UNDECIDABILITY AND THE STRUCTURE OF THE TURING DEGREES

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Abstract. This paper explores the structure and properties of the Turing degrees, or degrees of undecidability. We introduce the Turing machine, an abstract model of computation, in order to develop the concepts of undecidability and Turing reduction. We demonstrate the technique of proof by reduction through a series of examples of undecidable problems related to context-free grammars. We then employ reducibility to consider a partial ordering on the set of Turing degrees, $\mathcal{D}$. Finally, we prove a variety of theorems related to the structure of $\mathcal{D}$.

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1. Introduction

Historical overview of the field, motivation for study [1] [19] [18] [11]

2. Turing Machines

[21]

3. Undecidability

Problem 3.1 (The Halting Problem)**.** *[21] [17]*

Theorem 3.2 (Rice's Theorem)**.** *[10]*

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3.1. **Incompleteness.**

Theorem 3.3 (Gödel's Incompleteness Theorem)**.** *[7]*

4. Reducibility

[17] [12]
[9]

5. Context Free Grammars

Definition 5.1. A Context Free Grammar is...

Definition 5.2. A Pushdown Automata is...

Theorem 5.3 (Equivalence of CFGs and Pushdown Automata)**.** *[6]*

Theorem 5.4 (Closure Properties)**.** *[17] Union, Concat, Star, intersection with regular, substitution*

Problem 5.5. *Empty, finite*

5.1. **Undecidable Problems.**

Problem 5.6 (The Post Correspondence Problem)**.** *[6]*

Problem 5.7. *Undecidability of ambiguity*

[4] [5]

Problem 5.8. *Universality*

Problem 5.9. *Equality, inclusion*

Problem 5.10. *Disjointness*

5.1.1. *Problems decidable for Deterministic CFGs.* [3]

Definition 5.11. A Deterministic Context Free Grammar is... [17]

Theorem 5.12 (Closure Properties)**.** *[17] Complement*

Problem 5.13 (The Equivalence Problem for Deterministic CFGs)**.** *[15]*

Problem 5.14. *Universality*

6. Turing Degrees

Definition 6.1. A Turing degree... [12] [8]

Definition 6.2. The jump operator...

Definition 6.3. Computably Enumerable

Definition 6.4. Completeness

6.1. **Properties and Structure.**

Problem 6.5 (Post's Problem)**.** *[12] [2]*

Lemma 6.6. *D does not form a lattice [8]*

Theorem 6.7. *D forms an upper semi-lattice [8]*

Theorem 6.8. *Existence of minimal degrees [20] [16]*

Theorem 6.9. *The c.e. degrees are dense [14]*

Homogeneity problems
Theory of D Undecidable

6.2. **Turing Degrees of Problems related to CFGs.** [13]

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