# UNDECIDABILITY AND THE STRUCTURE OF THE TURING DEGREES

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ABSTRACT. This paper explores the structure and properties of the Turing degrees, or degrees of undecidability. We introduce the Turing machine, an abstract model of computation, in order to develop the concepts of undecidability and Turing reduction. We demonstrate the technique of proof by reduction through a series of examples of undecidable problems related to context-free grammars. We then employ reducibility to consider a partial ordering on the set of Turing degrees,  $\mathcal{D}$ . Finally, we prove a variety of theorems related to the structure of  $\mathcal{D}$ .

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## 1. Introduction

Historical overview of the field, motivation for study [1] [19] [18] [11]

### 2. Turing Machines

[21]

## 3. Undecidability

Problem 3.1 (The Halting Problem). [21] [17]

**Theorem 3.2** (Rice's Theorem). [10]

Date: July 18, 2018.

## 3.1. Incompleteness.

**Theorem 3.3** (Gödel's Incompleteness Theorem). [7]

## 4. Reducibility

[17] [12] [9]

#### 5. Context Free Grammars

**Definition 5.1.** A Context Free Grammar is...

**Definition 5.2.** A Pushdown Automata is...

**Theorem 5.3** (Equivalence of CFGs and Pushdown Automata). [6]

**Theorem 5.4** (Closure Properties). [17] Union, Concat, Star, intersection with regular, substitution

Problem 5.5. Empty, finite

#### 5.1. Undecidable Problems.

**Problem 5.6** (The Post Correspondence Problem). [6]

Problem 5.7. Undecidability of ambiguity

[4] [5]

Problem 5.8. Universality

Problem 5.9. Equality, inclusion

Problem 5.10. Disjointness

5.1.1. Problems decidable for Deterministic CFGs. [3]

**Definition 5.11.** A Deterministic Context Free Grammar is... [17]

**Theorem 5.12** (Closure Properties). [17] Complement

**Problem 5.13** (The Equivalence Problem for Deterministic CFGs). [15]

Problem 5.14. Universality

## 6. Turing Degrees

**Definition 6.1.** A Turing degree... [12] [8]

**Definition 6.2.** The jump operator...

**Definition 6.3.** Computably Enumerable

**Definition 6.4.** Completeness

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## 6.1. Properties and Structure.

**Problem 6.5** (Post's Problem). [12] [2]

Lemma 6.6. D does not form a lattice [8]

**Theorem 6.7.** D forms an upper semi-lattice [8]

**Theorem 6.8.** Existence of minimal degrees [20] [16]

**Theorem 6.9.** The c.e. degrees are dense [14]

Homogeneity problems Theory of D Undecidable

## 6.2. Turing Degrees of Problems related to CFGs. [13]

#### Acknowledgments

It is a pleasure to thank my mentor, Ronno Das, for supervising this project and providing valuable feedback and advice. I would also like to thank Peter May for organizing this REU, and Daniil Rudenko for running the Apprentice Program.

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