Category Theory Club

Lecture notes 1 - Max Johnson

1 Definitions

1.1 For definitions of Category and Functor, see the readings for meeting 1.

1.2 Contravariant functors

Normally we consider covariant functors, where if $F : C \to D$ is a functor, and $x \xrightarrow{h} y$ is in C, then F gives us the following diagram:

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, phantom, "\Downarrow F"] & y \\ F(x) \arrow[r, "F(h)"'] & F(y) \end{tikzcd}$$

A contravariant functor G reverses this action giving us

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, phantom, "\Downarrow G"] & y \\ G(x) & G(y) \arrow[l, "G(h)"] \end{tikzcd}$$

As G reverses all arrows, it is easy to check that G can still respect composition by reversing the order of composition. That is, G obeys $G(hk) = G(k)G(h)$. The second composition makes sense as we reversed domain and codomain through G.

1.3 Opposite Categories as functors

The readings for the first meeting defined C^{op}, the category obtained by taking C and "reversing all the arrows." As you might have guessed,

$$-^{OP} : \textbf{Categories} \to \textbf{Categories}$$

is a contravariant functor (often times we will use $-$to denote a functor defined for arbitrary categories). In fact, it is the contravariant version of the identity functor. This can be seen by the parallel notions of "reversing the arrows".

2 Defining Hom

2.1 Local smallness

For any category C, we say that C is *locally small* if for any $A, B \in C$ we have that the collection of morphisms from A to B forms a set.

For example, in the language given in the readings, the category **Sets** is large because there are too many objects for **Sets** to have a set of objects (recall that the set of all sets cannot exist). However, the set of functions between any two sets does exist, so **Sets** is locally small.

All of the categories we will work with will (probably) be locally small.

2.2 Hom sets:

We can now define one of the central concepts of category theory. For any two objects A, B in a category C, we define

$$\mathrm{Hom}_C(A, B)$$

to be the set of all arrows $A \to B$.

Clearly, this only makes sense in a locally small category. More generally, we say that a category is locally X if all of its $Hom - sets$ have property X.

3 Sets as a category

There is a full re-axiomitization of **Sets** that is equivalent to the ZFC axioms called ETCS (Elementary Theory of the Category of Sets). ETCS is interesting in its own right (and is also the subject of my REU paper) but for our purposes we will be working with Categories in the world of ZFC, as opposed to sets in the world of ETCS. Therefore the following is a limited explanation of some of the ways that category theory thinks about sets.

3.1 Initial and Final objects. For the definitions of initial and final objects, see the readings for meeting 1.

Due to the unique (empty) function $\emptyset \to S$ for any set S, we have shown that $\emptyset$ is initial. Let 1 denote a set with one element. Then $S \to 1$ is the unique function sending every element of S to the single element of 1. Therefore all final objects in **Sets** are singleton sets. Category theory doesn't really care about specific elements, so we might as well pick some singleton at random and call it 1.

3.2 Elements

Category theory is mainly concerened with arrows rather than the details involved in the objects. In fact, one of the most important results tells us that *all* we need to be concerned about is the arrows.

There is an obvious bijection between $\mathrm{Hom}(1, S)$ and the elements of S. Namely, if $s \in S$ we consider the function $1 \to S$ such that the one element of 1 gets mapped to s. Therefore we can think of the elements of S as "all the ways a singleton can be mapped into S."

4 The Natural numbers

The Peano axioms are one of the best-known axiomatizations of the Natural Numbers $\mathbb{N}$. Briefly, they are:

1. There exists a set $\mathbb{N}$ and an injective function $s : \mathbb{N} \to \mathbb{N}$.
2. There exists an element $0 \in \mathbb{N}$ such that $s(n) \neq 0$ for all $n \in \mathbb{N}$.
3. If $A \subset \mathbb{N}$, and $0 \in A$, and $s(a) \in A$ for all $a \in A$, then $A = \mathbb{N}$.

Notably, the last of these is known as the recursion axiom, and requires that every element of $\mathbb{N}$ be obtainable by applying s to 0 some number of times.

The categorical interpretation of $\mathbb{N}$ is as follows: There exists is a triple $(\mathcal{N}, s, 0)$ such that $1 \xrightarrow{0} \mathcal{N}$ is an element of $\mathcal{N}$ in the sense described above, $s : \mathcal{N} \to \mathcal{N}$ is an arrow, and if A is a set with a fixed $a \in A$ and $f : A \to A$ is an function, there exists a **unique** induced function h such that the following commutes:

$$\begin{tikzcd}
 & N \arrow[r, "s"] \arrow[dd, dashed, "\exists! h"] & N \arrow[dd, dashed, "\exists! h"] \\
1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\
 & A \arrow[r, "f"'] & A
\end{tikzcd}$$

Because it commutes, we can imagine extending this diagram to the right with repeated applications of s, f respectively.

5 Equivalency ($\mathcal{N} = \mathbb{N}$)

Normally to show equivalency you prove it in both directions, but we will only show that the categorical definition implies the peano axioms. Namely, we will prove (3), and then prove s is injective.

5.1 Axiom (3)

Let A be a subset of N (the notion of subset is more complicated in category theory, but for now just assume it functions like a normal subset, ie for every arrow $1 \xrightarrow{n} A$ there exists an arrow $1 \xrightarrow{n} \mathbb{N}$.)

Let $s\mid_A$ be the sucessor restricted to A. Assume further that $a \in A \implies sa \in A$. Finally, let $0 \in A$. Then we have a diagram of the form:

$$1 \xrightarrow{0} A \xrightarrow{s} A$$

By the axioms for $\mathcal{N}$ there must exist a unique function $h$making the following commute:

$$
\begin{tikzcd}
 & N \arrow[r, "s"] \arrow[d, dashed, "\exists! h"] & N \arrow[d, dashed, "\exists! h"] \\
1 \arrow[ur, "0"] \arrow[r, "a"'] & A \arrow[r, "s|_A"'] & A
\end{tikzcd}
$$

Then let $i : A \to N$ be the inclusion of the subset into the superset. Extend our diagram:

$$
\begin{tikzcd}
 & N \arrow[r, "s"] \arrow[d, dashed, "\exists! h"] & N \arrow[d, dashed, "\exists! h"] \\
1 \arrow[ur, "0"] \arrow[r, "a"'] \arrow[dr, "0"'] & A \arrow[r, "s|_A"'] \arrow[d, "i"] & A \arrow[d, "i"] \\
 & N \arrow[r, "s"] & N
\end{tikzcd}
$$

We know that the bottom of this diagram commutes because i is basically the identity function, but on the subset. Therefore we have a commutative diagram

$$
\begin{tikzcd}
 & N \arrow[r, "s"] \arrow[d, dashed, "ih"] & N \arrow[d, dashed, "ih"] \\
1 \arrow[ur, "0"] \arrow[r, "0"'] & N \arrow[r, "s"'] & N
\end{tikzcd}
$$

But these diagrams are unique by our axiom! And we know that $id_{\mathcal{N}}$ would make this same diagram commute instead of ih. So by unqiueness, $ih = id$. Clearly that means $h = id$ as i is already an identity map for A. Thus $A = N$.

See you next week!