Problem Set 1

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Problems are taken or inspired by

- Aluffi's Algebra Chapter 0
- Category Theory In Context
- * means that it would be awesome if someone could present it during meeting.
- ** means it strikes me as harder than the rest of the problems in this set. Also means *.

This is pretty long, to be honest I'd probably just convince myself of the ones without stars rather than write anything down and then spend time on any *-problems that seem interesting.

1. Fun with morphisms/arrows
   (a) Show that inverse morphisms are unique.
   (b) Let $f : x \to y$ be a morphism in some category C. Prove that if we also have $h, g : y \to x$ satisfying$hf = 1_x$ and $fg = 1_y$ that $g = h$ and $x \cong y$.
2. Learning more about **Sets**. There are a number of different conventions for how to refer to categories. For clarity in these problem sets I will try to use bold font and very clear names. (For example, **Ab** is normally used to mean the category of abelian groups. I will just use **AbelianGroups**.)
   (a) Prove that if a Category $\mathcal{C}$ has final objects that they are unique up to isomoprhism.
      i. The phrase "unique up to isomorphism" should be read, if F, F' are final objects then $F \cong F'$. And extension of this is the phrase "unique up to unique isomorphism" which imposes that in some sensible category, the isomorphisms from $F \to F'$ are themselves objects, and isomoprhic in that category.
   (b) Prove that a final object in $\mathcal{C}$is initial in $\mathcal{C}^{op}$ and visa versa.
   (c) Prove that $\emptyset$ is the only initial object in **Sets**. (Hint: an ismorphism in **Sets** is a bijection).
   (d) Let 1 denote a terminal object in **Sets**. Classify the isomorphism class of terminal objects in this category. (An isomorphism class is a collection of objects isomorphic to eachother).

(e) ** We denote by $Hom(A, B)$ the set of functions from A to B. Define a bijection between $Hom(1, S)$ and S. That is, every element of a set can be thought of as a morphism from the terminal object. Convince youself why this makes sense from the structural focus of category theory. From now on, elements of sets will be discussed in this way.

3. Proving things are categories.

   (a) Let T be a partially ordered set. Let the elements be objects and the relation $t \leq t'$ be the arrows. Prove that this forms a category. (Recall that a partial order is weaker than a normal order (formally a total order) as some objects cannot be compared. For example, if you take $\mathbb{R} \times \mathbb{R}$ and only order it by the first element of every ordered pair, then you cannot say that $(a, b) \leq (a, b')$ or visa versa).

   (b) * Let X be a topological space, and $\mathcal{P}(X)$ be its powerset. Let $A \hookrightarrow B$ be an inclusion of A into B. Prove that $(\mathcal{P}(X), \hookrightarrow)$ forms a category vis a vis part (a).

   (c) ** Let $\mathcal{C}$ be a category and $x \in \mathcal{C}$. We now define a slice category of $x/\mathcal{C}$. The objects are all morphisms in $\mathcal{C}$with domain x. If $f : x \to y$ and $g : x \to z$ are two *objects* in $x/\mathcal{C}$ we add arrows $f \to g$ if there exists a $h : y \to z$ such that $hf = g$.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

   Prove that this actually a category.

   (d) Prove in one line that if we define C/x as the set of morphisms with codomain $x$and morphisms h defined analagously, that this also forms a category.

   (e) ** Let G be a group. Prove that we can think of G as a category with one object, and elements as endomorphisms (endo = to itself, so if $g \in G$ as a group, we think of $g : G \to G$ in a category).

4. Proving some things are Functors

   (a) Prove that if $F \circ G$ is a composition of functors with sensible domains/codomains, then $F \circ G$ is itself a functor.

   (b) Let **Groups** be the category of groups with homomorphisms as monomorphisms. Give an (obvious) example of a functor **Groups** $\to$ **Sets**. (Note, the most obvious example here is known as a "Forgetful Functor.")

   (c) ** Recall (or learn!) that given a set S the free group $Free(S)$ generated on S is defined as follows:

i. A *word* in S is any finite "word" of elements in S, adding inverses. For example, if $a, b, c \in S$, then $abc, cb^{-1}a, a^{-1}abcc^{-1}$ are all words. But note that the last word can be reduced by cancelling inverses. Therefore the Free group is the set of all irreducible words in S.

ii. Prove that F : **Sets** →**Groups** defined by $S \mapsto Free(S)$ is a functor. Where does Fmap functions between sets? This one is called the "Free Functor."

(d) * Let $-^{op}$: **Categories** → **Categories** represent the transformation of a category into its opposite (ie $C \mapsto C^{op}$). Prove this is a functor.

(e) Show that functors preserve isomorphisms.