The Failure of Equivariance for Real Ensembles of Bohmian Systems

Abstract

The continuity equation has long been thought to secure the equivariance of ensembles of Bohmian systems. The statistical postulate then secures the empirical adequacy of Bohm's theory. This is true for ideal ensembles—those represented as continuous fluids in configuration space. But ensembles that confirm the experimental predictions of quantum mechanics are not ideal. For these non-ideal ensembles neither equivariance nor some approximation to equivariance follows from the axioms of Bohm's theory. Approximate equivariance (and hence the empirical adequacy of Bohm's theory) requires adding to Bohm's axioms or giving a detailed account of the behaviour of Bohmian ensembles.

1 Introduction

In 1952 David Bohm published a deterministic theory and proved it to be empirically equivalent to non-relativistic quantum mechanics. His proof of empirical equivalence uses three assumptions (Bohm 1952, 171):

> (1) That the ψ-field satisfies Schroedinger's equation.
>
> (2) That the particle momentum is restricted to $\mathbf{p} = \nabla S(\mathbf{x})$.
>
> (3) That we do not predict or control the precise location of the particle, but have, in practice, a statistical ensemble with probability density $P(\mathbf{x}) = |\psi(\mathbf{x})|^2$.

From the very start the third assumption, the statistical postulate, has come under scrutiny (Keller 1953). Bohm himself saw the need to give reasons for why the statistical postulate must hold for ensembles of Bohmian systems. By deriving it from the other assumptions of his theory he sought to demote it from the status of postulate. There have since been many attempts to do so (Bohm 1953), (Bohm and Vigier 1954), (Bohm and Hiley 1989), (Dürr *et al* 1992), (Valentini 1991).

These diverse attempts share one thing in common. They all hold that once a suitably isolated ensemble of Bohmian systems in our universe is distributed according to the statistical postulate, that ensemble remains distributed according to the statistical postulate so long as it remains isolated. The distribution specified by the statistical postulate is said to be *equivariant* with respect to Bohm's equations of motion.

Equivariance is a mathematical consequence of conditions (1)–(3). In this paper I question its application to real ensembles containing a finite number of systems. The aim of the paper is to show that conditions (1)–(3) are not together sufficient to secure the equivariance of a real ensemble of Bohmian systems. And it is these real ensembles that confirm the experimental predictions of quantum mechanics.

Dürr *et al* have shown that real ensembles are equivariant by adding to conditions (1)–(3) (Dürr *et al* 1992). Their analysis leaves open the question of whether it is *necessary* to add to these conditions in order to secure equivariance. The aim of this paper is to show that it is necessary to do so, if Bohm's theory is to account for the experimental statistics predicted by quantum mechanics.

I begin with a sketch of the received view of the role of continuity equation in Bohm's theory. Then I draw a distinction between real and ideal ensembles and explain how this difference poses a problem for the empirical adequacy of Bohm's theory.

2 The role of the continuity equation

It is widely accepted that conditions (1)–(3) are together sufficient for the empirical adequacy of Bohm's theory. Why so? The usual story runs like this.

It follows from Schrödinger's equation that

$$\frac{\partial(\psi^*\psi)}{\partial t}+\nabla\cdot J=0 \qquad (1)$$

where $\mathbf{J}(x, t)$ is defined as

$$\mathbf{J}(x,t) \equiv \frac{1}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*). \tag{2}$$

Now equation ?? is identical in form to the continuity equation for fluids,

$$\frac{\partial \rho(x, t)}{\partial t} = -\nabla \cdot (\rho(x,t)\mathbf{v}(x,t)). \tag{3}$$

Equation ?? is just a way to express the conservation of mass in the case of fluids.

As many others have, Bell noticed the similarity between equations ?? and ??. Moreover he posited that equation ?? is the continuity equation for $\psi^* \psi$ (Bell 1987, 127–128). As a consequence we obtain from equation ?? an expression for the velocity field that is exactly that of specified by Bohm. When equation ?? is rewritten to reflect Bell's posit it becomes

$$\frac{\partial(\psi^* \psi)}{\partial t} + \nabla \cdot (\psi^* \psi\, \mathbf{v}) = 0. \tag{4}$$

It then follows that the expression for the velocity must be

$$\mathbf{v}(x,t) = \frac{\mathbf{J}(x,t)}{\psi^*(x,t)\psi(x,t)}. \tag{5}$$

This is the velocity of the representative point x of a Bohmian system whose wavefunction is $\psi(x, t)$ and is precisely the velocity fixed by Bohm's theory. In his formulation of the theory, Bohm arrived at this expression for velocity in a different way. The connection between the two very different approaches of Bell and Bohm is interesting and perhaps profound. See (Baublitz and Shimony 1996) for more detail.

The continuity equation ?? is an equation governing the wavefunction-intensity field and the velocity field over configuration space. Now suppose a real ensemble of BCI systems with probability density function $f(x, t)$ evolves on this configuration space with a velocity field over this configuration space fixed by equation ??. If we assume that systems are neither created nor destroyed, a continuity equation holds for this ensemble probability density:

$$\frac{\partial f(x, t)}{\partial t} = -\nabla \cdot (f(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)). \tag{6}$$

The story is completed by noting that if at some time t_0, $f(x, t_0) = \psi^*(x, t_0)\psi(x, t_0) = | \psi(x, t_0) |^2$, then it follows from equation ?? that $f(x, t) = | \psi(x, t) |^2$ for all $t > t_0$. The continuity equation ?? is responsible for the equivariance of the ensemble distribution. The continuity equation is derived from axioms (1) and (2) of Bohm's theory. Adding condition (3), the statistical postulate, is thought to secure the empirical adequacy of Bohm's theory.

I argue that the equivariance of a real ensemble of Bohmian systems does not follow from conditions (1)–(3) alone. That is, it doesn't follow from the continuity equation ?? even when the statistical postulate is assumed. More is needed and any plausible addition threatens the relevance of the continuity equation for the equivariance of real ensembles of Bohmian systems. Hence I argue that even when the statistical postulate is assumed, (1) and (2) are not enough to secure the empirical adequacy of Bohm's theory.

3 Real and ideal ensembles

For an idealised ensemble of systems, one that can be thought of as a continuous fluid in configuration space, there is no doubt that the continuity equation entails equivariance of the ensemble distribution. But no actual ensemble contains an infinite number of systems, much less an uncountably infinite number of such systems. It is a fact that for these actual ensembles prepared in laboratories, the experimental statistics predicted by quantum mechanics are extremely well confirmed.

It is these real-world experimental statistics that Bohm's theory must account for. The continuity equation describes the evolution of a continuous field over the configuration space. It is impossible, due to a simple consideration of cardinality, for the systems in the real ensemble to occupy every point in the allowed region of configuration space.

Later I will be making use of coarse-grained regions of a configuration space. The actual size of the coarse-graining will not be important but its lower bound is set by the resolution of the measuring instruments or the sort of measurement performed on the individual systems in the ensemble. Also, the size of a region of the configuration space R must be sufficient to capture the fact that the real ensemble is distributed according to the statistical postulate. If the size of R is small enough to contain no real systems, and the wavefunction is non-negligible in R, then the real ensemble cannot be said to be distributed according to the statistical postulate. Given that real ensembles contain a large but finite number of systems, this number also sets a lower bound on the size of R. The difference in cardinality between real and ideal ensembles makes it the case that within each coarse-grained segment of the configuration space, there are points that are not occupied by any system in the real ensemble.

It is precisely this gap that blocks the continuity equation alone from entailing the equivariance of $|\psi|^2$ for real ensembles. It is important to note that I question the idealised representation of a real ensemble as a fluid not because of an idiosyncratic view about how the probability density $|\psi|^2$ should be interpreted or a distaste for infinitary probabilistic models. Rather, it is because the experimental facts must be explained by BCI. The real-world ensembles that generate these experimental facts cannot be represented as continuous fluids in configuration space. They cannot because we will see that not every set of trajectories of the systems in the real ensemble will mimic the dynamical behaviour of the idealisation. Additional conditions are needed to rule out these problem sets of trajectories; the continuity equation is not enough.

I am not questioning all instances in which inferences are made for finite real-world cases from an infinitary probabilistic model. There is no reason to reject wholesale this useful kind of inference. But I do call into question a specific kind of inference from an infinite to a finite case.

The particular case can be described as follows. There is an infinitary (or continuous) probability distribution that evolves according to a certain equation. This distribution is governed by a continuity equation. Suppose an inference about the distribution of a finite case is made at t_1. Then, a similar inference at a later time t_2 from the infinite to finite case might not be justified. The correctness of this inference depends on the equivariance of real ensembles of systems between the times t_1 and t_2. My aim is to show that the equivariance of real ensembles does not follow from the fact that the infinitary distribution is governed by a continuity equation. Applying this to Bohm's theory, my aim is to show that the equivariance of real ensembles of Bohmian systems, and hence the empirical adequacy of Bohm's theory, does not follow from the continuity equation ?? and conditions (1)–(3) alone.

4 The equivariance problem for real ensembles

The statistical postulate is formulated in terms of the intensity of the wavefunction over configuration space. This quantity, $|\psi(x)|^2$, is standardly interpreted as a probability density function defined over the whole configuration space. The ensemble distribution is to be understood in Bohm's theory as the ratio of the number of systems in the ensemble that are in the region Δx of the configuration space to the total number of systems in the ensemble. The intensity of the wavefunction field is defined at every point in the configuration space. The probability of finding systems in the ensemble in a certain region of the configuration space is roughly the ratio of the systems in that region to the total number of systems in the ensemble. If the ensemble is distributed according to the statistical postulate at time t, then this ratio at time t is roughly equal to $\int_{\Delta x} |\psi(x', t)|^2 \, dx'$.

Each system in the real ensemble is at some precise point in the configuration space. But no real ensemble can fill all points of the allowed configuration space. Hence any region of the configuration space will have points that are not occupied by any of the systems in the real ensemble. Let's focus on any non-empty region R of the configuration space. Assume that conditions (1)–(3) hold and the wavefunction intensity is non-negligible throughout R. Let the size of R be greater than or equal to the size of the coarse-graining adopted in order for the statistical postulate to be satisfied for the real ensemble under consideration.

Since the ensemble is distributed according to the statistical postulate, it follows that $\int_R |\psi|^2 dx$ gives (roughly) the ratio of the number of systems in R to the total number of systems in the ensemble. Due to the difference in cardinality between the number of points in R and the number of real systems in R, there are points in R that are empty—i.e. not occupied by any system in the real ensemble.

The systems in the ensemble are following trajectories in and out of the region R. Let's focus on a duration $\Delta t = t_1 - t_0$ and assume that at the start, t_0, and end, t_1, of this duration the real ensemble is distributed according to the statistical postulate. I take the real ensemble to be equivariant as long as this condition is satisfied. Equivariance of the real ensemble distribution either depends on the particular positions of systems within R at t_0 or it's insensitive to the precise position of these systems. Let's consider each of these mutually exclusive options.

If equivariance of the distribution depends on the precise positions of the systems within R, then Bohm's conditions (1)–(3) cannot be together sufficient for the equivariance of the real ensemble. Let's see why.

Suppose at t_0 we put test systems at each and every point in R. Then, at t_1 we can expect some of these systems to have left R and some to remain in R. The dynamics of Bohm's theory is responsible for this behaviour of test systems. The continuity equation, in conjunction with conditions (1)–(3) guarantees that this dynamical behaviour will result in the test ensemble being distributed according to the statistical postulate at t_1.

But real ensembles are not like the test ensemble; we just saw that a subregion r of R does not contain any real systems. Whether a real system is carried out of R or not at the end of Δt depends in Bohm's theory only on its position at t_0. If all the real systems are, for example, at points in R that are not carried out of R, then it is not clear how the real ensemble can manage to be equivariant, i.e. manage to be distributed according to the statistical postulate at t_1.

By hypothesis the real ensemble is distributed according to the statistical postulate at t_0. So, if the equivariance of the real ensemble depends on the precise positions of the real systems within R, the equivariance of the real ensemble does not follow from the conditions (1)–(3).

Alternatively,

> (H) the precise position of real systems within a region R plays no role in the equivariance of the real ensemble distribution.

To see what must hold for (H) to be satisfied let's start by asking what *cannot* hold if (H) is to be satisfied. The following cannot hold:

> (H1) At each duration some points inside R are swept out of R while some others remain in R. (A point that is swept out of R is a point such that if a system starts out at that point, it is carried out of R by the end of the duration.)

It is easy to see that (H1) conflicts with (H). A real ensemble does not fill up every point inside R. If some of the points inside R are not swept out while some are swept out in order to maintain the distribution given by the statistical postulate, then it is clear that not every arrangement of the positions of the systems in the real ensemble that are in R will do. Again, for example, if all the systems are at points in R that are swept out, then the distribution is not likely to remain equivariant.

If (H1) cannot hold, then the following must hold as a necessary condition for (H):

> (H2) By the end of the duration Δt all points within each region R are either swept out of R or remain in R.

The idea is that if all the points in a region are swept out or remain in the region, then it wouldn't matter how the real systems in the region were positioned. They would all be swept out or remain regardless of their precise positions within R.

(H2) may be true in one of two ways:

(H3) By the end of the duration Δt, for each region R, all points within it are swept into one and only one region R_1. (R might equal R_1; if it does then we get the situation in which all in the region remain in the same region for the duration.)

(H4) By the end of the duration Δt, for each region R, all points within it are swept out of it, some of these points enter region R_1 while some enter region R_2, and $R \neq R_1 \neq R_2$. (If this last condition fails, then either (H4) = (H1) or (H4) = (H3).)

Consider (H4). It conflicts with (H) for the same reason that (H1) does. This leaves us with (H3). Using a transition matrix to model this hypothesis it is easy to check that it is not in general strong enough to generate convergence to the distribution specified by the statistical postulate. For the matrix contains only ones and zeros and such a matrix does not converge to anything other than the original matrix itself. But this is in general not the distribution called for by the statistical postulate.

5 Conclusion

If the equivariance of real ensembles follows from the continuity equation and the statistical postulate alone(i.e from conditions (1)–(3) alone), it must be because the equivariance of real ensembles is insensitive to the exact positions of the real systems within regions of the configuration space such as R. This can be due only to (H3). But (H3) is too weak to generate convergence to the distribution specified by the statistical postulate. Again, conditions (1)–(3) are not together sufficient for the equivariance of real ensembles.

It remains to notice that any strengthening of (H3) turns it into (H4). Instead of each row in the transition matrix having all zeros except for a single 1, each row now has more than one element that is non-zero. So if (H) is assumed, the best that can be done is (H3). But (H3) is not in general enough to maintain equivariance of the real ensemble. So if the real ensemble is to be equivariant, then (H) must be abandoned.

This puts us back where we started: the equivariance of a real ensemble depends on the precise positions of the systems within regions R of the configuration space. So even if initially a real ensemble satisfies the statistical postulate, the continuity equation for $|\psi|^2$ alone does not ensure that the real ensemble distribution will be equivariant. A detailed dynamical or statistical analysis of the trajectories generated by Bohm's equations of motion is needed in order to secure equivariance for real ensembles of Bohmian systems. These details are likely to render superfluous the continuity-equation account of the equivariance of real ensembles.

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