

UNDECIDABILITY AND THE STRUCTURE OF THE TURING DEGREES

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ABSTRACT. This paper explores the structure and properties of the Turing degrees, or degrees of undecidability. We introduce the Turing machine, an abstract model of computation, in order to develop the concepts of undecidability and Turing reduction. We demonstrate the technique of proof by reduction through a series of examples of undecidable problems related to context-free grammars. We then employ reducibility to consider a partial ordering on the set of Turing degrees, \mathcal{D} . Finally, we prove a variety of theorems related to the structure of \mathcal{D} .

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1. INTRODUCTION

Historical overview of the field, motivation for study [1] [19] [18] [11]

1.1. Decision Problems.

2. TURING MACHINES

[21]

3. UNDECIDABILITY

Problem 3.1 (The Halting Problem). [21] [17]

Theorem 3.2 (Rice's Theorem). [10]

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3.1. Incompleteness.

Theorem 3.3 (Gödel's Incompleteness Theorem). [7]

3.2. Reducibility. While it is possible to prove that a problem is undecidable directly, as in problem 3.1, it is often more convenient to prove undecidability through comparison to problems which are already known to be undecidable. This comparison takes place through the technique of Turing reduction.

Definition 3.4. Given two decision problems A and B , we say that A is Turing reducible to B if, given a machine D_B that decides B , it is possible to construct a machine D_A that decides A .

So, we have that if A is reducible to B , then A can be no harder than B , because any solution to B also leads to a solution of A . So, if we have some problem B that we would like to prove is undecidable, we can do so by showing that some problem A which is already known to be undecidable is reducible to B . Since A cannot be harder than B , it follows that B must also be undecidable. Alternatively, if we would like to show that some problem A is decidable, it is sufficient to show that it is reducible to some problem B that is known to be decidable. [17] [12]

[9]

4. CONTEXT FREE GRAMMARS

Definition 4.1. A Context Free Grammar is...

Definition 4.2. A Pushdown Automata is...

Theorem 4.3 (Equivalence of CFGs and Pushdown Automata). [6]

Theorem 4.4 (Closure Properties). [17] *Union, Concat, Star, intersection with regular, substitution*

Problem 4.5. *Empty, finite, containment within regular*

4.1. Undecidable Problems.

Problem 4.6 (The Post Correspondence Problem). *Consider some alphabet Σ and two finite lists of words over Σ denoted $A = a_1, \dots, a_n$ and $B = b_1, \dots, b_n$. Then, does there exist some sequence of indices (i_k) , for $1 \leq k \leq K$ and for $K \geq 1$, and with $1 \leq i_k \leq n$ for all k such that*

$$a_{i_1} \dots a_{i_K} = b_{i_1} \dots b_{i_K}?$$

Solution. This problem is shown to be undecidable through a reduction to the halting problem. The proof involves [17] □

Describe proof in detail

The Post correspondence problem is a useful tool, because it allows us to demonstrate the undecidability of problems without doing the complex reasoning about Turing machines that the halting problem requires. We will now show the undecidability of a variety of problems about context-free grammars through reduction to the Post correspondence problem.

Problem 4.7 (Disjointness). *Given two context-free grammars P, Q , is $L(P) \cap L(Q) = \emptyset$?*

Proof of Undecidability. Consider the Post correspondence problem for two lists of words A, B . We construct two context free grammars from these lists as follows:

$$\begin{array}{ll} G_A \rightarrow a_1 1 & G_B \rightarrow a_1 1 \\ \vdots & \vdots \\ G_A \rightarrow a_n n & G_B \rightarrow a_n n \\ G_A \rightarrow a_1 G_A 1 & G_B \rightarrow a_1 G_B 1 \\ \vdots & \vdots \\ G_A \rightarrow a_n G_A n & G_B \rightarrow a_n G_B n \end{array}$$

Then, we can observe that for some string s to exist in both $L(G_A)$ and $L(G_B)$, it must be a solution to the Post correspondence for A, B . So, if $L(G_A) \cap L(G_B) = \emptyset$, then there are no solutions to the Post correspondence for A, B . So, it follows that the Post correspondence problem is reducible to the problem of the disjointness of context-free grammars, so since the Post correspondence problem is undecidable, it follows that the disjointness problem is undecidable. \square

Problem 4.8 (Universality). *Take some context-free grammar G over an alphabet Σ . Then, is*

$$L(G) = \Sigma^*?$$

Proof of Undecidability. \square

Definition 4.9. An ambiguous grammar is...

Problem 4.10 (Ambiguity). *Take some context free grammar G . Is G ambiguous?*

Proof of Undecidability. Consider the Post correspondence problem for two lists of words A, B . Construct their corresponding grammars G_A, G_B . Now, consider the grammar

$$G \rightarrow G_A | G_B.$$

It follows that the ambiguity of G implies that a solution to the Post correspondence problem for A, B exists, so the Post correspondence problem reduces to the ambiguity problem, so the ambiguity problem is undecidable. \square

[4] [5]

Problem 4.11. *Regularity*

Problem 4.12 (Equality). *Take context free grammars G_1, G_2 . Is $L(G_1) = L(G_2)$?*

Proof of Undecidability. Note that Σ^* is regular, so it is also a context-free. So, we have that Problem 4.8 reduces to the equality problem. It follows that the equality problem is undecidable. \square

Problem 4.13 (Inclusion). *Take context free grammars G_1, G_2 . Is $L(G_1) \subseteq L(G_2)$?*

Proof of Undecidability. As before, note that Σ^* is regular, so it is also a context-free. Additionally, observe that for any grammar G , $L(G) \subseteq \Sigma^*$. So, we have that Problem 4.8 reduces to the inclusion problem, since $\Sigma^* \subseteq L(G) \implies \Sigma^* = L(G)$. It follows that the inclusion problem is undecidable. \square

4.1.1. *Problems decidable for Deterministic CFGs.* [3]

Definition 4.14. A Deterministic Context Free Grammar is... [17]

Theorem 4.15 (Closure Properties). [17] *Complement*

Problem 4.16 (The Equivalence Problem for Deterministic CFGs). [15]

Problem 4.17. *Universality*

5. TURING DEGREES

Definition 5.1. A Turing degree... [12] [8]

Definition 5.2. The jump operator...

Definition 5.3. Computably Enumerable

Definition 5.4. Completeness

5.1. Properties and Structure.

Problem 5.5 (Post's Problem). [12] [2]

Lemma 5.6. *D does not form a lattice* [8]

Theorem 5.7. *D forms an upper semi-lattice* [8]

Theorem 5.8. *Existence of minimal degrees* [20] [16]

Theorem 5.9. *The c.e. degrees are dense* [14]

Homogeneity problems

Theory of D Undecidable

5.2. Turing Degrees of Problems related to CFGs. [13]

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