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Category Theory: The Language of Mathematics

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Within the current literature on the status of category theory, three positions can be discerned. The first, espoused by Lawvere [1966], is that category theory, or the category of categories[1], provides a *foundation* for mathematics. On the other hand, Mayberry [1990] argues that category theory cannot provide a foundation because at bottom it, like all other branches of mathematics, requires set theory as a foundation. Against such foundational debates, Mac Lane assigns category theory an *organizational role*: it picks out the common structural elements of all branches of mathematics in such as way as to allow it to be systematized and unified.

In this paper, I set out to situate my claim that category theory provides the *language* for mathematical discourse. Against the foundational approaches of both Mayberry and Lawvere, I argue that there is no need to *reduce* either the content or structure of mathematical theories —category theory included— to either the universe of sets or the category of categories. Of the content of mathematics, I reiterate the model-theoretic claim that we get all the content we require from within an interpretation of a given mathematical theory. With regard to the structure of mathematical discourse, I assign category theory the role of organizing *our talk* of the structure of mathematical concepts and theories. Finally, I argue that category theory, seen as the *language* of mathematics, provides a framework for mathematical structuralism.

Category Theory As A Foundation

Before taking up the question of whether, and in what sense, category theory can be said to provide a foundation for mathematics, it is first necessary to provide an account of what we mean by the term “foundation”. There appears to be two competing views of what a foundation is. On the one hand, a foundation is taken to be a *theory* with primitive objects and/or relations, and some criteria for both definition and proof such that all other mathematical theories can be cast in its terms. On the other hand, a foundation is taken as providing those criteria that are needed to capture the essence of the axiomatic method. This second meaning differs from the first in that insofar as a foundation must provide the criterion for the axiomatic method it cannot itself be an axiomatic theory. This latter is noted by Mayberry, who uses it to argue that since “no axiomatic theory can be used to explain what the axiomatic method is” (Mayberry, [1990], p. 35) and since “only set theory can provide the needed semantics for mathematics” (Mayberry, [1990], p. 28), only an ‘intuitive’ version of set theory can provide a foundation for mathematics.

In either case, whatever the account of foundation we accept, the question of what we take the *role* of a foundation to be also arises. This question considers whether we mean for a foundation to capture the structure or the content of mathematics. Though one might be tempted to rephrase this question in terms of the traditional form/content debate, this would be a mistake, at least as regards the current foundational debate. For, at least among some philosophers of mathematics, there appears to be agreement that what a foundation must capture is the essence of *mathematical structuralism.* (See Mayberry, [1990], p. 19) Perhaps, then, more important for understanding the current foundational debate is the distinction, made by Mayberry, between *classificatory* theories and *eliminatory* theories. He says:

> [t]he purpose of a classificatory theory is to single out an otherwise disparate species of structures by fixing certain features of morphology. Among classificatory systems we may cite the axiomatic definitions of group, . . . categories, toposes . . . The essential aim here is to single out common features of otherwise quite dissimilar structures . . . Reference to such ‘things’ [as the ‘common abstract form’] occurs, not *in* mathematics, but only in peripheral discourse *about* mathematics . . . Classificatory axioms lie at the heart of modern mathematics; they provide its subject matter. (Mayberry, [1990], p. 20-21)

Eliminatory theories, on the other hand,

> are of considerable philosophical interest, for they provide the means by which the traditional problem of mathematical *objects* can be given

> a satisfactory technical solution in mathematics . . . the purpose of 'eliminatory' axiomatic theories is precisely to eliminate from mathematics those peculiar ideal [e.g., geometric figures] and abstract [e.g., numbers] objects that, on the traditional view, constitute its subject matter. . . We eliminate the real numbers, for example, by giving an axiomatic definition of the species of complete ordered fields. (Mayberry, [1990], p. 21&23)

These two types of theories are brought together under the umbrella of structuralism. A foundation, then, must provide a means for *talking about* structures and their morphology by classifying them in a way that eliminates the non-structural features from the subject matter of mathematics.

Again, lest one be tempted to reformulate this acceptance of structuralism in terms of the traditional form/content debate, this is done only at the expense of deflating the value of structuralism. Mayberry warns:

> Someone may be tempted to say 'Ah yes, that's not surprising. It's form not content that counts in mathematics.' I have no quarrel with this inexact manner of speech as long as we do not fall into the absurdity of believing that because there are no properties, in particular, that an object is required to possess in order to belong to a complete field, it follows that such a field could be composed of abstract elements which have no properties are all . . . It is one thing to be indifferent to these peculiarities, quite another to suppose that they do not exist. The former is harmless, the latter simply unintelligible. (Mayberry, [1990], p. 24]

So what impact do all of these distinctions have on the question: What does it mean for something to provide a foundation for mathematics? They show that a foundation need not be seen as providing either the form or the content of mathematics, rather it ought to be seen as accounting for both the classificatory and eliminatory role of theories. As such a foundation need only provide the means for *talking about* the axiomatic method insofar as this method has structures and their morphology as its subject matter. This is the account of foundation that I will henceforth refer to.

Mayberry and I, then, agree that a foundation need not, in a strictly 'eliminatory' sense, capture what mathematics *is about*, i.e., it need not eliminate ideal and/or abstract objects, in favor of 'primary' objects. In this we both disagree with traditional set-theoretic foundationalism. We both hold that a foundation must capture, in a manner which reflects *both* the classificatory and eliminatory role of theories, what mathematics *talks about*, i.e., it must capture the claim that the subject matter of mathematics is structures and their morphology. Thus, regardless of whether set theory or category theory provides a foundation, we both agree that structuralism[2] provides the best account of the subject matter and method of mathematics.

Why then does Mayberry hold that *only* set theory can provide such a foundation, while I hold that *neither* set theory *nor* category theory can? To answer this question requires that we attend to yet another distinction, also made by Mayberry, between the Fregean tradition of logic and the Boolean tradition. Whereas the Fregean tradition "saw mathematics as a branch of logic, . . . [the] Boolean tradition, saw logic as a branch of mathematics". (Mayberry, [1990], p. 26) Insofar as Mayberry holds that "the foundations of mathematics are to be found in the logic of mathematics" (Mayberry, [1990], p. 18) but not in mathematical logic, he further holds that *only* set theory can provide a basis for such a logic.

While Mayberry is right to claim that one cannot, in the Fregean tradition, capture the notion of multiple interpretations that is needed to talk about structures and their morphology, he is wrong to assume that following the Boolean tradition *requires* set theory (intuitive or otherwise). That is, he is right to claim that the Fregean tradition requires us to talk about objects (as domains of concepts) as opposed to structures, and that consequently,

> [i]n the Fregean tradition . . . the question of multiple interpretations doesn't arise . . . Its quantifiers are interpreted as ranging over a fixed domain, namely everything there is, at least, everything that corresponds to the type of quantifier to be interpreted. (Mayberry, [1990], p. 26)

He is also right to note that the Boolean tradition does not require either *structure as objects*, or objects to 'make up' structures, and so,

> [i]n the other, Boolean, tradition mathematical logic deals with whole families of formal languages, each designed to accommodate an infinity of different interpretations. In these languages, the domain of the quantifiers are not fixed in advance, but differ from interpretation to interpretation . . . A language, on this view, is just, so to speak, *a syntactic assemblage waiting for a structure of the appropriate sort to give its formulas meaning*. (italics added: Mayberry, [1990], p. 26)

Mayberry errs, however, in assuming that when speaking of structures themselves, only set theory can account for “the essentially semantic . . . character of the central notion of logical validity and logical consequence”. (Mayberry, [1990], p. 28). Indeed, Lambek and Scott [1989], have demonstrated that category theory, or more precisely topos theory, can offer this needed semantic account. Despite his recognition of the value of topos theory, what Lambek [1994] has failed to realize is that it is only the Fregean tradition that has the ties to foundations. And these ties are necessary to warrant the claim that topos theory has any philosophical value, let alone that it provides us with ‘the real world of mathematics’. Thus, although I agree that “when we employ the axiomatic method we are dealing with structures”, I disagree that “when we are dealing with mathematical structures, we are engaged in set theory”. (Mayberry, [1990], p. 19)

While it seems clear that neither set theory nor category theory can be a foundation in the sense of providing a *theory* which captures the idea that the subject matter of mathematics is mathematical structures, it should also be clear that neither can they provide a foundation in the sense of providing a “sea in which structures swim”. (Mayberry, [1990], p.35) And this fact cannot be altered by claiming that set theory stands along the shore of these issues since it is needed to provide a semantics for mathematics. As McLarty notes,

> Mayberry . . . has simply confused his own head with Lawvere’s. [By claiming that “the idea of denying intuitive set theory its function in the semantics of the axiomatic method never entered Lawvere’s head in his treatment of the categories of categories”. (Mayberry, [1977]) Lawvere believes ‘intuitive’ categories, and spaces, and other structures are just as real (or, more accurately, just as ideal) as ‘intuitive’ sets. (McLarty, [1990], p. 364.)

Thus, even though it is right to conclude that category theory cannot provide a foundation for mathematics, this is not because it requires an ‘intuitive’ notion of set. It is because to talk about the structure of *general* categories themselves requires some notion of structure that category theory itself cannot provide. Consequently, it is in this sense that we are to understand why it is that the category of categories cannot provide a foundation for mathematics: its eliminatory role is diminished so it cannot be used to eliminate reference to categories as ideal (or abstract) objects in favor of categories as positions in structures by giving a definition of them in terms of the category of categories.

Category Theory As An Organizational Tool

In this section I show how Mac Lane's observation that category theory provides a tool for organizing the various branches of mathematics in terms of form can be used to capture the idea that mathematics has as its subject matter structures and their morphology. The first step in this demonstration will be to investigate, in greater detail, what it means to say that category theory, because of its diminished eliminatory role, cannot provide a foundation for mathematics. Instead, however, of concerning ourselves with the problem of the categories of categories, I consider Mac Lane's claim that the "protean" nature of mathematics explains why there can be no foundation for mathematics.

The second step is to recognize category theory's 'foundational significance' (Bell, [1981]) by noting that category theory reflects the protean character of mathematics by providing the means for organizing *what we say* in such a manner that allows us to talk about both mathematical structures and the structure of such structures. Finally, I conclude this section with the suggestion that, in virtue of this privileged organizational role, we should view category theory's 'foundational significance', as arising from the fact that it provides us with *the language of mathematical structure*. This claim, together with the claim that structures and their morphology characterize the subject matter of mathematics, provides the basis for my conclusion that category theory ought to be understood by philosophers as *the language of mathematics*.

Mac Lane [1992] sets out to characterize the relation between mathematics and the physical sciences with the aim of demonstrating that mathematics is about form. In so doing he further provides us with the means for characterizing the relation between category theory and the various branches of mathematics. Of the relation between mathematics and the physical sciences he says:

> [t]his [the fact that mathematics is protean] means that one and the same mathematical structure has many different empirical realizations. Thus, mathematics provides common overarching forms, each of which can and does serve to describe different aspects of the external world. This places mathematics in relation to the other parts of science; mathematics is that part of science which applies in more than one empirical context. (Mac Lane, [1992], p. 3)

This characterization, although explicitly about the relation between mathematics and the world, implicitly provides us with an account of the relation between category theory and mathematics. If we accept that mathematics *in relation to itself* is also protean, then we can view this as an implicit recognition that all the branches of mathematics can be organized according to their structure by *specific* categories, and, furthermore, that the structure of such specific categories can be organized according to their structure through the notion of a *general* category[3].

I propose, then, that category theory, in virtue of its ability to organize our talk about both structures and the structure of structures, ought to be taken as a framework for mathematical structuralism. With reference to the previous quote, category theory 'provides the common overarching mathematical forms (or structures), each of which can and does serve to describe different aspects of *mathematical discourse*'. This is what allows us to see in what manner mathematics is protean with respect to itself and, thus, is what allows us to privilege category theory over other branches of mathematics. Again, to paraphrase Mac Lane, 'this places category theory in relation to the other parts of mathematics; category theory is the part of mathematics which applies in more than one mathematical context'.

Mac Lane appears to have this idea —that mathematics *with respect to itself* is protean— in mind when he directs us to

> [o]bserve that the natural numbers have more than one meaning. Such a number can be an ordinal; first, second, or third. . . Or it can be a cardinal; one thing, two things, . . . The natural number two is neither an ordinal nor a cardinal; it is the number two, with these two different meanings to start with. It is the form of "two", which fits different uses, according to our intent. As a result, the formal introduction of these natural numbers can be made in different ways —in terms of the Peano postulates (which describe not unique numbers, but the properties which such numbers must have) or in terms of cardinals. . . or in terms of ordinals. . . (Mac Lane, [1992], p. 4)

One consequence, then, of taking mathematics as protean, both in its relation to the world and in relation to itself, is that we realize that mathematics is not *about* objects, either empirical or mathematical. It is about the axiomatic presentation of the structure of such objects in general, but it is not about any object in particular. Or, as Mac Lane notes:

> natural numbers are not objects, but forms, variously described with a view to their various practical meanings. Put differently, an axiomatic description of number, as with Peano, does not define THE NUMBERS but only numbers up to isomorphism. (Mac Lane, [1992], p. 4]

Yet another consequence is that, given this structuralist view of mathematics, we come to see the value of seeing category theory as providing a means of talking about the structure of structure. Again, I refer to Mac Lane:

> [t]he recognition of the prevalence of mathematical descriptions "up to isomorphism" has recently been reemphasized in category theory, where products, adjoints and all that are inevitably defined only "up to isomorphism". (Mac Lane, [1992], p. 4)

Thus, we see in what sense mathematical structuralism, or more specifically the notion of description up to isomorphism, as a characterization of the relation between mathematics and the world can also be used to characterize the relation between category theory and the various branches of mathematics themselves.

In the latter case, we appeal to the protean nature of mathematics to conclude that a category-theoretic presentation of mathematical structuralism captures both the subject matter and the method of mathematics. Structuralism, then, as Mac Lane tells us,

> [i]s a consequence of the protean character because mathematics is not about this or that actual thing, but about a pattern or form suggested by various things or by previous patterns. Therefore, mathematical study is not study of the thing, but of the pattern — and this is intrinsically formal. Properties of things many suggest theorems or provide data, but the resulting mathematics stands there independently of these earlier suggestions . . . (Mac Lane, [1992], p. 8-9)

More than allowing us to characterize the subject and method of mathematics, the protean nature of mathematics demonstrates why it is that mathematics cannot have a foundation. That is, it explains why neither its method nor its subject matter can be characterized by any one theory (intuitive or formal) which seeks to describe (or prescribe) what mathematics *is about*. Of its method, Mac Lane notes,

> [t]here are many models of this system of axioms . . . [and] no set theory and no category theory can encompass them all —and they are needed to grasp what mathematics does. (Mac Lane, [1968], p. 286 &

> 287)

Of its subject matter, he further remarks that

> mathematics does not need a “Foundation”. Any proposed foundation purports to say that mathematics is about this or that fundamental thing. But mathematics is not about things but about form. In particular mathematics is not about sets . . . Real numbers live in mathematics precisely because of their multiple meanings. No one meaning is “it”. (Mac Lane, [1992], p. 9)

To appreciate the linguistic relationship between category theory and mathematics, recall our previous analogy, namely, that the relationship that mathematics bears to the world is ‘like’ the relation that category theory bears to mathematics. Not ‘like’ in the eliminatory sense that the subject matter of mathematics can be *reduced* to the subject matter of category theory. Rather, ‘like’ in the sense that just as mathematics, in virtue of its ability to classify empirical and/or scientific objects according to their structure, presents us with those generalized structures which can be variously interpreted. So, category theory, in virtue of its ability to classify mathematical objects and relations according to their structure, presents us with those generalized structures which can be variously interpreted. It is in this sense, then, that specific categories act as linguistic frameworks for organizing our talk of the subject matter (or content) of various theories in terms of structure, because

> [i]n this description of a category, one can regard “object”, “morphism”, “domain”, “codomain”, and “composites” as undefined terms or predicates. (Mac Lane, [1968], p. 287)

Likewise, general categories act as linguistic frameworks for organizing our talk of the structure of various theories in terms of structure. That is, they can be used to *talk about* the structure of the various branches of mathematics in the same manner in which the various branches of mathematics are used to *talk about* the structure of their objects.

At this point it should be clear why category theory, though foundationally significant, cannot provide a foundation for mathematics. In virtue of its diminished eliminatory role, it cannot provide the means for talking about *all* the objects of mathematics in structural terms. Because it cannot capture the intended reference behind the claim that the category of categories is a mathematical structure, it cannot be used to claim that mathematics *is about* categories. Yet, it can, in virtue of its privileged classificatory role, provide a tool for organizing our *talk of* structure in terms of category-theoretical notions in a manner which does not violate the belief that mathematics itself is protean and has as its subject matter structures and their morphisms. That is, it can provide the means for *talking about* both the structure of mathematical objects (in terms of specific categories) and the structure of the structure of mathematical theories (in terms of general categories).

Category Theory As The Language Of Mathematics

Given all that has been said, it remains to consider in what sense this privileged organizational role can provide the basis for the claim that category theory is *the language* of mathematics. I have thus far shown that category theory does not require set theory as a foundation, yet, neither can it provide a foundation. If, however, we accept the structuralist claim that mathematics, in virtue of its protean nature, has as its subject matter structures and their morphology, and if we accept that category theory allows us to organize what we say about the content and structure of mathematical concepts and theories, then we have good grounds for accepting that category theory provides us with the language of mathematics.

What category theory does, as far as our talk of mathematical concepts and relations is concerned, is provide a means for organizing and classifying our talk of ‘the structure of the relationship’ between various mathematical concepts in various mathematical theories. More specifically, we can represent our talk of mathematical concepts and relations by representing them as objects and arrows in a *specific* category, wherein such terms are taken as syntactic assemblages waiting for a structure of the appropriate sort to give their formulas meaning. We say that category theory is the language of mathematical concepts and relations because it allows us to talk about their specific structure in various interpretations, that is, independently of any particular interpretation. Likewise, at the level of mathematical theories themselves, our talk of ‘the structure of the relationship’ between mathematical theories and their relations is represented by *general* categories. We say that category theory is the language of mathematical theories and their relations because it allows us to talk about their general structure in terms of objects and functors, wherein such terms are taken as syntactic assemblages waiting for a structure of the appropriate sort to give them formulas meaning.

Our lesson then is this: just as mathematics is protean with respect to the empirical or scientific world, so category theory is protean with respect to mathematical discourse. Just as mathematics can be seen as providing the language for the world —it allows us to *talk about* physical objects in structural terms without having to *be about* those objects— so category theory can be seen as providing the language for mathematics —it allows us to *talk about* objects in structural terms without having to *be about* those objects. It is in this sense that category theory ought to be taken as the framework for mathematical structuralism —it is the language that allows us analyze mathematical existence, meaning and truth on the basis of what can be said in, and about, mathematical structures[4]. This, then, is where we find the analogy between mathematical and physical discourse: if God writes in the language of mathematics, then Dieudonné[5] writes in the language of category theory!

1 For a full characterization and discussion of the category of categories, see McLarty, [1995], p. 110-111.

2 I intend this characterization to include the idea that objects (or relations), as interpreted concepts, i.e., as objects (or arrows) in specific categories, are positions in structures. And furthermore, that theories, as objects of mathematical study, i.e., as objects in general categories, are likewise positions in structures.

33A *specific* category C is a two-sorted system, the sorts being called *objects* A of C and *morphisms* *f* of C. The *undefined terms*, in context, are “A is the *domain* (or codomain) of *f*”, “*k* is the *composition* of *g* with *f*”, and “*f* is the *identity* morphism of A”. (See Mac Lane, [1971], p. 231) A *general* category can be defined functorially in the tradition of Lawvere [1966]. For simplicity, however, I will refer to McLarty’s [1995] account. A *general category* C can be thought of as an object in the category of categories **C**, whereby “[o]bjects and arrows within C are thought of as functors 1®C and 2®C respectively, . . . [where] 1 is a terminal category . . . [and] . . . 2 is a category with exactly two global elements 0: 1® 2 and 1: 1® 2” (McLarty, [1995], p. 110) This approach to defining categories, begins with an axiom that says that categories and functors collectively form a category; that is, functors have domains and codomains, and compose and so on. (McLarty, [1995], p. 110) Thus, we have a connection between general and specific categories in the sense that 'A category has objects and arrows. . .' becomes shorthand for 'We begin by looking at the functors to a category from 1 and 2' (McLarty, [1995], p. 110) and likewise we have established a connection between the eliminatory and classificatory roles of category theory.

4 One possible question that may remain is: If the only claims of existence possible are those from made from within a local discourse (from within an interpreted theory), then what basis do we have for calling these objects linguistic *entities*, are they not equally linguistic *fictions*? For example, one might ask: what distinguishes the mathematical claim "the number 2 exists" from the claim that, from within a given (fictional) novel, "Santa Claus exists"? In response, I note that the literary language of a novel is intended to capture the idea that its story *is about* its objects, where "is" is taken in the figurative sense of the term. While novels may have a fixed structure they *do not* have multiple interpretations. Mathematical language (presented category-theoretically), however, is intended to capture the idea that its theories *talk about* their objects as "positions in structures". So the claim "the number 2 exists" says more than it is meaningful to talk about "the object 2 as posited in *this* interpretation" it also says "any theory of the same structure, be it a physical or mathematical theory, regardless of its particular interpretation, *must* posit the object 2". On the other hand, one can image a novel whose structure allows for talk about a man (albeit it a fictional man) from the north pole, who knows who is naughty and who is nice, who rides a sleigh with reindeers, etc., which does not posit the object "Santa Claus". This *forcing* of existence by virtue of *what can be said* in, and about, structures is the essence of what I have elsewhere termed "the semantic realist" interpretation of mathematical structuralism — objects exist as position in structures, nothing more (contra the platonist) *and* nothing less (contra the post-modern or 'literary fictions' approach).

5 I mean to use Dieudonné here as a general example of a structuralist, that is, without reference to the fact that he was not a category-theoretic structuralist. (Though, as McLarty has pointed out, he sometimes did praise category theory and finally came to see it as allowing for a notion of structure that was different from Bourbaki's.)

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