Methods of Proof

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1 Direct Proof

We want to prove that $P \implies Q$, so we assume that P is true and, by using a sequence of logical steps, we reach the conclusion that Q is also true.

Example 1.1.

Theorem. Let $n \in \mathbb{N}$. If n is odd, then n^2 is also odd.

Proof. P is the proposition n is odd and Q is the proposition n^2 is odd. By using the direct method we assume that P is true, then we have that $n = 2m+1$, where $m \in \mathbb{N}$. So we can rewrite n^2 as follows: $n^2 = (2m+1)^2 = 4m^2+4m+1 = 2(2m^2+2m)+1$. The expression in parentheses is also a natural number, which we can denote as $p = 2m^2+2m$. Thus $n^2 = 2p+1$, which means n^2 is also odd. Then from the fact that P is true we reached the conclusion that Q is true, and that completes the proof.

2 Contrapositive Proof

We want to prove that $P \implies Q$. Notice that $(P \implies Q) \iff (\neg Q \implies \neg P)$. By using this method, we assume that Q is false and, by means of a sequence of logical steps, we get to the conclusion that P is also false, which is enough to prove that $P \implies Q$ is true.

Example 2.1.

Theorem. Suppose m and b are real numbers and $m \neq 0$. Let f be the linear function denoted by $f(x) = mx + b$. If $x \neq y$ then $f(x) \neq f(y)$.

Proof. Here we have $x \neq y$ as proposition P and Q is $f(x) \neq f(y)$. We want to prove that $P \implies Q$, but we are going to use the contrapositive method. So we go from $\neg Q$ to $\neg P$. $\neg Q$ means $f(x) = f(y)$ and so $mx + b = my + b \implies mx = my \implies x = y$, by first adding $-b$ and then dividing by m on both sides. Now $x = y$ is just $\neg P$, so $\neg Q \implies \neg P$, and since this is equivalent to $P \implies Q$, the proof is complete.

3 Proof by Contradiction

Again, we want to prove that $P \implies Q$. This implication is false only when we have P true and Q false, that is, $\neg(P \implies q) \iff (P \wedge \neg Q)$. To prove by contradiction, we are going to use the *false* case and show that it leads to $(R \wedge \neg R)$ for some proposition R which will eventually pop up. Since $(R \wedge \neg R)$ is an absurd, the *false* path cannot be right, then $P \implies Q$ must be true. The advantage of this method is that we have two propositions (P and $\neg Q$) from which to start reasoning. It is recommended to use this method when $\neg Q$ gives us new information.

For the example bellow, $r \in \mathbb{R}$ is *rational* when there are $m, n \in \mathbb{Z}, n \neq 0$, such that, $r = m/n$. If r is not rational, then it is *irrational*.

Example 3.1.

Theorem. If $r \in \mathbb{R}$ such that $r^2 = 2$, then r is irrational.

Proof. P is the proposition $r^2 = 2$ and Q is "r is irrational". By contradiction, let us assume that P is true and Q is false, that is, $r^2 = 2$ is true and "r is rational" is also true. So there exist numbers $m, n \in \mathbb{Z}, n \neq 0$, such that, $r = m/n$. Suppose m/n is such that there are no common factors between m and n that are greater than 1. If there are such factors, we can just simplify the fraction dividing both numerator and denominator by these same factors. It follows that $r^2 = 2 \iff (m/n)^2 = 2 \iff m^2/n^2 = 2 \iff m^2 = 2n^2$. Since n^2 is some integer, $2n^2$ is even and the last equation tells us that m^2 is even. Now we use the fact that any even number squared is still even (which can easily be proved by contrapositive on example 1.1) to conclude from m^2 is even that m is also even. Now, because m is even, we can rewrite $m^2 = 2n^2$ as $(2x)^2 = 2n^2$, for some $x \in \mathbb{Z}$, then we get $(2x)^2 = 2n^2 \iff 4x^2 = 2n^2 \iff 2x^2 = n^2$. Since x^2 is some integer, $2x^2$ is even, and so the last equation tells us that n^2 is even, which implies that n is an even number. Therefore we have that both m and n are even numbers, so they do have a common factor greater than 1 (2 divides all even numbers). We can see that, if we assume that m and n do not have a common factor greater than 1 we reach the conclusion that they do have a common factor greater than 1, which is an absurd. Therefore we cannot have $r^2 = 2$ and "r is rational", that is, $(P \wedge \neg Q)$ cannot be true, which means the only case where $P \implies Q$ is false just broke apart. The conclusion is that $P \implies Q$ is true and that completes the proof.