Much of the current thought concerning mathematical ontology and epistemology follows Quine and Putnam in looking to the indispensable application of mathematics in science. In particular, the Quine/Putnam *indispensability* approach is the inevitable staging point for virtually all contemporary discussions of mathematical ontology. Just recently serious challenges to the indispensability approach have begun appearing (Maddy 1992, Sober 1993, Vineberg 1996, Peressini 1997). At the heart of this debate is the notion of an indispensable application of (pure) mathematics in scientific theory. To date the discussion has focused on *indispensability*, while little has been said about the process of *application* itself.[1] In this paper I focus on the process of applying (pure) mathematical theory in physical theory.

1. Applying Pure Mathematics

There is an obvious and natural distinction between mathematical theories and mathematical scientific theories. A mathematical theory is a theory whose (apparent) subject matter is some sort of mathematical object; for example, number theory, real analysis, functional analysis, group theory, set theory, etc. These are the theories that occupy mathematicians; I will refer to them as *pure* mathematical theories. On the other hand there are scientific theories which, to varying degrees, make *use* of pure mathematical theories, i.e., mathematical (or mathematized) scientific theories. Prominent examples of mathematical scientific theories are found in quantum mechanics, population genetics, and general relativity.[2]

More precisely, the distinction between pure mathematical theories and mathematical scientific theories is underwritten by the latter's deployment of a physical interpretation of part of the mathematical vocabulary that mathematical theories lack.[3] At least some of the sets, numbers, functions, vectors, groups, and manifolds of scientific theories have an associated (operationally defined) physical interpretation evinced by their units; in this sense the mathematics present in the theory is applied. Mathematical theories, on the other hand, are pure in that they lack this physical interpretation. Consider the scientific theory of quantum mechanics which makes use of pure mathematical theories, for example, group theory.[4] How are we to understand the distinction between pure and applied group theory in this context? It is clear enough that pure mathematical group theory is not the *same* theory as the applied theory of groups that is present in the theory of the spin of quantum particles: they are about different things. The groups/members that appear in quantum theory are taken to be specific groups/members that are interpreted as the physical properties (spin) of physical objects (particles); the propositions of *pure* group theory, on the other hand, lack any such physical interpretation. This physical interpretation is far from trivial, as is illustrated by the details of how the pure theory is applied.

Consider first how the pure mathematical theory may be applied within the realm of pure mathematics. From pure group theory we have:

$$\text{(GT)} \qquad (\forall a \in G)\ (\exists b \in G)\ \ni\ (a \cdot b = e),$$

where G is a group. The equation states that for any element in a group, there is another element in the group such that the group product of the elements is the identity element. As such, this proposition has implications for particular pure groups, e.g., $\mathbb{Z}_{4,}$ integers modulo 4; or $P_{3,}$ the permutation group for 3 elements; or SU(2), the group of 2x2 unitary matrices with determinant equal to 1. In particular, the general pure proposition (GT) gives rise to the particular pure proposition ($GT_{SU(2)}$) in which the free variable G is replaced by the particular pure group SU(2).

Formally speaking, ($GT_{SU(2)}$) follows from (GT) and the additional premise

(AP) SU(2) is a group.

The relationship between (GT) and ($GT_{SU(2)}$) is purely mathematical and *a priori* in that (AP) is pure mathematical and *a priori*; this is true of pure applications in general and is in marked contrast to *physical* applications of pure mathematics.

As an example of a physical application, consider again quantum mechanics and group (representation) theory. The pure group SU(2) is used along with (some of) the pure propositions concerning it, which follow from the general pure theory in the sense just described. In the physical application, however, the members of SU(2) are further interpreted as (components of) the physical property of spin and the group operation is taken to correspond to the composition of spin states, thus allowing the mathematical terminology to fall under the physical principles which relate the property of spin to the rest of the physical theory. This physical interpretation of (part of) the pure theory is such that the resulting applied mathematical propositions of the physical theory imply claims about the physical world. Let the applied quantum group be denoted by ŜÛ(2). The crucial point here is that even though the physical theory propositions concerning ŜÛ(2) constitute an *application* of the pure theory, it is a qualitatively different kind from the pure application.

Let p be a proposition of the pure theory and $\hat{p}$ the corresponding proposition in the physical theory. Proposition $\hat{p}$ does not follow from p (and its attendant theory) conjoined with a pure mathematical premise like (AP); this is because the physical application requires empirical bridge principles to underwrite the physical interpretation These principles distinguish pure mathematics from mathematized physical theory and enable claims about the physical world to be deduced from the latter. That the composition of two particular elements, a and $b \in$ SU(2), yields the identity element is not related in any obvious way to the empirical fact that inducing two particular rotations on a particle will leave the particle in the same spin state. The latter is not merely a special case of the former; it is not merely a case of going from the universal to the particular. Any relationship between the two propositions involves substantive empirical bridge principles linking the pure mathematical vocabulary to the physical object/property vocabulary. It is only within the physical theory itself, which contains propositions corresponding to at least some of those of the pure theory, that certain of its propositions imply that spin components $\hat{a}$ and $\hat{b} \in \hat{S}\hat{U}(2)$ induced on a particle will leave its spin unchanged.

Another way of characterizing the difference between mathematical and physical applications of pure mathematics focuses on the "immediacy" of the application. Pure mathematical propositions apply "immediately" to other pure mathematical settings: the implicit range of the free variable(s) in the propositions like (GT) include particular pure groups like P_3, $\mathbb{Z}_4$, and SU(2); in this sense the application is immediate relative to physical applications. Physical applications are (qualitatively) less immediate in that they require substantive auxiliary premises which take the form of empirical bridge principles between the language of pure mathematics and the language of physical theory; it is only by way of such auxiliary premises that the mathematics of the physical theory says anything about the physical world.

It is possible to render formal treatments of applications in such a way that both pure and physical applications will include a premise of the form "X is a group," where the "only" difference is that in physical applications this premise is a physical claim rather than a mathematical claim. While this formal similarity allows for a succinct abbreviation of the details of an application, it may also be misleading. In the case of pure applications this "premise" is little more than a "formality," since the free variables of the pure theory range over the appropriate pure mathematical "objects." In physical applications, however, this "premise" is much more complicated; at the very least it encodes the substantive (empirical) bridge premises required for the pure theory to have physical implications. This abbreviated way of talking may misleadingly suggest that the physical application of mathematics is essentially the same as pure application—that it involves *nothing more* than replacing some mathematical terminology with physical terminology. As I think is clear, this does not characterize genuine physical application; rather, it describes a trivial change in notation.[5]

2. The Pure/Applied Relationship

Suppose that theories *s* and *m* bear the "applied" relationship to one another, that is that *s* is an applied version of *m* (*sRm*). One naturally wonders whether, in general, it is necessary that a pure theory *m* be worked out before we can have an applied theory *s*. If one were to form a naive opinion based only on pure mathematics texts, then it would appear that pure mathematical theories are worked out by elegantly deducing consequences from various mathematical postulates—a triumph of *a priori* reasoning. Then, only after the pure theory has been worked out, would it be applied to real problems. But this picture is seriously distorted. Historically this is rarely how theories develop, and even today, mathematics texts offer little insight into how pure mathematical research proceeds. Nor would it be right to suppose that progress in pure mathematics is always due to developments in the scientific use of mathematics. As it turns out, neither the pure theory nor the applied theory are in all cases epistemically prior. I illustrate this below by considering several historical episodes in the development of science and mathematics. Before doing this, however, consider further the formal character of the applied relationship.

As stated above, *sRm* is a two-place relationship, where *m* is a pure mathematical theory and *s* is an applied mathematical theory or mathematized scientific theory. Until now, I have been taking “applied mathematical theories” and “mathematized scientific theories” to be the same thing; however, they must be distinguished. Because not every mathematized scientific theory is also an application of a (pure) mathematical theory. There are mathematized scientific theories that do not bear the “applied” relationship to any pure mathematical theory, and so, strictly speaking, should not be considered applied *mathematical* theories. In these examples a scientist develops a technique in order to solve certain physical problems by what appear to be mathematical methods (e.g., evaluating a certain type of integral, multiplying an integrand by a certain “function,” dividing by a certain mysteriously small quantity, etc.). But in fact the new “mathematical” method makes no sense mathematically, and hence is not a physically interpreted version of a pure theory.[6]

Such cases in which the mathematized scientific theory is worked out first, and then only later, if ever, a pure mathematical theory is worked out, we have the inverse of the operation of application—call it *abstraction*. The history of science and mathematics abounds with examples going in each direction, as we will see in the next section.

Notice that until now the discussion has been focussed on *physical* applications of pure mathematical theories: the applications considered involved only physical interpretations of the pure theory. Pure mathematical theories, however, can be applied in settings other than the usual scientific/physical setting. One such setting is linguistics. In the modern study of linguistics, mathematics plays a conspicuous role (e.g., Gross 1972, Wall 1972, Gladkii 1983, and Partee 1990). Pure mathematical theories may be applied to a linguistic theory in essentially the same way as they are in physical theory. In this case, however, some of the pure mathematical vocabulary is given a linguistic interpretation rather than a physical interpretation. For example, a class of grammatical properties might be represented by the elements of a finite group, with the group operation being a certain grammatical transformation.

As discussed above, pure mathematical theory is often applied within pure mathematics itself. Analytic number theory is a prominent example. Number theory deals with integers, while analysis deals with continuous sets of numbers such as the reals or their extension, the complex numbers. In pure analytic number theory, methods and results from the pure theory of complex analysis are used to express and prove facts about the integers. The area of mathematics known as *numerical analysis* provides a wealth of other such examples. Numerical analysis deals with computation; it focuses on computational means of obtaining numerical results for mathematical expressions. For a mundane example, the methods by which your hand-held calculator computes its square roots, sines, and cosines are not what you might expect; the highly theoretical tools of the theory of numerical analysis have been employed to design accurate and efficient algorithms to compute these functions. Numerical methods ingeniously and indirectly arrive at the actual numbers implied by the formal solution of a physical problem. Although this (pure) application of pure mathematics is rarely seen in text books or classrooms, it is essential to prediction and confirmation.

3. Some Historical Illustrations

The following examples of applied mathematical theories can be loosely categorized by whether they are (primarily) examples of (1) moving from a mathematized scientific theory to a pure mathematical theory (abstraction), or conversely, (2) moving from a pure mathematical theory to a mathematized scientific theory (application). Of course these "directions" are only approximations; history, as usual, resists such neat categorization.

Only relatively late in the history of mathematics do we begin to see clear examples of the application of pure theories; this is because it was late in the history of mathematics that mathematical theories reached the level of abstraction that we have today. Early mathematical breakthroughs often took place in theoretical environments in which the mathematics was not clearly divorced from the physical problems it was developed to solve. The case of Euclidean geometry is a well-known example of the transition from mathematized scientific theory to pure mathematical theory. Initially (Euclidean) geometry was taken to be about physical space itself; proving geometrical theorems amounted to deducing facts about physical space. The development of non-Euclidean geometries, however, forced people to revise these views. These internally consistent alternatives to Euclidean geometry were considered to be on the same *mathematical* footing as Euclid's geometry. If Euclidean geometry is in some sense "more true," it would have to be so in a non-mathematical sense, i.e, true of the physical world.

Newton's work on the calculus is another example. In his second and preferred presentation of the *Methodus Fluxionum et Serierum Infinitarum,* Newton presents his version of the derivative (fluxion) in dynamical terms—based on the idea of rate of change with respect to time. In response to Berkeley's criticism of infinitesimal quantities, Maclaurin's authoritative presentation of Newton's calculus sought to base this calculus on our intuitions of space, motion, velocity, and time. As Newton himself wrote, the theorems of the calculus do not deal with "fictions" or "ghosts of departed quantities," but rather with things that have an "existence in nature" (Guicciardini 1989, 51).

Finally, Steiner (1992) gives examples of mathematical devices used by present-day physicists that still lack a consistent pure mathematical underpinning. Quantum field theory makes use of an integral called the Feynman integral, which unlike any integral in pure mathematics, is taken over an *infinite* dimensional space. As Steiner demonstrates, a general pure theory of such an integral has not yet been worked out. Another such device used in quantum electrodynamics is called "renormalization;" see Steiner (1992, 164 ff.) for details. In these examples, the mathematics-like devices employed by the scientific theory are motivated by physical considerations—they make sense given the physical interpretation. Equally important, these techniques accurately describe and predict the physical phenomena. What lacks, however, is a corresponding pure theory in which the techniques make sense. In the context of pure mathematical theories, these techniques make as much sense as "dividing by zero." If and when these techniques are given a pure mathematical foundation, the move from mathematized scientific theory to pure mathematical theory will be complete.

Consider now the opposite direction, moving from pure mathematical theory to mathematized scientific theory. Recently striking examples of scientists making use of previously developed pure mathematical theories to formulate their scientific theories have arisen. As Steven Weinberg puts it:

The mathematical structures that arise in the laws of nature . . . are often

> mathematical structures that were provided for us by mathematicians long before any thought of physical application arose. It is positively spooky how the physicists finds the mathematician has been there before him or her. (Weinberg 1986, 725)

The development of Einstein's general theory of relativity is a case in point. His theory identifies the effects of gravity with structural features of a curved space-time (Riemannian geometry). Einstein, however, unlike Newton, did not need to invent the mathematics to go along with his physical insight. The calculus of four-dimensional Riemannian manifolds requires a special calculus of tensors (tensor analysis), which had been developed years earlier by Ricci and Levi-Civita, but had not yet been noticed by physicists. Einstein studied these results and used them as a basis for formulating his general theory of relativity. In just about the same way, the pure mathematical theory of Lie algebras was discovered by the physicist Murray Gell-Mann as just what he needed to describe the unitary spin properties of elementary particles. Yet another example can be found in abstract algebra. Abstract group theory grew out of the work of Evariste Galois on the solution of polynomial equations by radicals. Much later physicists discovered it as the mathematics needed for describing the symmetries of elementary particles and incorporated it into the physical theory of symmetries.

It must be stressed that the distinction between pure and applied mathematics is a logical distinction; we should not expect to be able to definitively place work done in the actual development of mathematics and science precisely into one of these two categories. I offered the development of the calculus from Newton to the present as an example of the epistemic process of moving from mathematized science to pure mathematics. Does this mean that Newton was doing only applied mathematics? Not exactly. The theory of the calculus, as Newton left it, certainly was not a pure theory since it was still conceptually tied to physical concepts like motion, velocity, time, and space. At the same time, however, Newton's techniques and insights were instrumental in the development of the calculus as a *pure* theory and in this sense, were works of pure mathematics as well.

4. An Objection and Reply

Michael Resnik's (1990) objection to the pure/applied distinction has to do with the distinction between the mathematical and the physical. He questions the clarity of the mathematical/physical distinction by considering the ontology of theoretical physics. He argues that quantum particles are neither physical nor mathematical, but rather something in between. Resnik starts with the observation that from the standpoint of just about all theoretical realists (including nominalists), "quantum particles count as real and clearly physical." Resnik attempts to show that these physical quantum particles are also mathematical.

Resnik offers two arguments for the claim that quantum particles are mathematical, a suggestive but inconclusive first effort and a conclusive second effort. The basic line of the first argument shows that quantum particles cannot be characterized as "tiny object[s] located in spacetime."

> Most quantum particles do not have definite locations, masses, velocities, spin or other physical properties most of the time. Quantum mechanics allows us to calculate the probability that a particle of a given type has a given "observable" property. But that does not even imply that if we, say, detect a photon in a given

> region of spacetime then the photon occupied that position prior to our attempts to detect it or that the photon would have been in that region even if we had not attempted to detect it. Prior to its detection a photon is typically in a state that is a superposition of definite (or pure) states, and quantum theory contains no explanation of how a photon or any other quantum system goes from a superposition into a definite state. (Resnik 1990, 370)

We are urged to conclude that quantum particles are too peculiar to be "tiny objects located in spacetime." It is not clear how, even if this were right, it would entail that these physical objects were also mathematical objects. At most it might lead us to rethink our characterization of quantum particles as "tiny objects located in spacetime."

Resnik is aware of this problem; thus, he offers his second argument. As he states, one could still maintain that quantum particles are "tiny bits of matter with very weird properties—ones that are only partially analogous to classical physical properties." His second argument makes use of the fact that "sophisticated" forms of quantum theory are stated in terms of fields, not particles. Resnik begins by pointing out more "weird" properties of quantum particles; he uses these to motivate his claim that:

> . . . it is better to think of particles as features of spacetime—more like fields—rather than as bodies traveling through spacetime. (Resnik 1990, 371)

He quotes a physicist who agrees that the way to think of particles is as fields. Next Resnik tells us what a quantum field is not,

> . . . quantum fields are not distributions of physical entities, rather they are roughly distributions of probabilities. (Resnik 1990, 371)

Finally, we are told how to think of quantum fields:

> How, then, are we to think of quantum particles and fields? My proposal is that we take the mathematics as descriptive rather than as "merely representational." Fields and particles are functions from spacetime points to probabilities. (Resnik 1990, 371)

At bottom, the argument looks something like the following:

(1) quantum particles are (manifestations of properties of?) quantum fields, and

(2) quantum fields are functions from spacetime points to probabilities, hence

(3) quantum particles are (manifestations of properties of?) functions (from spacetime points to probabilities).

The main problem I find in this account concerns the identification of quantum fields with mathematical functions. Why think that fields are functions? Recall that this is a "proposal" by Resnik. After making his proposal he gives one advantage to his proposal and one disadvantage. I will not attempt to sort out these advantages and disadvantages; instead I will argue that the proposal itself is a mistake, regardless of its advantages.

First, confusion exists surrounding the conception of a field. Resnik conflates two uses of "field," mathematical and physical. A mathematical field, e.g., a vector field, is a function from the domain space to a vector space: if **F** is a field from $\mathbb{R}\text{x}\mathbb{R}$ to a vector in 3-space, then it would map ordered pairs of real numbers (x,y) to vectors in 3-space, $\mathbf{F}(x,y) = \mathbf{v} := f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} + f_3(x,y)\mathbf{k}$. Thus, *mathematical* fields are indeed defined in terms of mathematical functions.[7] There are also physical fields: electromagnetic and gravitational force fields, which are arrays of forces across a region of space. Any physical thing located in or sufficiently near this region of space will be acted upon by the component forces of the field corresponding to each point in space in which the field is non-zero. Physicists describe or represent electromagnetic fields using mathematical fields. They do not, however, *identify* or *define* electromagnetic fields as mathematical fields. It is a mistake to think that our physical theory entails that the electromagnetic field surrounding an x-ray machine is *literally* a mathematical function. Physical theory no more asserts the identity of physical fields and their mathematical representations than it asserts the identity of an object's velocity and the mathematical function used to represent it.

So what is different about quantum fields? Nothing, really. As I described above, the mathematical fields used by quantum theory are functions from $\mathbb{R}^4$ into a function space. The functions in the function space are the probability density functions for the particle. A quantum field itself is (roughly) an arrangement of probabilities of interacting in certain ways with the field in that region of space. Any physical thing (usually a measurement arrangement) located in or sufficiently near this region of space will have a certainly probability of being affected in a certain way corresponding to each point in space in which the field is non-zero. Admittedly these quantum fields are harder to get a feel for than electromagnetic fields, but they are still physical—we can physically interact with them in essentially the same way.

Resnik's proposes to take the mathematics as descriptive rather than as "merely representational." His use of this dichotomy, however, is nonstandard. The dichotomy is typically understood to mean that the mathematical conceptualization of a physical situation may either *actually describe*, in some sense, the physics of the situation, or else *merely model* or represent the observables in a way that enables us to make successful predictions. As a simple example of the first kind, consider celestial mechanics. Our mathematical model of the solar system, in which the planets are in motion relative to the sun due to the force of gravity, is understood to actually describe the physics of planetary motion. It is not merely a device for predicting where we will see bright spots in the night sky; the various components of the model (planets, motion, gravity, etc.) are taken to correspond to some real component of the physical situation. In contrast, consider “probability flow” in quantum mechanics. The idea of a probability flow is used to model and solve certain problems in quantum mechanics; it is based on the analogous idea of the flow of electrical current. However, unlike in the case of electrical current, probability flow is not taken to correspond to a real component of the actual physical process (see Morrison 1990, 216 for discussion).

Normally when someone asserts that a scientific theory (including its mathematics) actually describes a physical situation, s/he does not mean that the physical objects or properties are literally the mathematical objects used to described them. This is not a case of taking the mathematics to be “descriptive” of the physics, but rather as literally being constitutive of the physics. For example, the motion of a projectile may be described by Newton’s laws (expressed mathematically as functions). We understood (prior to relativity theory) Newton’s laws and the associated mathematics as actually describing a fundamental law of nature. But by this we did not mean that the motion of the projectile is actually identical with the mathematical function used to describe it.

In light of these problems in the second argument, reconsider Resnik’s remarks concerning his first argument. His claim that quantum particles are “tiny bits of matter whose properties are only partially analogous to classical physical properties” seems right. This, however, is no cause for alarm, since classical physical properties are themselves often only *partially* analogous to the properties of everyday experience. The characteristic of quantum particles or fields that unifies them with other more familiar physical things is that they have effects or manifestations that participate in the causal nexus.

5. Concluding Remark

The distinction between pure and applied mathematics is an intuitive one, yet its philosophical significance has been largely neglected of late. I have here attempted to flesh-out and defend this distinction. Given its coherence, the relevance of this distinction to the current reassessment of the Quine/Putnam indispensability account should be clear.

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Notes

1. Some discussion of the process of application can be found in Resnik (1992), Steiner (1995) and Peressini (1997).

2. Notice that the applications of mathematics mentioned so far have been *physical* applications. This sort of application of mathematics should not be confused with a another sort that will come up below; namely, an application of pure mathematics to other pure mathematics. I will indicate this specialized sense of an "application" of pure mathematics when it arises.

3. Attempts have been made to blur the distinction between the physical and abstract (Shelton 1980, Resnik 1990). My purposes here, however, require not that this distinction be sharp, but only that it be the case that *some* of the mathematically characterized entities in the scientific theory are interpreted as something that fits into the "causal nexus." I consider the specifics of Resnik's (1990) critique of the physical/abstract distinction below in section 4.

4. Actually, it is the mathematical theory of group *representations* that is applied in quantum mechanics (group theory itself is of course made use of by group representation theory). In the interest of clarity and brevity I simplify the details of how group theory works in quantum theory. See Sudberry (1986, esp. 138 ff.) for a discussion of the details by a physicist; Steiner (1989, esp. 460 ff.) contains a detailed discussion in a philosophical setting.

5. If space were not an issue, another example would be useful to illustrate more involved bridge principles. In arithmetic (number of apples on the table) and algebraic applications, (the possible permutations of players in a bridge tournament) the bridge principles are generally more simple, immediate, and involve less idealization than other sorts of applications. It is easier in such settings to miss the significance of the auxiliary bridge premises. Steiner (1978, 24 ff.) explicitly fleshes-out such auxiliary bridge premises for the application of simple arithmetic.

6. For example, the calculus had no genuine grounding in pure mathematics until Cauchy's work in the 19th century. Dirac's delta function is another such example. And to this day, Feynman path integrals in quantum field theory are without a pure mathematical foundation. See Peressini (1997, 914-16) for discussion and additional references.

7. The mathematical fields of the quantum theory are really no more exotic (mathematically) than vector fields. These fields are maps from $\mathbb{R}^4$ into a function space composed of (probability density) functions.