properties which satisfy the assumptions (C1) and (C2).
Before we go on, it may be useful to observe that similarity structures (S, ~) may be conceived as non−directed graphs without loops and multiple edges: the vertices are the elements of S, two different elements are the endpoints of an edge iff they are similar. Indeed, the concepts of a simple graph and a similarity structure are equivalent. Hence, according to the *Aufbau*, the world (or some part of it) may be conceived of as a (simple) graph: the vertices of this graph are the "elementary experiences" and the set of edges is the set of pairs of similar elements (cf. Dipert 1997).
The following graph−theoretical concepts will be frequently used in the following. If (S, ~) is a graph the complement graph (S, ~*) is a graph with the same set of vertices but "complementary" edges, i.e. for x - y we have x ~* y iff not (x ~ y). Obviously, a (simple) graph and its complement determine each other. For later use we need a certain class of subgraphs defined by the following definition:

<u>(2.2) Definition.</u> Let (S, ~) be a graph. A similarity circle T is a subgraph T ∏ S which satisfies the requirements

(i) (x) (y) (x,y T −−−−−> x ~ y)
(ii) (x) ∃y (x T −−−−−−> y T and x ~* y)

The class of similarity circles of (S, ~) is denoted by SC(S, ~). Informally, a similarity circle T is a maximal subgraph of (S, ~) all of whose elements are similar to each other. Hence, for all elements x not belonging to T there is a y of T such that x ~* y.
Consider the following finite graph (cf. Dipert 1997, 347) having the following unordered pairs as its edges (cf. Goodman 1951):

	(1,2), (1,3)
	(2,3), (2,5)
(2.3)	(3,4), (3,5)
	(4,5)
	(5,6)

This graph has four similarity circles: a = {1, 2, 3}, b = {2, 3, 5}, c = {3, 4, 5}, and d = {5, 6}. If we take an extensional stance (as Carnap did in the *Aufbau*) similarity circles may play the role of (quasi)properties by stipulating x has the property T iff x T. Thus for the graph (2.3) we get the following property list:

	1.	a	4.	c
(2.4)	2.	ab	5.	bcd
	3.	abc	6.	d

This list is to be read as "1 has the property a", "2 has the properties a and b", etc. In this way we see that 1 and 2 share the property a, 2 and 3 share the properties a and b etc. As is easily seen the property distribution provided by the list (2.4) satisfies Carnap’s requirements (C1) and (C2).

Such a property distribution for the elements of S is called a quasianalysis. As was already noted by Carnap himself, a quasianalysis is "a synthesis in the garb of an analysis”, since it synthetizes properties for the undivisible elements of the relational structure (S, ~). Denoting the power set of S by P(S) a quasianalysis may be considered as a map Q: S −−−−> P(P(S)) which maps each element of S to the set of its properties:

	Q(1) = {{1, 2, 3}}	Q(4) = {{3,4,5}}
(2.5)	Q(2) = {{1, 2, 3}, {2, 3, 5}} ,	Q(5) = {{2,3,5}, {3,4,5}, {5,6}}
	Q(3) = {{1,2,3}, {2,3,5}, {3,4,5}}	Q(6) = {{5,6}}

Thus, the quasianalysis Q is a representation which represents the elements of S by their quasiproperties (cf. Mormann 1994). A quasianalysis Q is said to be of the first kind iff for all x S one has Q(x) P(SC(S, ~)). In this case the representation Q actually has the form Q: S −−−−> P(SC(S)). In the following we will restrict our attention exclusively to this type of quasianalysis. In the representational framework Carnap’s condition (C1) becomes x ~ y iff Q(x) ↔ Q(y) - ÿ, and (C2) requires that no q Q(S) := {q; ∃ x S and q Q(x)} can be removed without violating (C1). In other words, a quasianalysis Q belongs to the class of the most parsimonious maps satis−fying the structural condition (C1).

3. Quasianalytical Representations and Incidence Structures . In the tradition of Carnap and Goodman the virtues and vices of the quasianalytical approach have been discussed almost exclusively in terms of small examples such as (2.4) (cf. Goodman 1956, Mormann 1994, Richardson 1998). The domain of applications of the quasianalysis approach is not, however, exhausted by these rather artificial and trivial cases. A new field of interesting instances is opened when we observe that quasianalytical representations of similarity structures are intimately related to incidence structures of in synthetic geometry (cf. Hilbert 1899, Goldblatt 1987). To put it bluntly: quasianalytical representations of similarity structures just are incidence structures. A particularly important example is the incidence structure of the familiar plane of Euclidean geometry.

The details are as follows. Let A be the Euclidean plane. Denote its points by x,y, z, .. and its lines by k, m, n, etc. As is well known, the geometric structure of A may be codified in terms of an incidence relation I ∏ A x PA. If (x, m) I this is to be interpreted as “the point x is a point of the line m” or “x lies on the line m”. Two lines m and n are called parallel, denoted by m || n, if either m = n or they have no point in common. A set of points is collinear if there is a single line passing through all of them. The parallelity asserts that any two different points x and y determine exactly one line. Thus, any line may be denoted by xy, x and y being two different points.

Now, our task is to characterize the Euclidean plane A (endowed with its standard incidence structure) as a similarity structure (A, ~). First, let us define an appropriate similarity relation. For this purpose, choose a class B of parallel lines of A. Depending on B we will define a similarity relation $\sim_B$ on A. Hence, the resulting similarity structure should be denoted by (A, $\sim_B$). In order not to overload denotation, however, we will denote it simply by (A, ~). This is justified since for different B the resulting similarity structures turn out to be canonically isomorphic. For reasons of intuitive vividness we may refer to the lines of B as horizontal lines. Having chosen B two points x and y are defined to be similar iff they are equal or are on a line m not belonging to B:

(3.1) x ~ y := (x - y and xy B) or x = y

The following diagram exhibits the geometrical meaning of this definition:

p. p ~ q
lines of B q ~* r
q. r. r ~ p

Obviously, the relation ~ is reflexive and symmetric, but not transitive. Hence, (A, ~) is a similarity structure. Note that the complementary similarity structure (A, ~*) is a very special similarity structure, to wit, it is an equivalence structure. Geometrically, two points x and y are in the relation ~* iff they lie on a horizontal line belonging to B, i.e., x ~* y := (xy B or x = y)). Obviously, (A, ~) and (A, ~*) determine each other, and all considerations dealing with ~* could be easily formulated in terms of ~, and vice versa. Hence, dealing with (A, ~) and (A, ~*) (instead of (A, ~) or (A, ~*) alone) does not add anything new. After these preparations we are able to construct the following quasianalysis of A:

<u>(3.2) Lemma .</u> Let (A, ~) be the similarity structure defined by (3.1). Then define Q: A −−−−−−> P(P(A)) by Q(x) := {xy; xy B}. Then the map Q is a quasianalysis of A of the first kind.

<u>Proof</u>: Geometrically, the quasiproperties attributed to x by Q are just the lines of A through x not belonging to B. First we show that Q satisfies (C1) and (C2). For x - y we have x ~ y iff xy B. Hence xy Q(x) ↔ Q(y) - ÿ. On the other hand, if m Q(x) ↔ Q(y) we have m B. Since x and y are on m we may write m = xy, hence x ~ y. Moreover, since any non−horizontal line m may be characterized as m = xy for some points x and y satisfying x ~ y, removing m would amount to a violation of C1. Hence, Q satisfies (C2),

In order to prove that Q is of the first kind we have to show that any line q not belonging to B is a similarity circle of the similarity structure (A, ~). Let m Q(x). For any y m and x - y we have xy = m. Hence, all points of m are similar to x. Suppose z m and z ~ x. Then there is a unique horizontal line k through z which meets m at, say, z'. Hence, by definition, z ~* z'. Hence m SC(A, ~) and Q is of the first kind.

In this proof we have used some properties of Q which deserve to be singled out, since they will be crucial for the following:

<u>(3.3) Lemma .</u> Let (A, ~) be the Euclidean plane endowed with the similarity relation defined by (3.1)

(i) If x and y are different similar points for all lines q, q' Q(A) one has x, y q, q' ==> q = q'.

(ii)The quasianalysis Q satisfies the <u>parallelity axiom</u> , i.e. for all x A and all q Q(A) one has: x q ==> ∃! q' Q(A) (x q' and q || q').

The lemma (3.3) may be geometrically explained as follows: (3.3)(i) asserts that the similarity circles of Q(A) are determined by two (different) of their elements. Two lines q and q’ having in common two different points x and y coincide. Hence we may call Q: A −−−−−> P(P(A)) a <u>linear</u> quasianalysis. (3.3)(ii) asserts that for every Q−property q and every x not having q there is exactly one Q−property q’ such that no y has both q and q’, or to put it still otherwise, for all q and all x not having q there is exactly one property q’ separating x from q. This is just the familiar parallel axiom in disguise. Q defines an incidence relation by I_Q ∏ A x P(A) by (x, m) I_Q := m Q(x). Q and I_Q determine each other, i.e. from I ∏ A x P(A) we may obtain a map Q_I: A −−−−−−> P(P(A)) by Q_I(x) := {m; (x,m) I}.

The relation I_Q defined by Q is not quite the incidence relation we are looking for. Some lines are missing, namely those of B. In order to include them we proceed as follows. First note that the complementary similarity relation ~* is an equivalence relation whose equivalence classes are just the lines of B. Hence, for the complement similarity structure (A, ~*) we have a canonical quasianalysis Q*: A −−−−−> P(P(A)) which maps x to the singleton {q}, q being the unique line of B with x q. Denote the incidence relation defined by Q* by I_{Q^*}. Then we may define the union I_{QQ^*} ∏ A x P(A) of I_{Q^*} and I_{Q^*} by

(3.4) (x, m) I_{QQ^*} := (x, m) I_Q or (x, m) I_{Q^*}

This is the relation we need for the construction of a coordination mapping r_{QQ^*}: A −−−−−> $\mathbf{R}^2$. Imposing the right axioms on I_{QQ^*} we may recover from it all the geometric structure of the affine plane A. These axioms ensure that A can be mapped onto $\mathbf{R}^2$ in such a way that the incidence structure I_{QQ^*} on A is isomorphically mapped onto the standard real affine structure I ∏ $\mathbf{R}^2$ x P($\mathbf{R}^2$). Instead of giving all of the well−known details, we are content to sketch the basic ingredients necessary for the construction of the coordinatisation mapping r_{QQ^*}. The minimal re−quirements for the type of coodinatisation we are looking for are collected in the following list of axioms:

<u>(3.5). Axioms for Affine Incidence Structures.</u> Let (S,~) be a similarity structure whose complementary structure (S, ~*) is an equivalence structure. Let Q: A −−−−> P(P(A)) be a quasianalysis of (S, ~) and Q*: A −−−−> P(P(A)) the standard quasianalyis of (S, ~*) with corresponding incidence relations I_Q and I_{Q^*}, respectively. Their union I_{QQ^*} ∏ A x P(A) is an affine incidence relation iff it satisfies the following axioms:

(1) There exist at least three non−collinear points. (Nontriviality)

(2) Any two distinct points lie on exactly one line. (Linearity)

(3) Given a point x and a line m, there is exactly one line k that passes through x and is parallel to m. (Parallelity axiom)

(4) If x, y, z is a triple of points on m, and x', y', z' points of m' such that xy' ll x'y and xz' ll x'z then yz' ll y'z (Pappus's axiom):

The axioms (3.5)(1) – (4) suffice to ensure that the lines af on affine incidence structure have a quite rich algebraic structure, to wit, they are fields **K**. That is to say, for collinear points x, y, z addition x + y and multiplication x•y can be defined which obey the laws of associativity, commutativity, distributivity etc. By some further axioms it can be ensured that **K** is indeed the ordered field of real numbers R.

We need not define these operations in detail, rather, we are content to recall the addition of collinear points. They are to be found in any textbook of synthetic geometry (cf. for example Goldblatt 1987). Choose two distinct lines m and n which intersect in a point 0. Fix some point w on m different from 0. The line through w parallel to n is denoted by n'. Let x, y n. Let the line through y parallel to m meet n' at y', and then let the line through x parallel to 0y' meet n' at z'. Then the line parallel to m through z' meets n at z. Declare x + y = z. Then it can be shown that this operation renders n a commutative group, i.e. addition on n is associative, commutative, has a neutral element 0 etc. In a similar way, using Pappus's axiom one can define a commutative multiplication • on n and show that it obeys the laws a multiplication of a field has to satisfy (cf. Goldblatt 1987). In sum, these geometrically defined operations + and • render n a field **K**. This can be carried in such a way that all lines are isomor−phic copies of **K**.

Then, the next step to get the real numbers is to impose some further axioms on the incidence relation I in order to ensure that the field is indeed **R.** Before we come to this task let us observe that the quasianalytical construction of an affine plane achieved so far is unique up to isomorphism. This is seen as follows: if we had chosen another family B' of parallels we had obtained a different similarity structure (A, ~'). But then the similarity structures (A, ~) and (A, ~') are isomorphic. This follows from the fact that for any pair B and B' one may find an affine map which maps B onto B' preserving the affine structure, i.e. incidence and parallelism. Hence, this map defines an isomorphism between the similarity structures (A, ~) and (A, ~').

4. The Real Plane . The crucial point in the construction of the field **R** of real numbers is the observation that **R** is distinguished from other fields in that it is a Dedekind complete ordered field. That is to say, the elements of a line of the real affine plane can be ordered in such a way that we may talk about positive and negative elements. In particular, this order allows us to define a triadic relation of betweenness for col−

linear points x, y, z. Thus, in order to constiruct the real affine plane from a similarity structure (S, ~) one has to construct an order on the similarity circles T   Q(SC(A,~)) ≈ Q*(SC(A, ~*)). In order to do this recall that in the *Aufbau* Carnap defined similarity ~ as the symmetrization of an even more basic asymmetric relation x < y of "recollection of similarity" ("Ähnlichkeitserinnerung") (cf. *Aufbau*, § 110). That is to say, he defined x ~ y := x < y or y < x or x = y for the fundamental relation <. Hence the following definition makes sense:

<u>(4.1) Definition .</u> Let (S, ~) be a similarity structure endowed with a quasianalysis Q: S −−−−> P(P(S)). Assume that the relation ~ is the symmetrization of an order relation < as defined above. Let x ² y be defined as x < y or x = y. Q is called an <u>ordered</u> quasianalysis iff on every similarity circle T Q(SC(S, ~)) the relation ² is a linear order on T, i.e. on T the relation ² is reflexive, antisymmetric, transitive and connex.

As is shown by the quasianalytical representations Q and Q* of the real plane ordered quasianalytical representations exist. As these examples show we may assume that the order relation is compatible with the field structure defined on the lines T, i.e., on T we may distinguish between positive elements (0 < x) and negative elements (x < 0) in such a way that addition and multiplication are compatible with the relation <, i.e., the sum and the product of positive elements are again positive etc. Hence we may assume that the similarity circles of a similarity structure (S, ~) which has an odered quasianalysis in the sense of (4.1) are ordered fields. Now we are almost done. The last requirement we need to obtain the real affine plane is to stipulate that the ordered fields of our lines are <u>Dedekind complete</u> (Hilbert 1899). As is well known, the structure of an ordered Dedekind complete field is categorical, i.e., up to isomorphism, there is only one type of Dedekind complete ordered field, to wit, the field of real numbers **R**.

Now the desired coordinatisation of the class of quality points is at hands, as is seen by reading backwards the constructions carried out so far: Let us start with a similarity structure (S,~) of quality points whose similarity relation ~ is the symmetrization of an order relation < and whose complement structure (S, ~*) is an equivalence structure. We further assume that (S, ~) and (S, ~*) have ordered quasianalysis Q: (S,~) −−−−−> P(SC(S,~)) and Q*: (S,~) −−−−−> P(SC(S,~*)), respectively, such that I_{QQ*} satisfies the axioms for affine incidence structures (3.6). Note that these axioms can be expressed in terms of the initial language of the relational base of the constitution system (S, ~). The incidence structure I_{QQ*} allows us to endow the similarity circles of Q(S) and Q*(S) with the structure of the real line **R**. Chosing three non−collinear points r,s, and t with the help of the intersecting lines rs and rt one may construct an internal coordinatisation of S which renders it canonically isomorphic to $\mathbf{R}^2$. Thus, the statement “Quality point x is at (p,q)“ has the meaning “with respect to the coordinatisation based on r, s, and t the quality point a is represented by the ordered pair of real numbers (p,q)".

Choosing another tripel of non−collinear points r’, s’ t’ amounts to an isomorphic coordinatisation that is related to the former by a unique linear isomorphism. Although all coordinatisations obtained in this way are linearly isomorphic to each other some are empirically more useful than others. Here, Carnap’s conventionalist considerations come into play (cf. *Aufbau* § 135, 136, Richardson 1998, pp.70ff). It may be that one coordinatisation is empirically more useful than another one.

5. Concluding Remarks. Being able to reconstruct mathematical and physical structures such as the affine Euclidean plane, the real numbers **R** and many others from the apparently very weak structural base of a binary similarity relation one may conclude that the concept of similarity should not be dismissed as a "quirk" or an "im−postor" as Goodman once put it. Rather, one may take the feasibility of these constructions as an evidence that Carnap’s quasianalytical approach is not thus dead as most philosophers use to think.

According to the *Aufbau* all scientific objects (except the base elements, of course) are quasiobjects, i.e. are to be constituted by the method of quasianalysis. After having constructed the above mentioned structures, this claim may regain some plausibility. Thus, Quine’s criticism that the failure of constructing the “is at”−connective in quasianalytical terms has to be considered as a fatal break in the *Aufbau’s* methodology, is ill−founded. The "is at"−connective is reducible to the language of elementary experiences and their relations. Thus, the constitution of the physical world in Carnap’s sense is feasible along the lines of the *Aufbau*.

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