CHAOS AND FUNDAMENTALISM[†]

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ABSTRACT

A tension exists between our conviction that certain physical systems are chaotic, and the fact that chaos is a much more uncommon feature in quantum mechanics than in classical mechanics. It is argued that this creates difficulty for fundamentalism, the doctrine that our less fundamental theories are in some sense dispensable.

1. It is natural to wonder what our multitude of successful physical theories tell us about the world—singly, and as a body. What are we to think when one theory tells us about a flat Newtonian spacetime, the next about a curved Lorentzian geometry, and we have hints of others, portraying discrete or higher-dimensional structures which look something like our familiar spacetime in the appropriate limits?

If we had The Final Theory then we would perhaps know what to believe. Such a theory would tell us the Truth, and from its millennial perspective the pronouncements of other theories could be read as hints—some helpful, some misleading. In the meantime our theories can be sorted into the more and the less fundamental. Here is neutral characterization: a theory is more fundamental the further removed its *raison d'être* and empirical support are from the subject matter of eighteenth century physics. This is a partial ordering. Neither electromagnetism nor statistical mechanics is more fundamental than the other. Rather than a single most fundamental theory we have a small set of maximally fundamental theories (perhaps: the electroweak theory, general relativity, quantum statistical mechanics).

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In the absence of a Final Theory, it is tempting to allow these most fundamental theories to stand in for *the* most fundamental theory. Of course, we can't simply affirm the conjunction of what our most fundamental theories tell us—they contradict one another about the structure of spacetime and the nature of matter. We have to be cautious about the content of these theories. That said, the following thought is alluring: if we had a most fundamental theory, we would have the deepest possible understanding of the physical world, and our less fundamental theories would, in some sense, be dispensable; in the absence of such a theory, our most fundamental theories represent our deepest insights into the nature of the physical world, and render our less fundamental theories, in some sense, dispensable. I will call this thesis *fundamentalism*, and its negation *pluralism* (see Cartwright 1996 for a related use of these terms).

The intuition which underwrites fundamentalism is that there is a close link between the notions of fundamentality and approximate truth. Toulmin employed a particularly perspicacious cartographic metaphor, by way of elucidating this connection (see §4.3 of his 1953). Both theories and maps are, Toulmin tells us, are representations of the world which enable us to find our way around. The most fundamental available theory dealing with a certain class of phenomena is likened to the most accurate available map—a topographic map, drawn to a large scale, including all features in the neighborhood. Less fundamental theories covering the same phenomena are like less detailed maps, containing much of the information represented in the most detailed map, but also omitting much. Think of the relationship between a detailed topographic map and an ordinary road map. This metaphor can't be taken too seriously. After all, the road map is merely a coarse-graining of the topographic map; as such the two maps can't really be said to disagree. But two theories treating overlapping ranges of phenomena *will* disagree—classical mechanics doesn't say *less* than quantum mechanics, it says *different* things! One can slightly alter the metaphor: typically, road maps may aim at coarse-graining, but inadvertently introduce inaccuracies, etc. In any case, the force of the comparison is clear:

a more fundamental theory is cast as a more accurate representation which renders redundant our less accurate representations.

Fundamentalism can assume a variety of forms, depending on how one construes the modality involved in the claim that less fundamental theories are dispensable. Dispensable for what purpose? The most-discussed forms of fundamentalism correspond to two of the most conspicuous purposes of physical theories: calculation and comprehension. Let us say that a less fundamental theory is *pragmatically dispensable* if, upto some appropriate degree of accuracy, its empirical predictions can be derived from a more fundamental theory; and that it is *epistemically dispensable* if whatever understanding of the world it yields can be gleaned from more fundamental theories.

No one suggests that our less fundamental theories are currently pragmatically dispensable: it is with good reason that more people know Newton's Laws than the Wightman axioms for quantum field theory, just as it is with good reason that most people keep relatively coarse-grained road maps in their cars. There is, however, room for disagreement about the source of this indispensability. Some fundamentalists may believe that it can be traced to rather uninteresting cognitive shortcomings on our part—a lack of patience or memory which prohibits the execution of the monumental feats of calculation which would be required to apply more fundamental theories in the domains of application of less fundamental theories. Attention to the details of particular cases reveals that this is far too optimistic. In fact, what is required to recover the results of less fundamental theories within the frameworks of more fundamental theories—if and insofar as they *can* be recovered—is typically not indefatigable calculation, but, rather, considerable mathematical and physical insight.[1] In the domain of calculation, pluralism enjoys a comfortable superiority over fundamentalism.

[1] See Batterman 1995 for some powerful examples which bear on epistemic as well as pragmatic dispensability.

It might seem obvious that this situation is inverted when we turn to the question of comprehension. If our deeper theories are indeed closer to the truth, then it seems clear that what understanding of the world we can glean from physics is, in principle, contained in them. So our less fundamental theories should be dispensable when it comes time to discuss what we know about our world.

Even here, however, fundamentalism has come under attack. Cartwright has argued that "nature is governed in different domains by different systems of laws not necessarily related to each other in any systematic or uniform way: by a patchwork of laws" (1996, p. 322). Her view is that to the extent that Newton's laws, for instance, are correct, they cover only their proper domain of application; outside of this domain we have no reason to think they describe phenomena; and, not describing them, can't really said to be false outside domain (see especially pp. 318 and 323). The same goes for other, more fundamental, theories: they tell us about a certain limited domain of phenomena. But even for our deepest theories, most of the world lies outside of this domain. Some of the remainder is covered by less fundamental theories. It would be a mistake to dispense with these when trying to suss out the structure of the world, since they are the only guides that we have.

All parties can agree that there are phenomena treated by our less fundamental theories which we don't know how to model using the resources of our more fundamental theories. Cartwright's argument focuses on such cases. But it seems to me that this is bound to lead to a debate which will quickly reduce to a clash of intuitions. Cartwright sees no reason, other than stodgy fundamentalist dogma, to maintain that there must be an adequate fundamental treatment of such phenomena. Her opponents will see no reason, other than trendy pluralist dogma, to doubt this. But neither side can adduce any convincing arguments, since by the nature of the case we will always be faced with such phenomena, and only faith of one sort or the other could make us confident about the prospects of fundamentalism in such contexts.

There is, of course, another option. We can focus our attention on situations where we have both a more fundamental and a less fundamental treatment of a given phenomenon. Then it will seem that the fundamentalist is bound to be vindicated: the more fundamental theory is typically introduced in response to certain inaccuracies in the predictions made by the less fundamental theory around the margins of its domain of successful application; and this is exactly the domain where the two theories overlap. In this case, the more fundamental theory is bound to present a more adequate account of the phenomena—just as a fine-grained map is a more accurate representation of a given region than is a coarser-grained one. Isn't it?

2. Speculation that certain physical systems—such as the solar system—are chaotic is over a century old. It seems safe to say, though, that this idea has more momentum now than ever. There is a paradox here. For it appears that the claim that the solar system is chaotic made more sense a hundred years ago than it does now, although there was then far less evidence in its favor.

Classical mechanics is the birthplace and natural habitat of chaos. There it has two complementary aspects: exponential divergence of nearby trajectories in phase space and long-time mixing behavior. Both aspects make prediction difficult for beings with finitely sensitive measurement techniques. We can make a good guess about the initial location of our system in phase space. But since the trajectories through nearby points diverge so rapidly, and some error is inevitable, our predictions quickly become nearly useless. Similarly, if we model our epistemic uncertainty using a probability density on phase space, then chaotic dynamics will quickly smear this density evenly over the entire space, and our predictive efforts again rapidly approach futility. In the tame context of finite conservative systems, one can show that these two types of behavior—exponential divergence and mixing—are in fact equivalent (see Belot and Earman 1997 for a review of the relevant results). In this context, one can argue that certain models of physical systems are chaotic. The level of rigor varies considerably: from solid proofs for mathematically tractable but physically unrealistic systems like Arnold's cat and geodesic motion on a manifold of negative curvature; to numerical simulations for physically motivated models like the kicked rotor.

The problem, of course, is that we no longer think that classical mechanics provides the best picture of our world. If we want a more adequate representation of a given physical system, it ought to be drawn from some deeper, more accurate, physical theory. Ultimately, it ought to be drawn from a theory capable of handling both the gravitational and the quantum. In the mean time, we tend to settle for models drawn from quantum mechanics and quantum field theory. It would seem to follow that if we want to establish that a given physical system is chaotic, it no longer suffices to argue that a certain classical model of that system satisfies the appropriate conditions. Rather, we ought to be able to show that our quantum model satisfies the requirements for *quantum chaos*.

What form would such requirements take? This is a much disputed question (see Belot and Earman 1997 for a survey). The basic situation is this. There are a large number of criteria of chaos in classical mechanics, many of which are equivalent in the context of finite conservative Hamiltonian systems. Some of these criteria can be generalized to apply to quantum systems. Although several of the classical equivalences are lost, there is still a great deal of agreement in the context of finite conservative systems: under most criteria there is *no* quantum chaos in this realm. Discussion of quantum chaos tends to focus on the relative scarcity of quantum chaos, and upon the search for traces of classical chaos in regular quantum systems.

This situation can be illustrated by focusing on the two central criteria of classical chaos: mixing behavior and exponential divergence of nearby trajectories.

Here is a characterization of mixing behavior which generalizes straightforwardly to quantum theories: no matter what the initial state of the system, the expectation value of any observable approaches the expectation value for the equilibrium state as t∅ .[2] There *are* conservative quantum systems which are mixing according to this criterion—but they all have infinitely many degrees of freedom (see Benatti 1993 for examples). According to this criterion, there is chaos in quantum field theory, but not in quantum mechanics.[3]

What happens if we follow the other route open to us, and look for quantum generalizations of the exponential divergence of phase space trajectories?

We could try the obvious: the mark of classical chaos is the divergence of dynamical trajectories in phase space; so the mark of quantum chaos should be the divergence of dynamical trajectories in Hilbert space. But because the Schrödinger time evolution operator is unitary, dynamical trajectories never diverge for any quantum system. So if divergence of Hilbert space trajectories is the mark of quantum chaos, there just isn't *any* quantum chaos—even in infinite quantum systems which display the approach to equilibrium which is characteristic of mixing. This result is far too strong and too easily had to be of much interest. Here is a consideration which suggests that it is largely irrelevant: the Hilbert space formalism of quantum theories is less similar to the standard formulations of classical theories, in which states are represented by individual points of a phase space, than it is to the Koopman formalism in which classical states are elements of the Hilbert space of square integrable probability densities on the phase space; but in the Koopman formalism, as in quantum theories, time evolution is implemented via a unitary operator, so that trajectories in the space of states do not diverge at all; but no one takes this to show that there isn't any chaos in classical mechanics.

We can pursue the analogy between quantum states and classical probability distributions on phase space. If we have a chaotic classical system, then we know that

[2] Here we are thinking of classical states as probability densities on phase space.

[3] More precisely: a necessary condition for this sort of quantum chaos is that the time evolution operator have 1 as a nondegenerate eigenvalue, and that this be its only eigenvalue; this rules out all standard quantizations of finite conservative systems, since the quantum time evolution operators of such systems have discrete spectra.

nearby trajectories in phase space will diverge rapidly. If we have a probability density, then we know that it will spread over the entire phase space over time. What happens to two nearby tightly focused probability densities at short times? Presumably, they mimic the exponential divergence of the trajectories through the points which they approximate, until they spread a bit, and begin to smear out very rapidly. That is: for short times, the expectation values of position and momentum for tightly focused probability densities should mimic those for point-states in phase space—we should get something which at least approximates exponential divergence for very short time-scales.

Let us focus on this symptom of exponential divergence of dynamical trajectories in phase space, and ask what happens when we construct a quantization of our classical system, and inquire after the behavior of the expectation values of position and momentum for wave packets which are tightly focused about classical states? Ehrenfest's theorem tells us that in the case of a free particle in Euclidean space, the quantum expectation values mimic those of a classical particle or probability density. But we know this cannot be the case when we start with a classically chaotic finite system: a classical probability density becomes uniformly smeared over phase space, so that the expectation values approach those for the equilibrium state; but because finite quantum states cannot be mixing, we know that we cannot expect the quantum expectation values to faithfully mimic the classical ones at long times. At short times, of course, we hope that they will exhibit some sort of exponential divergence.

Here we are in rather murky waters, where most of the results are numerical or based on heuristic treatments of simple model systems. But it seems that the situation is as follows. Whether we start with a regular or a chaotic classical system, we find that the quantum expectation values *initially* mimic the classical ones. It appears that different sorts of system have different characteristic time-scales at which this approximation breaks down (see Casati and Chirikov 1995b for a survey of the relevant results). For the most regular of classical systems, such as the free particle in Euclidean space, the time-scale is infinite. For generic regular systems, the time-scale is of the order ♒ $^{-1}$, while for generic chaotic systems it appears to be of the order ln ♒ .[4] For many macroscopic systems, ♒ $^{-1}$ is of the order of the age of the universe—which means that ln ♒ is much smaller. Beyond this time-scale, the behavior of the quantum expectation values will typically be far more regular than that of the classical expectation values—one can even have periodic quantizations of chaotic classical systems, such as Ford's quantum cat (Ford, Mantica, and Ristow 1991).

The search for quantum chaos in finite systems leads to a characterizations of the various senses in which finite systems fall short of being chaotic. Where does this leave our original question: What sense does it make to say that the Solar system is chaotic? Well, we can certainly construct models of the Solar system in classical celestial mechanics. These will be N-body problems in Newtonian gravitational theory. One can then ask: is this model chaotic? Despite a century of research, the answer is not forthcoming—although recent numerical work suggests that some classical models of the solar system may indeed be chaotic (see Wisdom 1987 and Sussman and Wisdom 1992).

For every model of classical celestial mechanics, we can construct a model of quantum celestial mechanics, by simply quantizing our classical Hamiltonian system. We know that such systems are not chaotic in the strict sense—they cannot exhibit mixing behavior, although the classical models that they are based upon may do so. Furthermore, if we set the initial conditions in our quantum model to be a wave packet which mimics the

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[4] Here ♒ stands for some dimensionless parameter which varies with the size of the classical action of the system.

classical initial positions of the celestial bodies, then we find that the quantum predictions diverge significantly from the classical predictions on a time-scale of the order of 700 million years—far less than the age of the Earth (see Zurek 1998 for the argument). If our classical and quantum models treat small, highly chaotic, subsystems of the Solar system, then the time-scale can be dramatically short: for the orbit of Hyperion around Saturn the magic number is twenty years!

This naive approach to quantum celestial mechanics cannot give us the mixing behavior characteristic of classical chaos; nor can it produce predictions which mimic the—*presumably empirically adequate*—predictions of classical celestial mechanics. A quantum treatment of the solar system simply cannot display the sort of unpredictability that is the hallmark of chaos. What sense, then, does it make to say that the solar system is chaotic? The content of any such claim must be underwritten by the chaotic features of the models of *classical* mechanics. We find ourselves in a situation where we have two theories which offer models of a given domain, and where what seems to be an important feature of our world is represented quite clearly in the models of the less fundamental theory, and is absent altogether in the models of the more fundamental theory.

3. One response to this situation would be to give up on the idea that finite systems can be chaotic. This is not entirely outlandish—how can we be certain that we observe the long-term unpredictability that is the hallmark of chaos, rather than the short-term mimicking of such behavior which is possible for quantum systems? Nonetheless, I bracket this possibility, since to consider its merits seriously would require a detailed discussion of a vast literature.

This leaves us with a genuine challenge to fundamentalism. At the level of calculation, we have a particularly nice illustration of the serious obstacles which stand in the way of the dispensing with classical mechanics. We never find infinite-time mixing behavior in finite quantum systems; and, for physically realistic values of

♒ for finite chaotic systems, we find that the quantum predictions depart unacceptably quickly from the classical predictions. The hope remains that we can recover the classical behavior in the limit ♒ ∅0. If this is possible then there is a *weak* sense in which we can dispense with classical mechanics as an autonomous discipline, since its predictions can be captured by taking limits within the quantum mechanical formalism. Indeed, since ln ♒ grows without bound as ♒ ∅0, we know that we can get quantum wave packets to mimic classical dynamical trajectories for as long as we like. The recovery of mixing in the limit turns out to be a difficult mathematical problem, which is open for even the simplest chaotic systems (see Degli Esposti, Graffi, and Isola 1995 and the references therein).

Even if these results could be established, epistemic fundamentalism would still be in danger. There we are interested those theories which are the sources of our knowledge of the world. And if we accept that some finite physical systems are chaotic, then it would seem that it is classical mechanics which gives us this insight—it is of little relevance that if we allow ♒ to take on unrealistically small values, we can recover behavior from quantum mechanics which approximates that which we observe. The only hope for fundamentalism here is to argue that the problem at hand is an artifact of the naive approach to quantum mechanics adopted in the previous section, and that we can find the desired degree of chaos in finite quantum systems if we adopt a more sophisticated approach. There are a number of programs along these lines (see Belot and Earman 1997). I will briefly sketch a few.

Zurek argues that taking the decoherence caused by interstellar dust, etc., into account allows one to produce quantum models of celestial motions whose dynamics exactly mimics that of the classical models.[5] Vitali and Grigolini 1998 supplement this work by arguing that the same effect can be achieved via a GRW or Penrose mechanism without invoking decoherence. The arguments of these authors depend upon heuristic calculations for toy models and, although interesting, remain less than compelling.

Another possibility is to employ more sophisticated techniques of quantization. Ford has constructed a quantization of Arnold's cat which lives in a finite-dimensional Hilbert space and has a periodic dynamics. The technique employed can be viewed as a generalization of geometric quantization (De Bièvre, Degli Esposti, and Giachetti 1996). Other techniques of quantization can lead to quite different quantum systems. Deformation quantization produces a quantum cat which exhibits mixing (and stronger properties as well), and which provides a good model of the quantum Hall effect (see Benatti 1993). Both quantizations begin from the *same* Hamiltonian system.[6] But one is regular, the other highly chaotic. Whether sophisticated techniques of quantization will in general produce physically interesting examples of quantum chaos is an open question.

It is, of course, impossible to rule out some such fundamentalist resolution of the quantum chaos crisis. But it would be imprudent not to consider another possible outcome: that we will be forced to recognize that there are phenomena for which classical mechanics provides *better* models than does quantum mechanics. These models are better in the sense that they display a qualitative feature absent in the corresponding models of the more fundamental theory, but present in the phenomena. I think it is likely that other examples of this sort can be found. In condensed matter physics, one often employs infinite and/or two-dimensional models to represent physical materials which are finite and three-dimensional (Liu 1998 and Neilson 1989). 'Less realistic' treatments of the

[5] Zurek 1998 and elsewhere. In particular, he argues that the evolution of the quantum Wigner distribution can be made to obey the classical equations. See Casati and Chirikov 1995a and Vitali and Grigolini 1998 for criticisms of Zurek's approach.

[6] The assertion to the contrary on p. 172 of Belot and Earman 1997 is mistaken. See Lesniewski, Rubin, and Salwen 1998 for an elementary treatment of the chaotic quantization; and for a proof that the regular quantization is a component part of the chaotic quantization, in the sense that the Hilbert space of the latter is naturally regarded as a direct integral of the Hilbert spaces of the former.

phenomena sometimes give one access to qualitative features of the world which are provably absent in ‘more realistic’ models.

Despite our belief that the systems we are dealing with are quantum, finite, and three-dimensional (or low-energy limits of higher-dimensional objects), we find that sometimes it is best to represent them as being otherwise—as classical, infinite, or two-dimensional. And it appears that this is not merely for convenience’ sake, but because these representations, although less fundamental, are for some purposes *more accurate*. If there really is a problem with representing chaos in quantum mechanics, and if there really are chaotic systems then we are driven to a serious form of epistemic pluralism: each of our theories, the more and the less fundamental alike, makes some indispensable contribution to our understanding of our world.

Toulmin’s cartographic metaphor must be revised. Fundamentalists recognize that, at the present stage of its development, physics does not break into theories which can be totally-ordered by the relation ‘more fundamental than.’ But they note that we do have certain chains of theories which can be so ordered—Newton’s theory of gravitation, general relativity, …. We expect the domains of adequacy of theories which are adjacent in such chains to overlap. And fundamentalists expect that within such chains, the ordering induced by ‘more fundamental than’ will coincide with that induced by ‘closer to the truth than.’ What I have been suggesting is that this latter expectation is not borne out by a consideration of certain pairs of adjacent theories. The less fundamental of such a pair of theories doesn’t typically just say less than the more fundamental—as a coarser-grained map just says less than a more fine-grained one—it says different things. Nor need it be misleading when it goes beyond what its more fundamental partner says. Thus, even theories that can be compared with respect to fundamentality typically cannot be compared with respect to verisimilitude. Indeed, even maps of the same region that differ in level of detail cannot typically be compared in these terms—the coarse-grained road map will typically include some accurate and enlightening information not represented by the more earnest topographic map (political boundaries, aircraft flight corridors, the quality of hotels).

One should perhaps supplement the cartographic metaphor with others which better highlight this feature. It is natural to compare scientific theories to other representations that we make of (bits of) our world. Sometimes the relation between two theories is compared to the relation between two media or genres of artistic representation (van Fraassen and Sigman 1992). And, of course, there is a sense in which photographs are more accurate representations of their subjects than are paintings; yet it is no surprise that some paintings can represent features of their subjects which cannot be pictured in any photograph. (The same point could be made using two styles of painting.) Ultimately, however, all such comparisons are very limited—scientific representation is *sui generis*, and issues of relations between representations in this domain can only be fully understood by attention to the details of scientific practice.

This has been a speculative paper, aimed at shaking people's faith in a widely accepted doctrine. I doubt that many who are antecedently committed to fundamentalism will be swayed. Let me close with a challenge to such people. What grounds do you have for believing that quantum mechanics provides, in principle if not in practice, a *more accurate* view of our world than does classical mechanics? Granting that the two theories diverge in their empirical predictions, it can only be that quantum mechanics has proven to be a much better way of approaching the microworld. But what grounds can we have for confidence that this superiority extends to the macroworld? The answer, I assert, can only be a philosophical preference for fundamentalism over pluralism.

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