

UNDECIDABILITY AND THE STRUCTURE OF THE TURING DEGREES

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ABSTRACT. This paper explores the structure and properties of the Turing degrees, or degrees of undecidability. We introduce the Turing machine, an abstract model of computation, in order to develop the concepts of undecidability and Turing reduction. We demonstrate the technique of proof by reduction through a series of examples of undecidable problems related to context-free grammars. We then employ reducibility to consider a partial ordering on the set of Turing degrees, \mathcal{D} . Finally, we prove a variety of theorems related to the structure of \mathcal{D} .

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1. INTRODUCTION

Historical overview of the field, motivation for study [1] [19] [18] [11]

1.1. Decision Problems.

2. TURING MACHINES

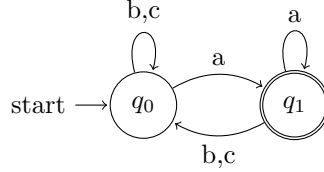
Upon consideration of different types of problems, it quickly becomes clear that some are much more difficult than others. For example, when considering words over the alphabet $\{a, b, c\}$, it seems much easier to determine whether a word ends in an a than whether a word is a palindrome. It seems even harder still to determine

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whether the word is of the form $a^n b^n c^n$. In order to formalize these notions of difficulty, we need to build abstract models of computation, and then test their ability to decide such questions. We observe that a machine that can solve the first problem only needs to “remember” the same amount of information no matter how long the input is: the last letter of the word. In contrast, the amount of memory that the second and third problems require is dependent on the size of the input, because the first half of the palindrome or n , respectively, can be arbitrarily large. The difference between the second and third problems is more subtle: the second problem can be solved moving only one direction in memory, while the third problem requires the ability to “look back” in memory.

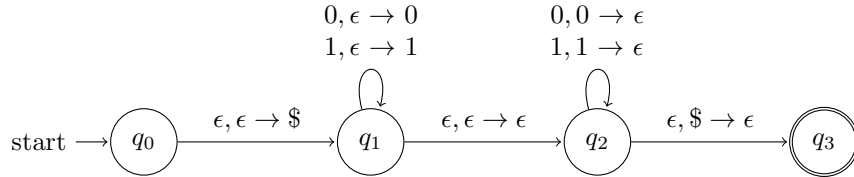
Based on these differences, we can begin to construct different models of computation that have just enough power to solve each problem. The first problem can be solved by a *Finite State Automaton (FSA)*, a collection of a finite number of states, a transition function, and a set of accepting states. The FSA accepts the input if the sequential application of each element to the input finishes in an accepting state, and rejects otherwise. Finite State Automata are often represented by charts as in Figure 1, a representation of an FSA that decides the first problem.

FIGURE 1. An FSA that determines whether a string ends in ‘a’.



The second problem can be solved by a machine called a *Pushdown Automaton (PDA)*, which is essentially a finite state automaton with the addition of a stack. Rather than transforming the current state and the input into a new state, as with a finite state automata, the transition function of a PDA also considers the stack, which it can pop and push from. Within the collection of PDAs, there is another division between *Deterministic Pushdown Automata (DPDAs)* and *Nondeterministic Pushdown Automata*. For DPDAs, the transition function only outputs one move, rather than a set of moves. Nondeterministic PDAs are able to recognize more languages than DPDAs. Figure 2 is a graphical representation of a PDA that decides palindromes. In this representation, ϵ is an empty input, and $\$$ is a special symbol that denotes the bottom of the stack. The moves are represented by a double of input and stack operation.

FIGURE 2. A Pushdown Automaton that decides palindromes over the alphabet a, b



Finally, we arrive at the Turing Machine (TM), devised by Alan Turing in his paper *On Computable Numbers, With an Application To the Entscheidungsproblem* [21]. Turing Machines offer an unrestricted model of computation,

Discussion of TMs

3. UNDECIDABILITY

Problem 3.1 (The Halting Problem). [21] [17]

Theorem 3.2 (Rice's Theorem). [10]

3.1. Incompleteness.

Theorem 3.3 (Gödel's Incompleteness Theorem). [7]

3.2. Reducibility. While it is possible to prove that a problem is undecidable directly, as in problem 3.1, it is often more convenient to prove undecidability through comparison to problems which are already known to be undecidable. This comparison takes place through the technique of Turing reduction.

Definition 3.4. Given two decision problems A and B , we say that A is Turing reducible to B if, given a machine D_B that decides B , it is possible to construct a machine D_A that decides A .

So, we have that if A is reducible to B , then A can be no harder than B , because any solution to B also leads to a solution of A . So, if we have some problem B that we would like to prove is undecidable, we can do so by showing that some problem A which is already known to be undecidable is reducible to B . Since A cannot be harder than B , it follows that B must also be undecidable. Alternatively, if we would like to show that some problem A is decidable, it is sufficient to show that it is reducible to some problem B that is known to be decidable. [17] [12]

[9]

4. CONTEXT FREE GRAMMARS

Definition 4.1. A Context Free Grammar is...

Definition 4.2. A Pushdown Automata is...

Theorem 4.3 (Equivalence of CFGs and Pushdown Automata). [6]

Theorem 4.4 (Closure Properties). [17] *Union, Concat, Star, intersection with regular, substitution*

Problem 4.5 (Emptiness). *Let G be a context-free grammar. Is $L(G) = \emptyset$?*

Problem 4.6 (Finite). *Let G be a context-free grammar. Is $L(G)$ finite?*

Problem 4.7 (Regular Containment). *Take some context-free grammar G and some regular language R . Is $L(G) \subseteq R$?*

Proof of Decidability. We have that $L(G) \subseteq R$ if and only if $\overline{L(G)} \cup R = \Sigma^*$. So, it follows that $L(G)$ is contained by R if and only iff $\overline{L(G)} \cup R = \emptyset$. By DeMorgan's laws, we have that this is equivalent to the statement $L(G) \cap \overline{R} = \emptyset$. Regular languages are closed under complement and context-free languages are closed under intersection with a regular language, so it follows that $L(G) \cap \overline{R}$ is a context free language. So the problem of containment by a regular language is reducible to Problem 4.5, the problem of emptiness, which is decidable. So, the problem of containment by a regular language is decidable. \square

4.1. Undecidable Problems.

Problem 4.8 (The Post Correspondence Problem). *Consider some alphabet Σ and two finite lists of words over Σ denoted $A = a_1, \dots, a_n$ and $B = b_1, \dots, b_n$. Then, does there exist some sequence of indices (i_k) , for $1 \leq k \leq K$ and for $K \geq 1$, and with $1 \leq i_k \leq n$ for all k such that*

$$a_{i_1} \dots a_{i_K} = b_{i_1} \dots b_{i_K}?$$

Solution. This problem is shown to be undecidable through a reduction to the halting problem. The proof is technical, and involves encoding the computation history of a Turing machine on an input in such a way that the encoding satisfies the PCP if and only if the Turing machine accepts the input. It is described in detail by Sipser in his book *Introduction to the Theory of Computation* [17]. \square

The Post correspondence problem is a useful tool, because it allows us to demonstrate the undecidability of problems without doing the complex reasoning about Turing machines that the halting problem requires. We will now show the undecidability of a variety of problems about context-free grammars through reduction to the Post correspondence problem.

Problem 4.9 (Disjointness). *Let P, Q be context-free grammars. Is $L(P) \cap L(Q) = \emptyset$?*

Proof of Undecidability. Consider the Post correspondence problem for two lists of words A, B . We construct two context free grammars from these lists as follows:

$$\begin{array}{ll} G_A \rightarrow a_1 1 & G_B \rightarrow a_1 1 \\ \vdots & \vdots \\ G_A \rightarrow a_n n & G_B \rightarrow a_n n \\ G_A \rightarrow a_1 G_A 1 & G_B \rightarrow a_1 G_B 1 \\ \vdots & \vdots \\ G_A \rightarrow a_n G_A n & G_B \rightarrow a_n G_B n \end{array}$$

Then, we can observe that for some string s to exist in both $L(G_A)$ and $L(G_B)$, it must be a solution to the Post correspondence for A, B . So, if $L(G_A) \cap L(G_B) = \emptyset$, then there are no solutions to the Post correspondence for A, B . So, it follows that the Post correspondence problem is reducible to the problem of the disjointness of context-free grammars, so since the Post correspondence problem is undecidable, it follows that the disjointness problem is undecidable. \square

Problem 4.10 (Universality). *Let G be some context-free grammar over an alphabet Σ . Then, is*

$$L(G) = \Sigma^*?$$

Proof of Undecidability. \square

Definition 4.11. An ambiguous grammar is...

Problem 4.12 (Ambiguity). *Let G be a context-free grammar. Is G ambiguous?*

Proof of Undecidability. Consider the Post correspondence problem for two lists of words A, B . Construct their corresponding grammars G_A, G_B . Now, consider the grammar

$$G \rightarrow G_A | G_B.$$

It follows that the ambiguity of G implies that a solution to the Post correspondence problem for A, B exists, so the Post correspondence problem reduces to the ambiguity problem, so the ambiguity problem is undecidable. \square

[4] [5]

Problem 4.13. *Let G be a context free grammars. Is $L(G)$ regular?*

Proof of Undecidability. Note that Σ^* is regular, so if $L(G)$ is not regular then it follows that $L(G) \neq \Sigma^*$. Additionally, universality is decidable within regular languages, because they are closed under complement and emptiness is decidable even within context-free grammars. So, we have that Problem 4.10 reduces to the regularity problem. It follows that the regularity problem is undecidable. \square

Problem 4.14 (Equality). *Let G_1, G_2 be context free grammars. Is $L(G_1) = L(G_2)$?*

Proof of Undecidability. Note that Σ^* is regular, so it is also a context-free. So, we have that Problem 4.10 reduces to the equality problem. It follows that the equality problem is undecidable. \square

Problem 4.15 (Inclusion). *Let G_1, G_2 be context free grammars. Is $L(G_1) \subseteq L(G_2)$?*

Proof of Undecidability. As before, note that Σ^* is regular, so it is also a context-free. Additionally, observe that for any grammar G , $L(G) \subseteq \Sigma^*$. So, we have that Problem 4.10 reduces to the inclusion problem, since $\Sigma^* \subseteq L(G) \implies \Sigma^* = L(G)$. It follows that the inclusion problem is undecidable. \square

4.1.1. *Problems decidable for Deterministic CFLs.* [3]

Definition 4.16. A deterministic context-free language is a language accepted by a deterministic pushdown automata.

Theorem 4.17 (Closure Properties). *If L is a deterministic context-free language, then \bar{L} is a deterministic context-free language.*

Problem 4.18. *Let L be a deterministic context-free language. Is $L = \Sigma^*$?*

The decidability of the following problem was an open problem in the field of computability theory from 1965, when it was introduced by Ginsburg and Greibach [3]. In 1997, when it was solved by Géraud Sénizergues [15]. A sketch of the proof follows.

Problem 4.19 (The Equivalence Problem for Deterministic CFGs). *Let L_1, L_2 be deterministic context-free languages. Is $L_1 = L_2$?*

5. TURING DEGREES

Definition 5.1. A Turing degree... [12] [8]

Definition 5.2. The jump operator...

Definition 5.3. Computably Enumerable

Definition 5.4. Completeness

5.1. Properties and Structure.

Problem 5.5 (Post’s Problem). [12] [2]

Lemma 5.6. D does not form a lattice [8]

Theorem 5.7. D forms an upper semi-lattice [8]

Theorem 5.8. Existence of minimal degrees [20] [16]

Theorem 5.9. The c.e. degrees are dense [14]

Homogeneity problems

Theory of D Undecidable

5.2. Turing Degrees of Problems related to CFGs. [13]

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